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A Generalisation of Gregus Fixed Point Theorem

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Abstract: Let C be a closed convex subset of a Banach space X and T: C → C a mapping that satisfies $||Tx - Ty|| \leq a||x - y|| + b||Tx - x|| + c||Ty - y||$ for all x, y ∈ C where $0 < a < 1$, $b \geq 0$, $c \geq 0$ and $a + b + c = 1$. Then T has a unique fixed point. The above theorem, proved by Gregus, is hereby generalized to when X is a metrisable topological vector space. In addition, we are able to use the Mann iteration scheme to approximate the unique fixed point.

Key words: Topological vector space, fixed point, Mann iteration scheme

Let C be a closed convex and bounded subset of a reflexive Banach space X and T: C → C such that

$$||Tx - Ty|| \leq ||x - y|| \text{ for all } x, y \in C \quad (1)$$

Kirk (1965) considered the existence of fixed point for T.

Kannan ((1969) and (1971)) and Wong (1975), among others, considered similar mappings, but the condition imposed is

$$||Tx - Ty|| \leq 1/2||Tx - x|| + 1/2||Ty - y|| \text{ for all } x, y \in C \quad (2)$$

The results are respectively generalized to when X is a metrisable locally convex space (Olaleru, 2006a, b).

Gregus (1980) combined these two conditions in the following manner:

$$||Tx-Ty|| \leq a||x-y||+b||Tx-x||+c||Ty-y|| \text{ for all } x, y \in C \quad (3)$$

where a, b, c are nonnegative constants such that $a + b + c = 1$.

In this study, we use Gregus technique to generalize his result to when X is a complete metrisable topological vector space. Examples of such spaces include uniformly convex Banach spaces, Banach spaces and complete metrisable locally convex spaces (Olaleru, 2002, 2006a; Shaeffer, 1999; Roberston and Robertson, 1980).

Note that if T satisfies (3), then it also satisfies:

$$||Tx - Ty|| \leq a||x - y|| + p||Tx - x|| + p||Ty - y|| \text{ for all } x, y \in C \quad (3')$$

where $a, p \geq 0$, $a + 2p = 1$ ($p = 1/2b + 1/2c$). If $a = 1$, we obtain condition (1) and if $a = 0$, we obtain (2).

Now, what happens if $0 < a < 1$? We show that in this case there holds a far more general result than those obtained for the extreme cases $a = 0$, $a = 1$.

Gregus (1980) pointed out that for uniformly convex Banach spaces and for arbitrary point x ∈ C, $x = x_0$ the sequence of iterates $x_n = 1/2Tx_{n-1} + 1/2x_{n-1}$ converges to a fixed point of T. We generalize this sequence to Mann iteration sequence and shows that this sequence converges for a more general space of complete metrisable topological vector space.

The most general Mann iteration scheme being studied is:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n,\ x_0 \in C,\ n \geq 0 \quad (4)$$

where $\{\alpha_n\}$ satisfy $0 \leq \alpha_n \leq 1$ for all n and $\sum_0^\infty \alpha_n = \infty$.

For more discussion on Mann iteration scheme, (Berinde, 2004; Chidume, 2003; Rhoades, 1976).

The following result will be needed for our result.

Theorem A: A topological vector space is metrisable if and only if it has a countable base of neighbourhoods of zero. The topology of a metrisable topological vector space can always be defined by an F-norm (Adasch *et al.*, 1978; Belluce *et al.*, 1968).

For the same result Kothe (1969).

Henceforth, unless otherwise indicated, F shall denote F-norm if it is characterising a metrisable topological vector space. Observe that an F-norm will be a norm if it is defining a normed space.

Theorem 1: Let C be a closed convex subset of a complete metrisable space X and T: C → C a mapping that satisfies $F(Tx - Ty) \leq aF(x - y) + bF(Tx - x) + cF(Ty - y)$ for all x, y ∈ C where $0 < a < 1$, $b \geq 0$, $c \geq 0$ and $a + b + c = 1$. Then T has a unique fixed point.

Proof: We already know, that (3) implies (3'). Take any point $x \in C$ and consider the sequence $\{T^n x\}_{n=1}^{\infty}$.

$$F(T^n x - T^{n-1}x) = aF(T^{n-1}x - T^{n-2}x) + pF(T^n x - T^{n-1}x) + pF(T^{n-1}x - T^{n-2}x)$$

And by simple calculation we obtain

$$F(T^n x - T^{n-1}x) \le F(Tx - x) \text{ for all } n \in N \text{ and all } x \in C \quad (5)$$

That is, the distance between two consecutive elements of $\{T^n x\}$ is less or equal to the distance between the first and the second element. Now let us consider the distance between two consecutive elements with odd (resp. even) power of T. It is clear, that it is sufficient to consider only the distance between Tx and T^3x.

$$F(T^3x - Tx) \le aF(T^2x - x) + pF(T^3x - T^2x) + pF(Tx - x)$$

$$\le aF(T^2x - Tx) + aF(Tx - x) + 2pF(Tx - x)$$

$$\le 2(a + p)F(Tx - x).$$

Hence $F(T^3x - Tx) \le 2(a + p)F(Tx - x)$ for all $x \in C$.
C is convex; therefore the midpoint $z = 1/2T^2x + 1/2T^3x$ is in C
and from the properties of the norm we have:

$$F(Tz - z) \le 1/2F(Tz - T^2z) + 1/2F(Tz- T^3x)$$
$$\le 1/2(aF(z - Tx) + pF(Tz - z) + pF(T^2x - Tx)) +$$
$$+ 1/2(aF(z - T^2x) + pF(Tz - z) + pF(T^3x - T^2x))$$
$$\le pF(Tz - z) + 1/2(aF(z - Tx) + aF(z - T^2x) + 2pF(Tx - x)),$$

i.e., $2(1 - p)F(Tz - z) \le aF(z - Tx) + aF(z - T^2x) + 2pF(Tx - x)$.

But we have also:

$$F(z - Tx) \le 1/2F(T^2x - Tx) + 1/2F(T^3x - Tx) \le 1/2F(Tx - x) + (a + p)F(Tx - x),$$

and

$$F(z - T^2x) \le 1/2F(T^2x - T^3x) \le 1/2F(Tx - x);$$

and we obtain: $2(1 - p)F(Tz - z) \le F(Tx - x) + a(a + p) F(Tx - x)$.

From $a + 2p = 1$ follows $p = (1 - a)/2$ and

$$F(Tz - z) = \left(1 - \frac{a(1-a)}{2(1+a)}\right) F(Tx - x) = \lambda F(Tx - x),$$

where $\lambda = 1 - a(1 - a)/2(1 + a)$.

From $0 < a < 1$ follows $a(1 - a) \ne 0$ and $0 < \lambda < 1$.

Now let $i = \inf\{F(Tx - x): x \in C\}$. Then there exists a point $x \in C$ such that
$F(Tx - x) < i + \varepsilon$ for $\varepsilon > 0$.

Suppose $i > 0$. Then for $0 < \varepsilon < (1 - \lambda)i/\lambda$ and $F(Tx - x) < i + \varepsilon$, we have

$F(Tz - z) = \lambda F(Tx - x) \le \lambda(i + \varepsilon) < i$, i.e., $F(Tz - z) < i$, which is a contradiction with the definition of i. Hence $\inf\{F(Tx - x): x \in C\} = 0$.

To prove that the infimum is attained is the easy part of the proof. Take the following system of sets:

$K_n = \{x: F(x - Tx) \le 1/2n(q + 1)\}$; $T(K_n)$ and $\overline{T(K_n)}$; where $n \in N$, $q = (a + p)/(1 - a)$ and $\overline{T(K_n)}$ is the closure of $T(K_n)$. Then for any $x, y \in K_n$:

$$F(Tx - Ty) \le qF(Tx - x) + qF(Ty - y) \le 1/n,$$

$$F(x - y) \le (q + 1)F(Tx - x) + (q + 1)F(Ty - y) \le 1/n,$$

i.e., $\text{diam}(K_n) \le 1/n$, $\text{diam}(T(K_n)) \le 1/n$ and therefore, since $\text{diam}(T(K_n)) = \text{diam}(\overline{T(K_n)})$ we have $\text{diam}(\overline{T(K_n)}) = 1/n$. It is clear that $\{K_n\}$ and $\{(\overline{T(K_n)}\}$ form a monotone sequences of sets and from (4) we have $T(K_n) \subset K_n$.

Suppose $y \in \overline{T(K_n)}$ then there exists $y' \in K_n$ such that $F(y - Ty') < \varepsilon$ for $\varepsilon > 0$ and

$$F(y - Ty) \le F(y - Ty') + F(Ty' - Ty) \le F(y - Ty') + aF(y - y') + pF(Ty' - y') + pF(Ty - y),$$

i.e., $(1 - p) F(y - Ty) \le (a + 1)\varepsilon + (a + p)F(Ty' - y')$,

and, in view of $a + p = 1 - p$ and $F(Ty' - y') \le 1/2n(q + 1)$:

$$F(y - Ty) \le \frac{a+1}{1-p}\varepsilon + \frac{1}{2n(q+1)}.$$

As $\varepsilon > 0$ is arbitrary, it follows that $F(y - Ty) \le 1/2n(q + 1)$ and we have $y \in K_n$. Hence $\overline{T(K_n)} \subset K_n$ too.
$\{\overline{T(K_n)}\}$ is a decreasing sequence of closed nonempty sets with $\text{diam}(\overline{T(K_n)}) \to 0$ as $n \to \infty$. Hence they have a nonempty intersection $\{x^*\}$ and T is a unique fixed point $Tx^* = x^*$.

Corollary 1: Let C be a closed convex subset of a Banach space X and $T: C \to C$ a mapping that satisfies $\|Tx - Ty\| \le a\|x - y\| + b\|Tx - x\| + c\|Ty - y\|$ for all $x, y \in C$ where $0 < a < 1$, $b \ge 0$, $c \ge 0$ and $a + b + c = 1$. Then T has a unique fixed point.

We now proceed to use Mann iteration scheme to approximate the fixed point of Gregus' mapping.

Theorem 2: Let C be a nonempty closed convex subset of a complete metrisable topological vector space X and let $T: C \to C$ be a mapping that satisfies $F(Tx - Ty) \le$

$aF(x - y) + bF(Tx - x) + cF(Ty - y)$ for all $x, y \in C$ where $0<a<1$, $b\geq 0$, $c\geq 0$ and $a+b+c = 1$. Suppose $\{x_n\}$ is a Mann iteration sequence defined by $x_{n+1} = (1-\alpha_n)x_n+\alpha_n Tx_n$, $x_0 \in C$, $n \geq 0$, where $\{\alpha_n\}$ satisfy $0 \leq \alpha_n \leq 1$ for all n and $\sum_0^\infty \alpha_n = \infty$ and $\alpha_n \leq \min\{1, \frac{1}{1-\delta}\}$ for each n, where $\delta = \max\{\frac{a+c}{1-c}, \frac{b+c}{1-c}\}$ and $c < \min\{a, b\}$. Then $\{x_n\}$ converges to the unique fixed point of T.

Proof: The fact that T has a unique fixed point is already shown in Theorem 1.

If $F(Tx - Ty) \leq aF(x - y) + bF(Tx - x) + cF(Ty - y)$, then

$$F(Tx - Ty) \leq aF(x - y) + bF(Tx - x) + c\{F(Ty - Tx) + F(Tx - x) + F(x - y)\}$$

which, after computation, gives

$$F(Tx - Ty) \leq \frac{a+c}{1-c} F(x - y) + \frac{b+c}{1-c} F(Tx - x)$$

If $\delta = \max\{\frac{a+c}{1-c}, \frac{b+c}{1-c}\}$, then

$$F(Tx - Ty) \leq \delta\{F(x - y) + F(Tx - x)\} \quad (6)$$

Also note that $\delta < 1$ since by assumption, $c < \min\{a, b\}$.

Suppose p is a fixed point of T, then, if $x_n = p$ and $y_n = x_n$, from (6), we obtain

$$F(Tx_n - p) \leq \delta\{F(x_n - p) \quad (7)$$

$$F(x_{n+1} - p) = F(1 - (1 - \alpha_n)x_n + \alpha_n Tx_n - (1 - \alpha_n + \alpha_n)p)$$

$$= F((1 - \alpha_n)(x_n - p) + \alpha_n(Tx_n - p)$$

$$\leq (1 - \alpha_n)F(x_n - p) + \alpha_n(Tx_n - p)$$

$$\leq [1 - (1 - \delta)\alpha_n] F(x_n - p) \text{ by (7).}$$

Since $1 - (1 - \delta)\alpha_n < 1$ by the choice of α_n in the theorem, then $\{x_n\}$ converges to p.

Corollary 2: Let C be a nonempty closed convex subset of a Banach space X and let $T: C \to C$ be a mapping that satisfies $||Tx - Ty|| \leq a||x - y|| + b||Tx - x|| + c||Ty - y||$ for all $x, y \in C$ where $0 < a < 1$, $b \geq 0$, $c \geq 0$ and $a + b + c = 1$. Suppose $\{x_n\}$ is a Mann iteration sequence defined by $x_{n+1} = (1-\alpha_n)x_n+\alpha_n Tx_n$, $x_0 \in C$, $n\geq 0$, where $\{\alpha_n\}$ satisfy $0\leq\alpha_n \leq 1$ for all n and $\sum_0^\infty \alpha_n = \infty$ and $\alpha_n \leq \min\{1, \frac{1}{1-\delta}\}$ for each n, where $\delta = \max\{\frac{a+c}{1-c}, \frac{b+c}{1-c}\}$ and $c < \min\{a, b\}$. Then $\{x_n\}$ converges to the unique fixed point of T.

Remarks

- A more general result cannot be obtained. When $a = 1$, then the mapping T becomes a nonexpansive map studied by Kirk (1965) and Olaleru (2006b), among others, in which case X must be assumed to a reflexive Banach space (reflexive metrisable locally convex space) and C must in addition have a normal structure in order for T to have a fixed point. For a discussion on normal structure Belluce *et al.* (1968). Gregus (1980) gave an example where $a =1$ and T does not have a fixed point.

  If $a = 0$, then T becomes Kannan maps studied by Kannan (1969, 1971), Wong (1975) and Olaleru (2006b) among others. For T to have a fixed point, C must be weakly compact in addition to the normal structure of C.
- The case $a + b + c < 1$ (with a, b, c nonnegative) is easy. This was already studied by Reic (1976). In this case T has a unique fixed point in any complete metric space.
- It is not yet known if this result can be generalized to the maps studied by Hardy and Rogers (1973) in which case map T satisfies

$$||Tx - Ty|| \leq a||x - y|| + b||Tx - x|| + c||Ty - y|| + d||Ty - x|| + e||Tx - y||$$

for all $x, y \in C$ where $0 < a < 1$, $b \geq 0$, $c \geq 0$, $d \geq 0$, $e \geq 0$ and $a + b + c + d + e = 1$.

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