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On Polar Moments of Inertia of Lorentzian Circles

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Abstract: In this study, we first compute the polar moment of inertia of orbit curves under planar Lorentzian motions and then give the following theorems for the Lorentzian circles: When endpoints of a line segment AB with length a +b move on Lorentzian circle (its total rotation angle is δ) with the polar moment of inertia T, a point X which is collinear with the points A and B draws a Lorentzian circle with the polar moment of inertia T_x. The difference between T and T_x is independent of the Lorentzian circles, that is, $T_x - T = \delta ab$. If the endpoints of AB move on different Lorentzian circles with the polar moments of inertia T_A and T_B, respectively, then $T_x = [aT_B + bT_A]/(a + b) - \delta ab$ is obtained.

Key words: Lorentzian motion, trigonometry in Lorentzian geometry, moment of inertia, lorentzian circle

INTRODUCTION

Holditch[1] and Blaschke and Muller[2] given the following theorem: If the endpoints A and B of a segment AB of fixed length a+b are rotated once on an oval in Euclidian plane, then a point $X(\overline{AX} = a, \overline{XB} = b)$ which is collinear with the points A and B describes a closed, not necessarily convex curve. The area of the ring-shaped domain bounded by the two curves is πab. Later, this classical result was generalized by different methods.

The computation of polar moment of inertia of closed orbit curves on planar kinematics is given by Müller[3]. Also, Müller[3] had given a Holditch-type theorem for polar moments of inertia.

Let IL^2 be the vector space $\mathbb{R}^2$ provided with Lorentzian inner product

$$<x, y> = x_1y_1 - x_2y_2, \text{ for } x = (x_1, x_1),\ y = (y_1, y_2) \in \mathbb{R}^2 \quad (1)$$

We say that a vector $x \in \mathbb{R}^2 - \{0\}$ is spacelike, timelike or null if and only if $<x, x>$ is positive, negative or zero, respectively. The norm $\|x\|$ is defined to be $\sqrt{|<x,y>|}$. A timelike vector $x = (x_1, x_2)$ is future-pointing (resp. past-pointing) if $<x, e> <0$ (resp. $<x, e> >0$) where $e = (0, 1)$. If x, y are future-pointing timelike vectors, then $<x, y> \le 0$, $x + y$ is a future-pointing timelike vector and $ch\theta = - <x, y>/\|x\|\|y\|$, where, θ is the hyperbolic angle between x and y[4]. If x and y are spacelike vectors, then we have the followings:

$$\left.\begin{array}{l} \text{i) } <x, y> \ge 0 \\ \text{ii) } x+y \text{ is a spacelike vector} \\ \text{iii) } ch\theta = <x,y>/\|x\|\,\|y\| \end{array}\right\} \quad (2)$$

Let A, B, C be three noncolinear points such that AB and AC are future-pointing timelike vectors and BC is a spacelike vector such that $<AB, BC> = 0$. Then,

$$chu = \|AB\|/\|AC\|,\ shu = \|BC\|/\|AC\|$$

where, u is the hyperbolic angle between AB and AC[1]. If AB and AC are spacelike vectors and BC is a timelike vector such that $<AB, BC> = 0$, then we have:

$$\|AC\|^2 = \|AB\|^2 - \|BC\|^2 \quad (3)$$

Let AB and BC be future-pointing timelike vectors (and so AC is future-pointing timelike vector). Then, a triangle VABC is called (timelike) pure triangle with vertices A, B, C. If AB and BC are spacelike vectors (and so is AC, as a consequence of Eq. 2-ii), then a triangle $VABC \leftrightarrow \Delta ABC$ is called spacelike triangle with vertices A, B, C. Also, for a spacelike triangle VABC, we have:

$$a^2 = b^2 + c^2 - 2bcch\hat{A}$$
$$c^2 = a^2 + b^2 - 2bcch\hat{C}$$
$$b^2 = a^2 + c^2 + 2bcch\hat{B}$$

These equations are called the Hyperbolic cosine law for a spacelike triangle VABC.

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Let P be a point in Lorentzian Plane and r>0. The curve $\{X \in IL^2 \mid \|PX\| = r\}$ has two branches and each of them is called a Lorentzian circle with center P and radius r. If PX is a spacelike vector (resp.,timelike) for $X = (x_1, x_2)$, $P = (p_1, p_2) \in IL^2$ then the curve is called a timelike (resp., spacelike) Lorentzian circle (that is, its tangent vectors are timelike (resp., spacelike)) with equation:

$$(p_1 - x_1)^2 - (p_2 - x_2)^2 = r^2 \text{ (resp., } (p_1 - x_1)^2 - (p_2 - x_2)^2 = -r^2)$$

Let A and B be two points on the timelike (resp., spacelike) Lorentzian circle. Then, the chord AB is a timelike (resp., spacelike) vector[4].

The planar Lorentzian motion: Let L and L′ ($L = L' = IL^2$) be moving and fixed Lorentzian planes and $\{O; l_1, l_2\}$ and $\{O'; l', l'\}$ be their coordinate systems, respectively. By taking

$$OO' = u = u_1 l_1 + u_2 l_2, \quad \text{for } u_1, u_2 \in \mathbb{R} \tag{4}$$

the motion defined by the transformation

$$x' = x - u = (x_1 - u_1)\, l_1 + (x_2 - u_2)\, l_2 \tag{5}$$

is called one-parameter planar Lorentzian motion and denoted by L/L′, where, OX = x and O′X = x′ are the position vectors with respect to the moving and fixed rectangular coordinate systems of a point $X = (x_1, x_2) \in L$, respectively. Taking $\varphi = \varphi(t)$ as the rotation angle between l_1 and l''_1, the equations:

$$\begin{aligned} l_1 &= \text{ch}\varphi\, l''_1 + \text{sh}\varphi\, l''_2 \\ l_1 &= \text{sh}\varphi\, l'_1 + \text{ch}\varphi\, l''_2 \end{aligned} \tag{6}$$

can be written[4]. Also, the rotation angle φ and the vectors x, x′ and u are continuously differentiable functions of a time parameter t. We suppose throughout this study $\varphi(t) = d\varphi/dt \neq 0$. From Eq. 4-6, we have the sliding velocity of L/L′as

$$V_f = \{-du_1 - (u_2 - x_2)\, d\varphi\}\, l_1 + \{-du_2 - (u_1 - x_1)\, d\varphi\}\, l_2$$

The solution of equation $V_f = 0$ is the pole point of L/L′, at that instant t. If we denote the Pole point by $P = (p_1, p_2) \in L$, then we have:

$$\begin{aligned} p_1 &= u_1 + du_2/d\varphi \\ p_2 &= u_2 + du_1/d\varphi \end{aligned} \tag{7}$$

The polar moments of inertia of the orbit curves on Lorentzian Kinematics: Let the Lorentzian motion L/L′ is restricted to time interval $[t_1, t_2]$ such that all points $X = (x_1, x_2) \in L$ moved only on timelike (resp., only on spacelike) region of L′. Then the polar moment of inertia of the orbit curve segment of the fixed point $X = (x_1, x_2) \in L$ is:

$$T_X = \int_{t_1}^{t_2} \|x'\|^2\, d\varphi \; [3]$$

Using the Lorentzian inner product, we can write:

$$T_X = \varepsilon \int_{t_1}^{t_2} (x_1'^2 - x_2'^2)\, d\varphi \tag{8}$$

$$\text{Where, } \varepsilon = \begin{cases} +1, & \text{if } x' \text{ is spacelike} \\ -1, & \text{if } x' \text{ is timelike} \end{cases}$$

The Eq. 4-6 permit us to write the followings:

$$\begin{aligned} u_1'^2 - u_2'^2 &= u_1^2 - u_2^2 \\ x_1'^2 - x_2'^2 &= x_1^2 - x_2^2 - 2u_1x_1 + 2u_2x_2 + u_1^2 - u_2^2 \end{aligned} \tag{9}$$

Then by Eq. 8 and 9, we get

$$T_X = \varepsilon \left\{ \begin{aligned} &(x_1^2 - x_2^2)\int_{t_1}^{t_2} d\varphi - 2x_1 \int_{t_1}^{t_2} u_1\, d\varphi \\ &+2x_2 \int_{t_1}^{t_2} u_2\, d\varphi + \int_{t_1}^{t_2} (u_1^2 - u_2^2)\, d\varphi \end{aligned} \right\}$$

So, from Eq. 7, we have

$$T_X = T_O + \varepsilon\,\delta(x_1^2 - x_2^2 - 2s_1x_1 + 2s_2x_2) \tag{10}$$

such that $\delta s_1 = \int_{t_1}^{t_2} p_1\, d\varphi + u_2(t_1) - u_2(t_2)$

$\delta s_2 = \int_{t_1}^{t_2} p_2\, d\varphi + u_1(t_1) - u_1(t_2)$, where, $\delta = \int_{t_1}^{t_2} d\varphi$ is the total rotation angle (Gesamtdrehwinkel) of L/L′, T_o is the polar moment of inertia of the orbit curve segment of the point O and $S = (s_1, s_2) \in L$ is the Steiner point of L/L′.

Finally, we can give the following theorem:

Theorem 3: During Lorentzian motion L/L′, all the fixed points $X \in L$ which have equal polar moment of inertia T_x lie on the same Lorentzian circle with the center S on L.

Remark 1: Since the property of being a timelike (resp., spacelike) vector is preserved under the motion L/L′, a timelike (resp., spacelike) vector of L must move on timelike (resp., spacelike) Lorentzian circle in L′.

Then, we can give the following theorems for the polar moments of inertia:

Theorem 4: Let the points A = (0, 0) and B = (0, a + b) ∈ L move on the timelike Lorentzian circle with center O′ and radius r on L′.Then, the point X = (0, a) which is collinear with points A and B draws the timelike Lorentzian circle with center O′ and the radius $\sqrt{r^2+ab}$ on L′. The difference between the polar moments of inertia of timelike Lorentzian circles is δab.

Proof: Let M be a midpoint of AB. Since the perpendicular bisector of a chord of a timelike (resp., spacelike) Lorentzian circle passes through the center of circle we can write <O′M, AB> = 0 (Fig. 1). Thus, VO′MX↔ΔO′MA, VO′MA↔ΔO′MA are right triangle. From Eq. 3 we have $\|O'X\|^2 = \|O'M\|^2 - \|MX\|^2$ and $\|O'A\|^2 = \|O'M\|^2 - \|MA\|^2$. Hence, we obtain $\|O'X\|^2 = r^2 + ab$.

Since a, b and r are constant, it is shown that the point X draws the timelike Lorentzian circle with the radius $\sqrt{r^2+ab}$.

Moreover, for the polar moments of inertia of the orbit curves of the points A, B, X, from Eq. 10, we obtain:

$$T_B = T_A + \delta\,[-(a+b)^2 + 2s_2(a+b)]$$
$$T_X = T_A + \delta\,[-a^2 + 2s_2 a]$$
or
$$T_X = [aT_b + bT_A]/(a+b) + \delta ab$$

Since $T_A = T_B$, we find

$$T_X - T_A = \delta ab. \qquad (11)$$

Remark 2: If the points A = (0, 0) and B = (a + b, 0) ∈ L move on the spacelike Lorentzian circle with center O′ and radius r on L′, then the point X = (a, 0) which is collinear with points A and B draws the spacelike Lorentzian circle with center O′ and the radius $\sqrt{r^2+ab}$ on L′. The difference between the polar moments of inertia of spacelike Lorentzian circles is δab.

Theorem 5: Let the points A = (0, 0) and B = (a + b, 0) ∈ L move on the timelike Lorentzian circles with center O′ and radii r_1 and r_2 on L′, respectively. Then, the point X = (a, 0) which is collinear with points A and B draws the timelike Lorentzian circle with center O′ and the radius

$$\sqrt{\frac{br_1^2+ar_2^2}{a+b}-ab}$$ on L′and the polar moment of inertia of the timelike Lorentzian circle of the point X is

$$T_X = [aT_B + bT_A]/(a+b) - \delta ab \qquad (12)$$

Proof: Since O′X, O′A, O′B and AB are spacelike vectors, VO′AB↔ΔO′AB is a spacelike triangle. Then we can write O′A = r_1 (chα, shα), O′B = r_2 (chβ, shβ) and

$$O'X = \frac{b}{a+b}O'A + \frac{a}{a+b}O'B$$ (Fig. 2). Thus, we obtain

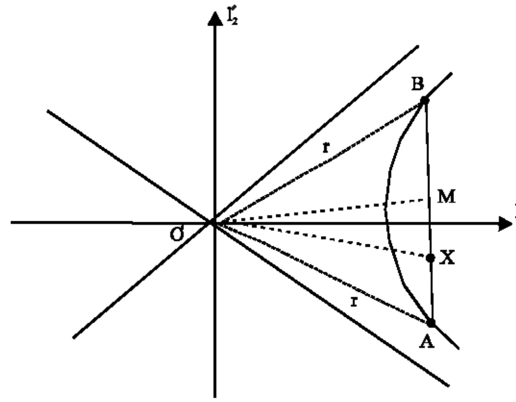

Fig. 1: Holditch-motion of a line segment AB

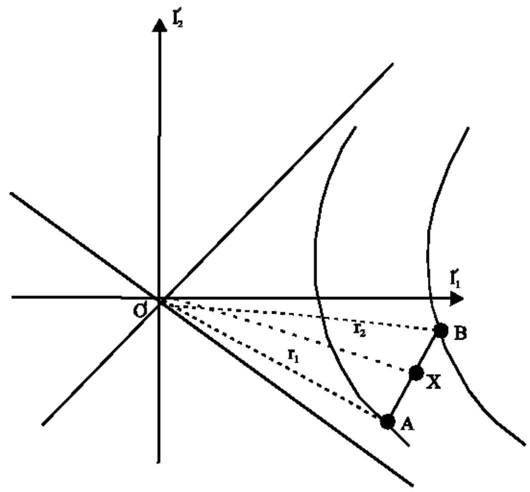

Fig. 2: The orbits of three collinear points

$$\|O'X\|^2 = \left(\frac{b}{a+b}\right)^2 r_1^2 + \frac{2ab}{(a+b)^2}\,ch(\beta-\alpha) + \left(\frac{a}{a+b}\right)^2 r_2^2$$

Using the hyperbolic cosine law for spacelike triangle, we have

$$ch(\beta-\alpha) = \frac{r_1^2 + r_2^2 - (a+b)^2}{2r_1r_2}$$

Then we get $\|o'X\|^2 = \frac{br_1^2+ar_2^2}{a+b} - ab.$

Since a, b, r_1 and r_2 are constants, it is shown that the point X draws the timelike Lorentzian circle with center O′ and the radius $\sqrt{\frac{br_1^2+ar_2^2}{a+b}-ab}$. In addition, from Eq. 10, for the polar moments of inertia of the orbit curves of the points A, B, X, we find

$$T_B = T_A + \delta[(a+b)^2 - 2s_1(a+b)],$$

$$T_X = T_A + \delta[a^2 - 2s_1 a]$$

From these equations, we get Eq. 12.

Remark 3: If the points A = (0, 0) and B = (0, a + b) ∈ L move on the spacelike Lorentzian circles with center O′ and radii r_1 and r_2 on L′, respectively, then VL′AB↔ΔO′AB is a pure triangle. Using the hyperbolic cosine law[4] for pure triangle, the point X = (0, a) ∈ L which is collinear with points A and B draws the spacelike Lorentzian circle with center O′ and radius

$$\sqrt{\frac{br_1^2 + ar_2^2}{a+b} - ab}$$

on L′. Using Eq. 10, for the polar moments of inertia T_A, T_B and T_X we have $T_X = [aT_B + bT_A]/(a + b) - \delta ab$

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