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The Fixed Points of Certain Discontinuous Operators on Locally Convex Spaces

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Abstract: The fixed point properties of four classes of operators mapping a metrisable locally convex space into itself are considered. These classes include contraction and nonexpansive mappings, discontinuous operators for certain parameter values of the classes. The existence of fixed points are proved for these classes of mappings under some conditions. Furthermore, a cone ordering scheme is devised for one of these classes, while another is shown to have open mapping properties. All these results generalise the results of Derrick and Nova from Banach spaces to metrisable locally convex spaces.

Key words: Locally convex space, fixed points, 2000 AMS subject classification, 47H10, 46A03

INTRODUCTION

Derrick and Nava (1992) studied the fixed point properties of four classes of operators mapping a Banach space into itself. In this study, we use their approach to generalise their results to Hausdorff locally convex spaces that are not necessarily normable. Such locally convex spaces which are complete and metrisable abound and are obvious generalizations of Banach spaces (Olaleru, 2002; Schaffer, 1999).

For example, the set of all real (or complex) valued indefinitely differentiable functions on the interval (a, b) becomes a metrisable locally convex space under the topology defined by the seminorms $p_m(f) = \sup_{a \le t \le b} |f^{(m)}(t)|$, (m = 0, 1,…..).

Also consider the set of all real (or complex) valued indefinitely differentiable functions on the interval (-∞, ∞). Under the topology of compact convergence for all the derivatives defined by the seminorms $p_m(f) = \sup_{a \le t \le b} |f^{(m)}(t)|$, (m =0, 1,….), the set is a metrisable locally convex space (Robertson and Robertson, 1980). In fact the duals of those spaces of distributions under their appropriate strong topologies are also metrisable locally convex spaces (Robertson and Robertson, 1980). Those spaces are also complete (Robertson and Robertson, 1980).

A locally convex space (X, u) with topology u is a topological vector space which has a local base of convex neighbourhoods of zero. It is metrisable if it is Hausdorff and has a countable zero basis. Consequently, it is metrisable if u can be described by a countable family of continuous seminorms (Robertson and Robertson, 1980; Schaffer, 1999). Under the topology determined by the set Q of seminorms, X is Hausdorff if and only if for each non-zero $x \in X$, there is some $p \in Q$ with $p(x)>0$ (Robertson and Robertson, 1980) (Nova, 1986). To each absolutely convex absorbent subset U of X corresponds a seminorm p, called the gauge of U defined by

$$p(x) = \inf\{\lambda : \lambda > 0, \quad x \in \lambda U\}$$

and with the property that

$$\{x : p(x) < 1\} \subseteq U \subseteq \{x : p(x) \le 1\}$$

U is a neighbourhood of zero if and only if p is continuous. In this case the interior of U is $\{x: p(x)<1\}$ and the closure of U is $\{x: p(x)\le 1\}$ (Robertson and Robertson, 1980).

We shall now state the following theorem which is fundamental to our results in this study.

Theorem A: (Robertson and Robertson, 1980): The topology of a metrisable locally convex space can always be defined by a metric $d(x, y) = f_c(x-y)$, which is invariant under translation, where

$$f_c(x) = \sum_{n=1}^{\infty} 2^{-n} \min\{p_n(x), 1\}$$

and $\{p_n\}$ is the set of seminorms characterising the locally convex topology.

It should be observed that if X is a normed linear space, then f_c satisfies the triangle inequality and will also be a norm. It is also easy to see that $f_c(x) = 0$ implies that $x = 0$ for any $x \in X$. It is also easy to prove that $f_c(\lambda x) \le f_c(x)$ for any x whenever $0 \le \lambda \le 1$ and X is a metrisable locally convex space. Henceforth f_c will denote the function as defined above.

Let T be a mapping of a complete metrisable locally convex space X, or some closed subset K of X, into itself satisfying one of the following conditions. For any two points x and y and a, b ≥ 0:

(A) $f_c\{(Tx - Ty) - b[(x - Tx) + (Ty - y)]\} \le af_c(x - y)$,

(B) $f_c\{(Tx - Ty) - b(x - Tx)\} \le af_c(x - y) + bf_c(y - Ty)$,

(C) $f_c(Tx - Ty) \le f_c(Tx - x) + f_c(x - y) + f_c(y - Ty)$

(D) $f_c\{(Tx - Ty) \le af_c(x - y) + b\{f_c(x - Tx) + f_c(y - Ty)\}$.

If X is a Banach space, these conditions are equivalent to those studied by Derrick and Nova (1992) and all become contraction mappings when we set b = 0 and let 0 < a < 1. We shall say that a mapping is of class A (a, b) when it satisfies condition A above and similarly for classes B(a, b), C(a, b) and D(a, b). Mappings of the type C(0, b) have been studied by Kannan (1969 and 1971), where it was proved that if X is a Banach space, then T has a unique fixed point whenever 0 < b < 1/2. Kannan also proved the uniqueness of fixed points with b = 1/2 in uniformly convex spaces under certain restrictions. The author also generalised some of his results to when X is a metrisable locally convex space Olaleru (2006). Derrick and Nova (1989) and Nova (1986) have examined the fixed points properties of class D(a, b) and have proved that if X is a Banach space and T is in D(a, b) with a + 2b < 1, then T has a unique fixed point (Derrick and Nova, 1992). Hardy and Rogers (1973) and Goebel *et al*. (1973) have studied a generalised version of D(a, b) on a complete metric space. The survey article by Rhoades (1977) list many other papers written on mappings of these types. In this study, we show the similarities and contrast the differences among these four classes. One common similarities is that if T has a fixed point, then this fixed point must be unique whenever 0≤a<1. Observe that any mapping of classes A(a, b), B(a ,b), or C(a, b) is of class D(a ,b) by a trivial application of the triangle inequality. We will show, however, that the former classes are smoother than class D(a, b), which includes discontinuous operators (the motivaton for using the letter D) whenever b > 0. In particular, class D(1,1) contains all operators from X onto itself, since

$$f_c\{(Tx - Ty) \le (Tx - x) + (x - y) + (y - Ty)$$

by the triangular inequality. Because all mappings of the first three classes are included in the fourth class, it is essential to discuss class D(a, b)'s properties first.

PROPERTIES OF CLASS D(A,B)

Observe that the class D(a', b') contains class D(a ,b) whenever a' ≥ a and b' ≥ b. This is a cone order with the first quadrant as the cone. Since all operators (continuous or not) are in D(1,1) we need only consider the pairs (a,b) in the first quadrant where either a or b < 1 or a = b = 1. If b = 0, then D(a, 0) is a contraction for 0 ≤ a < 1, nonexpansive mapping if a = 1 and a Lipschitz mapping if a > 1.

Suppose T belong to classes D(a, b) and D(c, b). Then, multiplying the D(a, b) inequality by t and the D(c,d) inequality by 1-t we obtain that T belongs to class D(ta+(1-t)c, tb+(1-t)d), for 0 ≤ t ≤ 1. We do not know whether this convexity result is best possible: if D(a, b) and D(c, b) cannot be improved upon, is this also the case for D(ta+(1-t), tb+(1-t)d)?.

Theorem 1: Let T be in class D(a, b).

- If 0 ≤ a < 1 and inf $f_c(x - Tx) = 0$, then there exists a convergent sequence $\{x_n\}$ for which $x_n - Tx_n \to 0$
- If 0 ≤ b < 1 and $x_n - Tx_n \to 0$, then T has a fixed point.

Proof: By hypothesis select a sequence $\{x_n\}$ for which $x_n - Tx_n \to 0$. Substitute x_m and x_n in the D(a, b) condition and apply the triangle inequality to the first term on the right side of the D(a, b) inequality to get

$$0 \le (1 - a)f_c(Tx_m - Tx_n) \le (a + b)\{f_c(Tx_m - x_m) + f_c(Tx_n - x_n)\},$$

from which it follows that $\{Tx_n\}$ is a Cauchy sequence. Hence $Tx_n \to p$ and

$$f_c(x_n - p) \le f_c(x_n - Tx_n) + f_c(Tx_n - p) \to 0, \text{ as } n \to \infty$$

Suppose $x_n \to p$, then by that triangular inequality and applying D(a, b) to the last term on the right side

$$f_c(p - Tp) \le f_c(p - x_n) + f_c(x_n - Tx_n) + f_c(Tx_n - Tp)$$
$$\le (1 + a)f_c(p - x_n) + (1 + b)f_c(x_n - Tx_n) + bf_c(p - Tp)$$

$$\text{Hence, } 0 \le (1-b)f_c(p-Tp) \le (1+a)f_c(p-x_n) + (1+b)f_c(x_n-Tx_n) \to 0$$

Example: The following example shows that Theorem 1 is sharp: let X be the set of all real (or complex) valued functions continuous on].-∞, ∞ [It forms a locally convex space under the topology defined by the seminorms $p_n(x) = \sup_{-n \le t \le n} |f(t)|$, (n = 1, 2,....). This space is Hausdorff, metrisable and complete but not normable (Robertson and Robertson, 1980). Define T such that T(f) = 1 + 1/2f. Then

T belongs to D(1,0) and inf $f_c(f - T(f)) = 0$ since $f(t) = 2$ for all $t \in]-\infty, \infty[$ satisfies $f - T(f) = 0$. But no converging sequence satisfies $f_n - T(f_n) \to 0$ and T has no fixed points in X.

It follows from (i) and (ii) that if $0 \le a, b<1$ and inf $f_c(x - Tx) = 0$, then T has a unique fixed point. We can omit the hypothesis that inf $f_c(x - Tx) = 0$ if we require that $a + 2b < 1$: select any x_o and define the Picards iterates $x_{n+1} = Tx_n$, for n = 0, 1, 2,.....Then

$$f_c(x_{n+1} - Tx_{n+1}) = f_c(Tx_n - Tx_{n+1}) \le af_c(x_n - x_{n+1}) + b\{f_c(x_n - Tx_n) + f_c(x_{n+1} - Tx_{x+1})\}$$

Or $(1 - b)f_c(x_{n+1} - Tx_{n+1}) = (a + b)f_c(x_n - Tx_n)$

But $a + b < 1 - b$, so that $f_c(x_n - Tx_n) \to 0$, implying that inf $f_c(x - Tx) = 0$.

We thus have

Theorem 2: If T is in D(a, b) with $a + 2b < 1$, then T has a unique fixed point.

PROPERTIES OF THE OTHER CLASSES

Although cone ordering is lost for classes A(a, b), B(a, b) and C(a ,b), it is trivial to verify that A(a', b) and B(a', b) contain A(a, b) and B(a, b) respectively, when $a' \ge a$ and C(a, b') contains C(a, b) for $b' \ge b$. The convexity result still hold: for example, if *T* belongs to B(a ,b) and B(c, d), then by the triangle inequality it follows that T belongs to class B(ta+(1-t)c, tb+(1-t)d), but is this best possible when B(a, b) and B(c ,d) are best possible?. The same question was posed by Derrick and Nova (1992) when X is a Banach space.

Theorem 2 now holds whenever $0\le a<1$, for any b = 0, for classes A(a, b) and B(a ,b). Note that (A) can be rewritten as

$$\text{(A)}' \quad f_c((1+b)(Tx - Ty) - b(x - y)) \le af_c(x - y),$$

so that

$$f_c(Tx - Ty) =\le \alpha f_c(x - y),\ a = (a+b)/(1+b) < 1,$$

implying that T is a contraction mapping and thus has a unique fixed point.

For the second class, apply B(a,b) to the Picard iterates getting

$$(1+b)f_c(x_{n+1} - x_n) = f_c\{(Tx_n - Tx_{n-1}) - b(x_n - Tx_n)\} \le af_c(x_n - x_{n-1}) + bf_c(x_{n-1} - Tx_{(n-1)}) = (a+b)f_c(x_n - x_{n-1})$$

from which it follows that $x_n - Tx_n \to 0$. Using the geometric series

$$f_c(Tx_m - Tx_n) \le \sum_{k=m+1}^{n} f_c(Tx_k - Tx_{k-1}) \le a^{m+1}((1 + b)/(1 - a))f_c(x_1 - x_o) \to 0,$$

we have that $\{Tx_n\}$ is a Cauchy sequence, $x_{n+1} = Tx_n \to p$ and

$$f_c((1 + b)(Tp-p) + (p-Tx_n)) = f_c(Tp-Tx_n)-b(p-Tp)) = af_c(x_n-p)-bf_c(x_n-Tx_n) \to 0,$$

implying that p is the unique fixed point.

Condition (A)' implies that A(a,b) operators are at least Lipschitz smooth when X is a Banach space. One can assume without loss of generality, that T0 = 0, since the operator T'x = Tx - T0 also satisfies A(a ,b). The next argument is adapted from Graves (1950).

Theorem 3: If T is in A(a, b) with a < b, then T is onto.

Proof. Select any y in X, let $x_o = 0$ and $y_o= y$ and define recursively $x_n = x_{(n-1)} + ((1-b)/b)y_{(n-1)}$, $y_n = y_{(n-1)}-(Tx_n - Tx_{(n-1)})$. We show that $Tx_n \to y$ and $y_n \to 0$. Note that

$$f_c(x_n - x_{(n-1)}) = ((1 - b)/b)f_c(y_{n-1}) = b^{-1}f_c((1+b)(Tx_{n-1} - Tx_{n-2}) - b(x_{n-1} - x_{n-2}) \le (a/b)\, f_c(x_{n-1}-x_{n-2}) \le .... \le (a/b)^n f_c(x_1 - x_o) \to 0,$$

hence $y_n \to 0$ and $\{x_n\}$ is a Cauchy sequence, so $x_n \to p$. Finally, by continuity and T0 = 0,

$$y = -\sum_{k=0}^{\infty} (y_{(k+1)} - yk) = \sum_{k=0}^{\infty} (Tx_{k+1} - Tx_k) = \lim_n Tx_n = Tp.$$

Theorem 4: If $T \in$ A(a,b) with a < b, then the open ball centred at the origin of radius (b-a)/(1+b) is contained in the image under T of the open unit ball.

Proof: Select any y in the open ball centered at the origin of radius (b-a)/(1-b). By Theorem 3 there is a point x such that Tx = y. Apply (A)' to x and 0 and use the triangle inequality:

$$(b-a)/(1+b)f_c(x) \le f_c(Tx) = f_c(y) < (b-a)/(b+a).$$

If b = 1 > a = ½, then the open ball centred at the origin of radius 1/4 is contained in the image under T of the open unit ball, so we have Koebe 1/4 Theorem if X is a metrisable locally convex space.

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