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Chi-square Mixture of Chi-square Distributions

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Abstract: In this study, Chi-square mixture of Chi-square distribution has been defined and determined some characteristics of the distribution. The distribution is always positively skewed and leptokurtic for any value of the parameters.

Key words: Chi-square mixtures, Chi-square distribution, positively skewed, Leptokurtic

INTRODUCTION

A mixture of distributions is a weighted average of probability distribution with positive weights that sum to one. The distributions thus mixed are called the components of the mixture. The weights themselves comprise a probability distribution called the mixing distribution. Because of these weights, a mixture is in particular again a probability distribution. Probability distributions of this type arise when observed phenomena can be the consequence of two or more related, but usually unobserved phenomena, each of which leads to a different probability distribution. Mixtures and related structures often arise in the construction of probabilistic models. Pearson[1] was the first researcher in the field of mixture distributions who considered the mixture of two normal distributions. After the study of Pearson[1] there was a log gap in the field of mixture distributions. Decay[2] has improved the results of Pearson[1], Hasselbled[3] studied in grater detail about the finite mixture of distributions. Now the following definition can be used:

Definition 1: A random variable X is said to have a mixture distribution if the distribution of X depends on a quantity that also has a distribution. A typical example of mixture distribution is as follows:
Let us consider the following hierarchy

$$X/K \sim \chi^2_{p+2k}$$
$$K \sim \text{Poison}\ (\lambda)$$

This indicates that
The marginal density of X is obtained as:

$$P[X=x] = \sum_{k=0}^{\infty} P(X = x, K = k)$$

$$= \sum_{k=0}^{\infty} P(X = x, K = k)\ .P[K=k] \qquad (1)$$

Now if we consider $X/(K=k)$ is χ^2_{p+2k} and K is Poisson (λ), the marginal density of X can be obtained as

$$f(x \mid \lambda;p) = \sum_{k=0}^{\infty} \frac{\chi^{\frac{p}{2}+k-1} e^{-\frac{x}{2}}}{2^{\frac{p}{2}+k}\Gamma(\frac{p}{2}+k)} . \frac{e^{-\lambda}\lambda^k}{k!} \qquad (2)$$

which is nothing but the non-central Chi-square distribution with degrees of freedom p and non-centrality parameter λ. Thus non-central chi-square distribution is considered as a Poisson mixture of Chi-square distributions.

PRELIMINARIES

Mixtures mostly occur when the parameter θ of a family of distributions, given by the density function $f(x;\theta)$, is itself subject to the chance variation.

The general formula for the finite mixture is:

$$\sum_{k=1}^{p} f(x;\theta_k)g(\theta_k) \qquad (3)$$

and the infinite analogue is:

$$\int f(x;\theta)g(\theta)d\theta \qquad (4)$$

where, $g(\theta)$ is a density function.

Poisson, Binomial, Negative binomial, t-moments mixtures of distributions have been defined as follows.

Definition 2: Roy *et al.*[4] defined that a random variable X is said to have a Poisson mixture of distributions, if its density function is given by:

$$\sum_{\theta=0}^{\infty} \frac{e^{-\lambda}\lambda^{\theta}}{\theta!} f(x;\theta) \qquad (5)$$

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Definition 3: Roy *et al.*[5] defined that a random variable X is said to have a binomial mixture of distributions, if its density function is given by:

$$\sum_{\theta=0}^{N} \binom{N}{\theta} p^{\theta}(1-p)^{N-\theta} f(x;\theta) \qquad (6)$$

where, $f(x;\theta)$ is a density function.

Definition 4: Roy and Sinha[6] defined that a random variable X is said to have a negative binomial mixture of distributions, if its density function is given by:

$$\sum_{k=0}^{n} \binom{k+r-1}{r-1} p^{r} q^{k} g(x;r,\theta) \qquad (7)$$

where, $g(x;r,\theta)$ is a density function.

Definition 6: Zaman *et al.*[7] defined that a random variable X is defined to be a chi-square mixture of binomial distributions with v d.f. and parameters n and p, if its density function is defined by:

$$f(x;v,n,p)=\int_0^{\infty} \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\,\Gamma\frac{v}{2}} \binom{n+\chi^2}{x} p^{x}(1\text{-}p)$$

$$n+\chi^2-x \quad d\chi^2 \qquad (8)$$

$$;x=0,1,2,\ldots\ldots\ldots\ldots, n+\chi^2 \quad n>0 \text{ and } 0<p<1$$

OBJECTIVES

The objective of the study was to derive the Chi-square mixture of Chi-square distribution and to have their properties.

CHI-SQUARE MIXTURED DISTRIBUTION

Definition 7: A random variable X having the density function

$$f(x;v,\theta,p)=\int_0^{\infty} \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\,\Gamma\frac{v}{2}} g(x;\theta,p)\, d\chi^2 \;;\; -\infty<x<\infty \qquad (9)$$

is said to have a Chi-square mixtured distributions with n d.f. where, $g(x;\theta,p)$ is a density function. The name Chi-square mixture of distributions is due to the fact that integral values $f(x;v,\theta,p)$ in the derived distribution in Eq. (1) is equal to one with weights equal to the ordinates of chi-square distribution having ν d.f.

Definition 8: A random variable X is defined to be a Chi-square mixture of Chi-square distributions with v and n d.f. if its density function is defined by :

$$f(x:v,n)=\int_0^{\infty} \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\,\Gamma\frac{v}{2}} \;\frac{e^{-\frac{x}{2}}x^{\frac{n}{2}+\chi^2-1}}{\Gamma\left(\frac{n}{2}+\chi^2\right).2^{\frac{n}{2}+\chi^2}} \; d\chi^2 \qquad (10)$$

$$0<x<\infty$$

and hence the total probability of this distribution is unity

$$\int_{x=0}^{\infty} f(x:v,n)dx = \int_{x=0}^{\infty}\int_0^{\infty} \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\,\Gamma\frac{v}{2}} \;\frac{e^{-\frac{x}{2}}x^{\frac{n}{2}+\chi^2-1}}{\Gamma\left(\frac{n}{2}+\chi^2\right).2^{\frac{n}{2}+\chi^2}}\, d\chi^2 dx$$

$$\int_0^{\infty} \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\,\Gamma\frac{v}{2}} \;\frac{\theta^{\frac{n}{2}+\chi^2}}{\Gamma(\frac{n}{2}+\chi^2)} \;\frac{\Gamma(\frac{n}{2}+\chi^2)}{\theta^{\frac{n}{2}+\chi^2}}\, d\chi^2$$

$$=\frac{1}{\Gamma\frac{v}{2}\;2^{\frac{v}{2}}} \bullet \frac{\Gamma\frac{v}{2}}{(\frac{1}{2})^{\frac{v}{2}}}$$

$$=1$$

RESULTS

The Chi-square mixture of Chi-square distributions has been defined in the light of the Chi-square mixtured distributions. The results of the study are presented in the form of theorems.

Chi-square mixture of Chi-square distributions

Theorem 1: If X follows a Chi-square mixture of Chi-square distributions with v d.f. and parameter λ then the characteristics function of the distribution is given by:

$$\Phi_x(t) = (1-2it)^{-\frac{n}{2}}\{1+2\log(1-2it)\}^{\frac{v}{2}}$$

and hence

Mean $= n+2v$ Variance $= 2n+12v$

$$\beta_1 = 8\frac{(n+16v)^2}{(n+6v)^3} \text{ and } \beta_2 = 3+\frac{12n+496v)}{(n+6v)^2}$$

Proof: The characteristics function of the distribution given by:

$$\Phi_x(t) = E[e^{itx}]$$

$$= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\,\Gamma\frac{v}{2}} \int_0^\infty e^{itx} \frac{e^{-\frac{x}{2}}x^{\frac{n}{2}+\chi^2-1}}{\Gamma\left(\frac{n}{2}+\chi^2\right).2^{\frac{n}{2}+\chi^2}}\, dx\; d\chi^2$$

$$= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\,\Gamma\frac{v}{2}} \frac{1}{\Gamma\left(\frac{n}{2}+\chi^2\right).2^{\frac{n}{2}+\chi^2}} \int_0^\infty e^{-\frac{x}{2}}x^{\frac{n}{2}+\chi^2-1} e^{itx} dx\, d\chi^2$$

$$= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\,\Gamma\frac{v}{2}} \frac{1}{\Gamma\left(\frac{n}{2}+\chi^2\right).2^{\frac{n}{2}+\chi^2}} \int_0^\infty x^{\frac{n}{2}+\chi^2-1} e^{-\left(\frac{1}{2}-it\right)x} dx\, d\chi^2$$

$$= \int_0^\infty \frac{e^{-\frac{\chi^2}{2}}(\chi^2)^{\frac{v}{2}-1}}{2^{\frac{v}{2}}\,\Gamma\frac{v}{2}} \frac{1}{\Gamma\left(\frac{n}{2}+\chi^2\right).2^{\frac{n}{2}+\chi^2}} \cdot \frac{\Gamma\left(\frac{n}{2}+\chi^2\right)}{\left(\frac{1}{2}-it\right)^{\frac{n}{2}+\chi^2}}\; d\chi^2$$

on simplifications,

$$\Phi_x(t) = (1-2it)^{-\frac{n}{2}}\{1+2\log(1-2it)\}^{-\frac{v}{2}}$$

Now the cumulant generating function is

$$K_x(t) = \log \Phi_x(t)$$

$$= \log\left\{(1-2it)^{-\frac{n}{2}}\{1+2\log(1-2it)\}^{-\frac{v}{2}}\right\}$$

$$= -\frac{n}{2}\log(1\text{-}2it) - \frac{v}{2}\log\{1+2\log(1\text{-}2it)\}$$

on simplifications,

$$K_x(t) = (n+2v)it + (2n+12v)\frac{(it)^2}{2!} + (8n+128v)\frac{(it)^3}{3!} + (48n+1984v)\frac{(it)^4}{4!} + \text{--------------------}$$

$\therefore$ mean $= \kappa_1 =$ Co-efficient of (it) in $K_x(t) = n+2v$

Variance $= \kappa_2 =$ Co-efficient of $\frac{(it)^2}{2!}$ in $K_x(t) = 2n+12v$

$\mu_3 = \kappa_3 =$ Co-efficient of $\frac{(it)^3}{3!}$ in $K_x(t) = 8n+128v$

$\kappa_4 =$ Co-efficient of $\frac{(it)^4}{4!}$ in $K_x(t) = 48n+1984v$

$$\mu_4 = \kappa_4 + 3\kappa_2^2$$

$$= 48n+1984v+3(2n+12v)^2$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(8n+128v)^2}{(2n+12v)^3} = \frac{8(n+16v)^2}{(n+6v)^3}$$

So the coefficient of skewness

$$\gamma_1 = \frac{\mu_3}{\sqrt{\mu_2^3}} = \frac{2\sqrt{2}(n+16v)}{(n+6v)^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^3} = \frac{48n+1984v+3(2n+12v)^2}{(2n+6v)^2} = 3 + \frac{12n+496v}{(n+6v)^2}$$

So the coefficient of kurtosis

$$\gamma_2 = \beta_2 - 3 = \frac{12n+496v}{(n+6v)^2}$$

Some important properties of the chi-square mixture of chi-square distribution

- The mean and variance of the distribution is *n+2v* and 2n+12v, respectively.
- If v=0, then the distribution reduces to chi-square distribution
- The distribution is always positively skewed.

The shape of the distribution is always leptokurtic for the any value of the parameter: In order to find

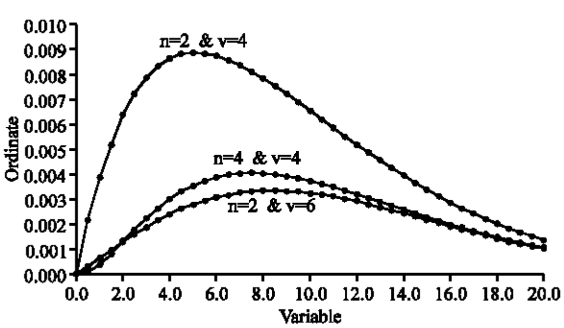

Fig. 1: Area curve of the chi-square mixture of Chi-square distribution

out the shape characteristics of the chi-square mixture of chi-square distributions, different area curves were constructed for different value of the parameters. These curves are presented in the Fig. 1. It is observed that the mixture distribution is positively skewed and leptokurtic.

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