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On the Solution of Linear Complementarity Problem by A Stochastic Iteration Method

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Abstract: An earlier study proposed a stochastic algorithm based on a modified Robbins-Monroe type for the solution of finite-dimensional variational inequality problem. In this study we describe a similar approach for the linear complementarity problem. This study show that the stochastic algorithm arising from this approach converges strongly to the non-zero solution of the linear complementarity problem when it exists.

Key words: Linear complementarity problem, stochastic iteration, monotone operator

INTRODUCTION

Let M be a given n×n real matrix and q is a given n-dimensional real vector. Then the linear complementarity problem is the of finding n-dimensional $x \in R^n$ such that

$$Mx+q\geq 0,\ x\geq 0,\ x^T(Mx+q) = 0 \qquad (1)$$

or conclude that no such vector exists.

Interest on LCP stems from the fact that many important Mathematical problems can be formulated as LCP (Cottle *et al*., 1992). This problem has been extensively studied by many authors including Murty (1988).

We formulate (1) into an equivalent minimization problem:

$$\min\{x^TMx+xq:\ x\geq 0,\ Mx+q\geq 0\} \qquad (2)$$

We observe that from (2)

$$f(x) = x^TMx+x^Tq$$

has zero as a feasible minimizer. Thus the problem reduces to that of searching for the global minimizer:

$$\{x\in R^n:\ x\geq 0,\ q+Mx\geq 0,\ x^T(Mx+q) = 0\}$$

But x = 0, f(x) = o

This suggests a reformation of the problem as a search for

$$V_{x^*} = \left\{x^*\in R^n : x^* > 0 : \partial f(x^*) = 0\right\} \qquad (3)$$

where

$$\partial f(x^*) = \frac{\partial f(x^*)}{\partial x} = Mx + q$$

In this study stochastic gradient type recursive sequence is suggest:

$$x_{j+1} = x_j-\rho_j d_j \qquad (4)$$

where d_j is the estimate of $\partial f(x)$ Mx+q and $\{\rho_j\}$ is a sequence of positive scalars to be specified.

The procedure is a way of stochastically locating the set (3) when it exists. The iteration method described in this study differs from most iterative methods mainly in the way the search direction at each iteration and the starting point of the search algorithm are estimated to determine the optimum direction and provide maximum rate of decrease of f(x).

DEFINITIONS AND NOTATIONS

Let us indicatate first the notations used in this study: n is a fixed interger, and R^n is the Euclidean n-dimensional space with the usual norm

$$\|x\|^2 = x^1x \text{ and inner product } \langle x,y\rangle = \sum_{j=1}^{n} x_j y_j$$

where x^1 denotes the transpose of the vector $x\in R^n$. The nonnegative orthant of R^n is denoted by $R^n_+ = \{x\in R^n: x\geq 0\}$. For a fixed $a\in R^n$ and random vectors z_1, z_2 in $R^n\|z_1\|$, $\langle z_1, z_2\rangle$, $\langle a, z_1\rangle$ are random variables in the usual sense.

Let E denote the expectation operator. Then Ez_1 is defined by the requirement that $E\langle a,z_1\rangle = \langle a,Ez_1\rangle$ for any random vector z and a fixed a in R^n if $E\|z\| < \infty$.

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Here $D(\partial f)$ denotes the domain of ∂f and by $N(x)$, a δ-neighborhood of a point $x \in R^n$.

Definition 1: The element $x \in D(\partial f)$ is an attractor of x^0 if there is a sequence of point s in $D(\partial f)$ starting at x^0 which converges to the point x.

Definition 2: The set $\varphi(x)$, of all points x^0 with attractor in $N(x)$ is known as the domain of attraction of $N(x)$.

MATHEMATICAL FORMULATION OF STOCASTIC ALGORITHM

$$\text{Denote } \partial f^k = \frac{\partial f(x^k)}{\partial x}, \frac{\partial^2 f(x^k)}{\partial x_r \partial x_s} = \partial^2_{r,s} f^k$$

As in (Okoroafor and Osu, 2004), we construct a sequence of random vectors $d^k \in R^n$ the strongly approximate $\partial f^k = \partial f(x^k)$ for each k in the sense that

$$E\left\| d_j^k - \partial f^k \right\| = 0$$

and their expected Euclidean distance

$$E\left\| d_j^k - \partial f^k \right\|^2$$

is minimum so that a search in the direction of the random sequence $\{d_j^k\}$ approximates a search through the true gradient ∂f^k and this is expected to lead to the non-zero global minimizing factor if it exists. To this end, we consider the natural Taylor's expansion of a quadratic function f about a point x_0 giving by

$$f(x)-f(x_0) = \langle \partial f(x_0), x-x_0 \rangle + 1/2(x-x_0)H(x_c)(x-x_0) \quad (5)$$

where x_c is on the line segment between x and x_0 and $H(x_c)$ is the Hessian of f at x_c.

Let $e(x_j)$ be a sequence of non-observable random errors satisfying

$$Ee(x_j) = 0 \text{ for each } j$$

and

$$Ee(x_j)e(x_j) = \sigma^2 \delta_{ij}, \; 0<\sigma^2<\infty$$

Let $y(x_1), y(x_2), \ldots, y(x_m)$ be real-valued independent observable random variables performed on $x_1, x_2, \ldots, x_n$, $n+2<m<1/2n(n+1)$ chosen in the neighborhood of x^k for a fixed k, then

$$\begin{aligned} yj = y(x_j) &= f(x+t_j)-f(x_j) \\ &= \langle \partial f(x^k), t_j \rangle + \frac{1}{2}\sum_{k=1}\sum_{r=1} t_{kj} t_{rj} \partial^2_{k\,r} f^k + e(x_j) \end{aligned} \quad (6)$$

is identifiable with (1) so the a fixed $t_j \in R^n$ satisfying $\sum_{i=1}^m t_{ij} = 0, \frac{1}{m}\sum_{j=1}^m t_{ij}^2$ linearizes f, (Okoroafor and Osu, 2005) and hence the least squares approximation

$$d^k = M^{-1}\sum_{j=1}^m t_j y_j \;\; M = \sum_{j=1}^m t_j t^1{}_j \quad (7)$$

exists and is adequate for approximating ∂f such that

$$E\|d^k - \partial f(x^k)\| = 0 \text{ for each } k \quad (8)$$

and also yields, by elementary calculation the minimum euclidean distance

$$E\|d^k - \partial f(x^k)\| = M^{-1}\sigma^2 \quad (9)$$

In the sequel we assume, without lost of generality that $\sigma^2 = 1$. $\{d^k\}$ is, thus, a sequence of independent and identically distributed random vectors and determines the direction of search.

It follows that by letting x^0 be an initial point, the sequence of path produced by $\{x^k\}_{k=0}^\infty$ through its definition

$$x^{k+1} = x^k - \rho^k d^k$$

by successive iteration, is the trajectory of the point x^0 and any limiting point of the sequence is therefore the attractor of x^0.

GETTING THE DOMAIN OF ATTRACTION

Let R_+^n-N(0) be partitioned into t exclusive segments, S_j j = 1, 2,..., t, $n<t\le 2^n$. Let x_j be chosen randomly in S_j, such that $f(x_j)>0, \forall j$

Let $p_j = P(x_j = \alpha)$ be the probability that $x_j = \alpha$ so that

$$p_j \ge 0, \sum_{j=1}^t p_j = 1 \quad (10)$$

Put

$$p_j = \frac{f(x_j)}{\sum_{j=1}^t f(x_j)}$$

so that

$$\bar{x} = \sum_{j=1}^t x_j p_j = \sum_{j=1}^t \frac{x_j f(x_j)}{\sum_{j=1}^t f(x_j)} \quad (11)$$

It is shown in (Okoroafor and Osu, 2004) that if

$$\hat{x} = \bar{x} - \rho d,\ \rho > 0 \qquad (12)$$

where d is as in Eq. (7), then $f(\hat{x}) = \min_j \langle f(x_j) : x_j \in S_j \rangle$. It follows that the segment S_T where $\hat{x} \in S_T$ contains x>0 for which f(x) is minimum and hence we have $\varphi(V_{\bar{x}}) \subset S_T$ so that if {0} is the attractor of the point $\bar{x}$ and $\varphi(\{0\}) \cap \varphi(V_{\bar{x}}) = \phi$ than $N(0) \cap N(V_{\bar{x}}) = \phi$ or else $N(0) = N(V_{\bar{x}})$ with global domain of attraction $\varphi(0) = \varphi(V_{\hat{x}})$.
Thus we have

Lemma 1: Suppose that $V_{\hat{x}} \neq \phi$. Thus there exists a neighbourhood $N(V_{\hat{x}}) \subseteq D(\partial f)$ of $V_{\hat{x}}$ such that for any initial guess $\hat{x} \in \varphi(V_{\hat{x}})$, the non negative minimizer $V_{\hat{x}}$ is obtained as the limit of iteratively constructed sequence $\{x^j\}_{j=1}^{\infty}$ generated form $\hat{x}$ by $x^{j+1} = x^j - \rho^j d^j$.

Then with $\hat{x}$ as our starting point we search for the minimizer of f as follows: starting at $\hat{x}$ as in Eq. (12).

- Compute the d^k as in Eq. (7)
- Compute the corresponding ρ as specified below
- Compute $x^{k+1} = x^k - \rho^k d^k$

Has the process converged ? i.e., $\|x^{k+1} - x^k\| < \sigma$, σ>0
If yes, then $x^{k+1} = x^k$, if no return to (1).
Here we prove the strong convergence of the sequence to the solution of (3)

Theorem 1: Let $\{\rho^k\}$ be a real sequence such that

(i) $\rho^0 = 1,\ 0 < \rho^k < 1\ \forall\ k > 1$

(ii) $\sum_{k=0}^{\infty} \rho^k = \infty$

(iii) $\sum_{k=0}^{\infty} \rho^{2k} < \infty$

Then the sequence $\{x^k\}_{k=0}^{\infty}$ generated by $\hat{x} \in \varphi(V_{\hat{x}}) \subseteq D(\partial f)$ and defined iteratively by $x^{j+1} = x^j - \rho^j d^j$ remain in $D(\partial f)$ and converges strongly to $V_{\hat{x}}$.

Proof: Let $b^k = \rho^k \|d^k - \partial f^k\|$
then $\{b_k\}_{k+1}^{\infty}$ is a sequence of independent random variabl and from Eq. (8) $Eb_k = 0$ for each K.

Noticing that the sequence of partial sums $\{S_k\}_{k=1}^{\infty}$, $S_k = \sum_{j=1}^{k} b_j$, for ms a Martingale. Therefore,

$$ES_k^2 = \sum_{j=1}^{k} Eb_j^2 = \sum_{j=1}^{k} \rho^{2j} E\left\|d^j = \partial f^j\right\|^2$$

$$= M^{-1}\sigma^2 \sum_{j=1}^{k} \rho^{2j}$$

and

$$\sum Eb_j^2 < \infty, \text{since} \sum_{j=1}^{\infty} \rho^{2j} < \infty$$

Hence by a version of Martingale convergence theorem (whittle, 1976), we have

$$\lim_{k\to\infty} S_k = \sum_{j=1}^{\infty} b_j < \infty$$

so that

$$\lim_{k\to\infty} \rho^k = \left\|d^k - \partial f^k\right\| = 0$$

Noticing that in (2), M is positive definite so that f(x) is convex and hence ∂f is monotone. But an earlier result in theory of monotone operators, due to (Chidume, 1990), shows that the sequence $\{x^k\}$ generated by $x^0 \in D(\partial f)$ and defined iteratively by:

$$x^{k+1} = x^k - \rho^k \partial f^k$$

remain in $D(\partial f)$ and converges strongly to $\{x^*: \partial f(x^*) = 0\}$. It follows from this result that our sequences converges strongly to V_{x^*} if $V_{x^*} \neq 0$.

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