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A Positive Linear Operator Using Probabilistic Approach

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Abstract: A positive linear operator, modifying the well-known Bernstein Polynomial (Positive Linear Operator)'s weight function has been proposed. It uses a new weight functions derived using a probabilistic approach. The resultant polynomial approximation operator has been studied analytically for its convergence rate, as also empirically using a simulation study to compare it with the Bernstein's polynomial approximation operator. The approximating polynomials have been computed for certain selected functions and with different number of node points in the interval of approximation C [0, 1], without any loss of generality. It has been observed that the proposed polynomial approximation formula produces significantly better approximating polynomials.

Key words: Polynomial approximation, positive linear operator, probabilistic weight function, maple

INTRODUCTION

Readers of this study, if not conversant with polynomial approximation; numerical analysis set-up, would do well to refer to Carothers (1998, 2000) and Lorentz (1986), Cheney and Kincaid (1994), Hartley and Wynn-Evans (1979) and Polybon (1992). In this study we have focused on the betterment of the well-known Bernsteins (Positive Linear Operator) Polynomial, using the probabilistic approach to modify the weight-function therein. The resultant polynomial approximation operator has been studied analytically for its convergence rate, as also empirically using a simulation study to compare it with the Bernstein's polynomial approximation operator. The approximating polynomials have been computed for certain selected functions and with different number of node points in the interval of approximation C [0, 1], without any loss of generality. It has been observed that the proposed polynomial approximation formula produces significantly better approximating polynomials.

THE PROPOSED PROBABILISTIC POLYNOMIAL OPERATOR

As mentioned earlier, likewise Bernstein Polynomial we have assumed that the limits of the interval of approximation are zero to one, without any loss of generality. As such, the interval of integration happens to be (0, 1). Also, while x0 = 0 and xn = 1 and we consider equidistant nodes (right-limits of n intervals (with x0 as the left-node of the first interval: the beginning of the beginning):

$$xi\ (= i/n),\ i = 1, 2, \ldots, n \qquad (1)$$

Thus, we have n equal-width intervals of points on the line (0, 1). Let us visualize a randomly sitting point x on this line. It is obvious to conclude that the probability of a point (in one of these equal-width intervals of points) on this line being less than x, i.e., on its left on the line is x; whereas the probability of a point (in one of these equal-width intervals of the points) on this line being more than x, i.e., on its right, on the line is 1-x; i.e.,

$$p(X < x) = x \text{ and } p(X > x) = 1\text{-}x \qquad (2)$$

Hence, equally obviously, the expected number of points out of say, n equidistant points/nodes on the line which are on the left of the point x (or smaller than x) will be nx and the expected number of points out of say n equidistant points on the line which are on the right of the point x (or greater than x) will be n-nx or equivalently n (1-x).

Now, to devise the weight function wk (x) associated with the node xk, we place it in the shoes of x.

However, we know that according to our choice of the n intervals in Eq. 1, for any node xk there are k nodes on the left of the node xk (including it, i.e., ≤xk) and that there are (n-k) nodes on the right of the node xk.

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Consequently, in this probabilistic setup the probability of our choice of the node xk is:

$$\frac{\binom{nx}{k}\binom{n(1-x)}{n-k}}{\binom{n}{n}} \qquad (3)$$

Observe that $\binom{n}{n} = 1$, here for each xk:

$$wk(x) = \binom{nx}{k}.\binom{n(1-x)}{n-k} \qquad (4)$$

It could well be noted here that in case of the Bernstein Polynomial we have:

$$\text{Say, } bwk(x) = \binom{n}{k} x^k.\ (1\text{-}x)^{n\text{-}k}.$$

Equation 4 might well be expressed in terms of the well-known Gamma functions for computational purposes to accommodate any real value of x in (0, 1).

Therefore, our proposed polynomial approximation formula (resultant upon using the aforesaid probabilistic perspective' for the weight function wk (x) therein) is:

$$P_n(f, x) = \sum_{k=0}^{k=n} wk(x).f\left(\frac{k}{n}\right) \qquad (5)$$

where xk (= k/n)'s are the nodes given in Eq. 1 and wi(x)'s are the weight functions given in Eq. 4.

ANALYTICAL STUDY OF PROPOSED PROBABILISTIC POSITIVE LINEAR OPERATOR

We could note the important but very simple facts from the following Lemma.

Lemma 1

$$\sum_{k=0}^{k=n} \binom{nx-m}{k} \binom{n-nx}{n-m-k} = 1$$

Proof: We start by noting the identity:

$$(1\text{-}\ \theta)^{n\text{-}m} = (1\text{-}\ \theta)^{nx\text{-}m}\ .\ (1\text{-}\ \theta)^{n\text{-}nx}$$

Now, if we expand the binomials on both the sides, we easily are led to the following equivalence by equating the coefficients of $\theta^{n\text{-}m}$ on both the sides:

$$\sum_{k=0}^{k=n} \binom{nx-m}{k} \binom{n-nx}{n-m-k} = 1$$

QED

Following straightforward 'Corollaries' will be of use in what follows (By putting m = 0, 1 and 2, respectively, in the above result).

$$\sum_{k=0}^{k=n} \binom{nx}{k} \binom{n-nx}{n-k} = 1.\text{, say, CR\#1}$$

$$\sum_{k=0}^{k=n} \binom{nx-1}{k} \binom{n-nx}{n-1-k} = 1.\text{, say, CR\#2}$$

$$\sum_{k=0}^{k=n} \binom{nx-2}{k} \binom{n-nx}{n-2-k} = 1.\text{, say, CR\#3}$$

The Corollary CR#1 is the basically important one for our proposed Probabilistic Positive Linear operator, say $PP_n(f)$ (x):

$$PP_n\ (f)\ (x) = \sum_{k=0}^{k=n} \binom{nx}{k} \binom{n-nx}{n-k} f(k/n)$$

Next, we proceed to prove the following theorem for our Probabilistic Positive Linear operator.

Theorem 1

For P_n (f) (x), we have:

- $P_n\ (f_0(x)) = f_0(x)$,
- $P_n\ (f_1(x)) = f_1(x)$,
- $P_n\ (f_2(x)) = f_2(x)$;

Wherein, $f_0(x) = x^0$, $f_1(x) = x$ and $f_2(x) = x^2$.
And also,

- $\sum_{k=0}^{k=n} (\frac{k}{n} - x)^2 \binom{nx}{k} \binom{n-nx}{n-k} = 0$

Proof: While the proof for 1 to 3 above is rather straightforward, the results therein are of immense importance as they lead to the wonderful result in 4 in the aforementioned theorem.

The result # 1 in the above theorem follows very conspicuously from the corollary CR # 1 of the preceding lemma.

The result # 2 in the above theorem follows equally simply when we use the identity: (k/n). (nx/k!) = x/(k-1)! together with the corollary CR # 2 of the preceding lemma.

The result # 3 in the above theorem follows analogously and equally simply when we use the identity k. (k-1) + k = k^2 together with the corollaries CR#2 and CR#3 of the preceding lemma.

Now, using the results # 1 to 3 leads us to the final result # 4 of the theorem.

QED

Now, to have this paper in a good stead, it would be topical to note that the proof of the well-known Bernsteins Theorem not only afforded the understanding and more so the insight into the problem of polynomial approximation, but also it heralded the era of very active interest of the researchers in this area. Improved estimates of the Error and Rate of Convergence: $||f(x)-B_n(f)(x)||$ for Bernstein Polynomial $B_n(f)(x)$ were successfully attempted at, with the help of modulus of continuity. The modulus of continuity of a bounded function f (x) on the closed interval (a, b) is defined by:

$$\omega_f(\delta) \equiv \omega_f([a,b];\delta) = \sup\{|f(x)-f(y)|: x, y \in (a, b), |x-y| \le \delta\}$$

Now, we could proceed to note as below:

$$\begin{aligned}
&||f(x)-PPn(f)(x)||\\
&\le [\sum_{k=0}^{k=n} |f(x)-f(y)| \binom{nx}{k} \binom{n-nx}{n-k}.\,]\\
&\le [\sum_{k=0}^{k=n} \omega_f |x-k/n| \binom{nx}{k} \binom{n-nx}{n-k}.\,]\\
&\le [\omega_f(1/\sqrt{n}) \sum_{k=0}^{k=n} [1+\sqrt{n}|x-k/n|] \binom{nx}{k} \binom{n-nx}{n-k}.\,]\\
&\le [\omega_f(1/\sqrt{n})\{1+\sqrt{n} \sum_{k=0}^{k=n} |x-k/n| \binom{nx}{k} \binom{n-nx}{n-k}.\,]\\
&\le [\omega_f(1/\sqrt{n})
\end{aligned}$$

Using the following Cauchy-Schwarz inequality and the last result#4 in the preceding theorem

$$\begin{aligned}
&[\sum_{k=0}^{k=n} |x-k/n| \binom{nx}{k} \binom{n-nx}{n-k}.\,]\\
&\le [\sum_{k=0}^{k=n} (|x-k/n|)^2 \binom{nx}{k} \binom{n-nx}{n-k}.\,]^{1/2}\\
&.[\sum_{k=0}^{k=n} \binom{nx}{k} \binom{n-nx}{n-k}.\,]
\end{aligned}$$

≤ 0. (By Result # 4 in the Theorem).

Hence, we have the error estimate using the modulus of continuity:

$$||f(x)-PPn(f)(x)|| \le \omega_f(1/\sqrt{n}) \qquad (6)$$

THE EMPIRICAL SIMULATION STUDY

To illustrate the gain in efficiency by using our proposed polynomial approximation formula (5), we have carried an empirical study. We have taken the example-cases of n = 2, 4, 7 and 10 in the empirical study to numerically illustrate the relative gain in efficiency in using the probabilistic approximation formula (5) vis-à-vis Bernstein Polynomial. Essentially, the empirical study is a simulation one wherein we would have to assume that the integrand f (x) is known to us. We have confined to the illustrations of the relative gain in efficiency with the polynomial approximation formula (5) for the following four illustrative functions:

$$f(x) = \exp(x), \ln(2+x), \sin(1+x) \text{ and } 10^x$$

We have considered numerical values of three quantities-two Percentage Relative Errors (PRE) corresponding to our probabilistic approximation Polynomial PREPPn (f) (x) and corresponding to the Bernstein Polynomial PREBn (f) (x) and the Percentage Relative Gains (PRG) in using our proposed probabilistic polynomial approximation formula (5) in place of Bernstein Polynomial. These quantities are defined as follows.

The Percentage Relative Error using Probabilistic Polynomial formula (5) with n intervals in (0, 1), i.e.,

$$\text{PREPPn}(f)(x) = \frac{\left|\int_0^1 f(x)dx - \int_0^1 PP_n(f)(x)dx\right|}{\int_0^1 f(x)dx} \times 100 \qquad (7)$$

And

The Percentage Relative Error using Bernstein Polynomial formula (5) with n intervals in (0, 1), i.e.,

$$\text{PREPPn}(f)(x) = \frac{\left|\int_0^1 f(x)dx - \int_0^1 B_n(f)(x)dx\right|}{\int_0^1 f(x)dx} \times 100 \qquad (8)$$

DISCUSSION

These aforesaid three numerical quantities from Eq. 7 and 8 have been computed using Maple for all the

Table 1: Fn: exp (x)/PREBn (f) (x) and PREPPn (f) (x)

Fn: exp (x) ↓ n: →	2	4	7	10
PREBn (f) (x)	4.115693	2.055768	1.173468	0.820997
PREPPn (f) (x)	0.033715	0.000490	0.000138	0.000123

Table 2: Fn: sin (1 + x)/PREBn (f) (x) and PREPPn (f) (x)

Fn: sin (1 + x) ↓ n: →	2	4	7	10
PREBn (f) (x)	4.219915	2.112224	1.208311	0.846292
PREPPn (f) (x)	0.035783	0.005400	0.001830	0.000649

Table 3: Fn: ln (2 + x)/PREBn (f) (x) and PREPPn (f) (x)

Fn: ln (2+x) ↓ n: →	2	4	7	10
PREBn (f) (x)	0.754127	0.376699	0.215027	0.150434
PREPPn (f) (x)	0.006095	0.000301	0.000256	0.000096

Table 4: Fn: 10^x/PREBn (f) (x) and PREPPn (f) (x)

Fn: 10^x ↓ n: →	2	4	7	10
PREBn (f) (x)	20.777220	10.339726	5.876746	4.102577
PREPPn (f) (x)	0.006529	0.000786	0.000283	0.000102

four illustrative functions mentioned in section 4 for four values of n, namely n = 2, 4, 7 and 10. Table 1 to 4 contain these quantities when the function f(x) has been taken as:

exp (x), ln (2 +x), sin (1 + x) and 10^x, respectively

The percentage relative errors with our probabilistic polynomial approximation formula (5) PREPPn(f)(x) is far too low as compared to percentage relative errors with Bernstein's polynomial in all the four functions and for all the columns, when n = 2, 4, 7 and 10.

In conclusion, on the basis of the above empirical study, one can assert that the new probabilistic polynomial approximation formula (5) is far better than Bernstein's Polynomial approximation.

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