Journal of Applied Sciences

ISSN 1812-5654

Journal of Applied Sciences 6 (1): 80-84, 2006
ISSN 1812-5654

The Lyapunov Exponents of the Impact Oscillator

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Abstract: This study concerns the application of computing Lyapunov exponents for impact oscillator. The impact oscillator is considered as non-smooth dynamic system due to the motion constraint; therefore the Jacobian matrix for impact map instead of Poincare map is derived for the calculation of Lyapunov exponents. The results show the spectrum of Lyapunov exponents agrees with the plot of bifurcation diagram for varying the driving frequency.

Key words: Impact oscillator, bifurcation diagram, Lyapunov exponents

INTRODUCTION

Impact dynamics is considered to be one of the most important problems which arise in vibrating systems. Such impacting oscillators may occur in the motion with amplitude constraining stops (Fig. 1). This problem has been researched in detail using bifurcation theory[1-7]. Different types of impacting response due to different ranges of driving frequency or control parameters can be predicted from bifurcation diagrams. In this study, calculation of the Lyapunov exponents for impact oscillator is considered. Lyapunov exponents measure the average divergence or convergence rate of nearby trajectories on a particular return map in space. If the largest Lyapunov exponent is positive the trajectory is chaos, whereas non-positive exponents indicate the trajectory is a stable motion. Thus the spectrum of Lyapunov exponents is one of the most useful diagnostics for systems.

Lyapunov exponents described the system's behavior and provide the stability or instability of an equilibrium point of the non-linear system of the differential equation. For m-dimension system, with the mapping P: $R^m, \rightarrow R^m$,

$$t_{n+1} = P(t_n) \qquad n = 1,2,3... \tag{1}$$

a small perturbation of this mapping $t_n \rightarrow t_n + \delta t_n$, the Taylor expansion of Eq. 1 and the linearization model is

$$\delta t_{n+1} \approx DP^{(n)}(t_n)\delta t_n \tag{2}$$

The stability at point t_n is decided by the eigenvalues of the Jacobian matrix $DP^{(n)}(t_n)$. The Lyapunov exponents measure the average rate of the convergence or divergence of the system and are defined by many researchers[8-10].

$$\lambda_m = \lim_{N\rightarrow\infty} \frac{1}{N} \ln(\mathrm{eig} \prod_{n=1}^{N} DP^{(n)}(t_n)) \tag{3}$$

Thus m exponents of the average rate λ_m can be calculated if the Jacobian matrix is available.

In general, if the underlying dynamic system is smooth, then the spectrum of Lyapunov exponents can be calculated using the Poincare return map[10]. For example, a three dimensions flow in (t, x, v) space, the solutions can be projected onto a particular section $t = \psi$, where ψ is a constant in $[0, T = 2\pi/\omega]$. Thus a two-dimensional Poincare map is defined by a difference equation,

$$x_{n+1} = f(x_n, v_n) \text{ and } v_{n+1} = g(x_n, v_n) \tag{4}$$

where, x, v are vectors and f, g are non-linear transformations. The Jacobian matrix is expressed as:

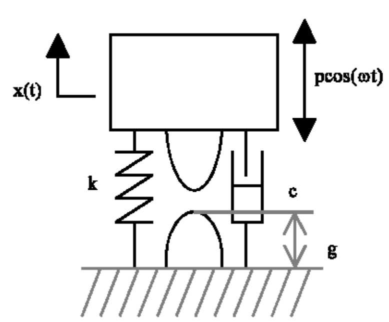


Fig. 1: Impact oscillator

$$DP = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial v} \end{bmatrix} \quad (5)$$

In the current problem of the impact oscillator under consideration is non-smooth due to the amplitude constraint. Thus the equation of the Poincare map is non-smooth when impact occurs and the derivative in Eq. 5 is not defined[11-14]. To overcome this problem the impact map is considered to develop the algorithms.

MATHEMATICAL MODEL

The impact oscillator is considered in Fig. 1, where the system is under a harmonic excitation with an amplitude constraint. The impact oscillator is governed by the following equation

$$\begin{aligned} &m\ddot{x} + 2c\dot{x} + kx = p\cos\omega t, && \text{for } x > g \\ &x(t^+) = x(t^-) \text{ and } \dot{x}(t^+) = -r\dot{x}(t^-), && \text{for } x \le g \end{aligned} \quad (6)$$

where, $\ddot{x}$, $\dot{x}$, x are the acceleration, the velocity, the displacement and m, c, k, p, ω, g are the mass, the damping, the stiffness, the forcing amplitude, the forcing frequency, the amplitude constraint, respectively. The impact occurs whenever x = g and the velocity is modeled by $\dot{x}(t^+) = -r\dot{x}(t^-)$, where the t^+ represents the time after impact, the t^- represents the time before impact and r is restitution coefficient. For simplicity, we set m = 1, c = 0, k = 1 and p = 1 then the system convert to

$$\begin{aligned} &\ddot{x} + x = \cos\omega t, && \text{for } x > g \\ &x(t^+) = x(t^-) \text{ and } \dot{x}(t^+) = -r\dot{x}(t^-), && \text{for } x \le g \end{aligned} \quad (7)$$

With initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = v(t_0) = v_0$ the solution of the displacement x and the velocity v between impacts are:

$$\begin{aligned} x = {} & [x_0 - z\cos(\omega t_0)]\cos(t - t_0) \\ & + [v_0 + z\omega\sin(\omega t_0)]\sin(t - t_0) + z\cos(\omega t) \end{aligned} \quad (8)$$

$$\begin{aligned} v = {} & [-x_0 + z\cos(\omega t_0)]\sin(t - t_0) \\ & + [v_0 + z\omega\sin(\omega t_0)]\cos(t - t_0) - z\omega\sin(\omega t) \end{aligned} \quad (9)$$

where, $z = 1/(1-\omega^2)$.

JACOBIAN MATRIX FOR IMPACT MAP

To obtain the difference equation of Eq. 1, the variables x_n, v_n and t_n are considered as the impact n, then the variables x_{n+1}, v_{n+1} and t_{n+1} are obtained from Eq. 8 and 9 for the initial conditions

$$(t_0, x_0, v_n) = (t_n, x_n, -rv_n) \quad (10)$$

Thus the difference equations can introduce as below:

$$\begin{aligned} x_{n+1} = {} & [x_n - z\cos(\omega t_n)]\cos(t_{n+1} - t_n) + \\ & [-rv_n + z\omega\sin(\omega t_n)]\sin(t_{n+1} - t_n) + z\cos(\omega t_{n+1}) \end{aligned} \quad (11)$$

$$\begin{aligned} v_{n+1} = {} & [-x_n + z\cos(\omega t_n)]\sin(t_{n+1} - t_n) + \\ & [-rv_n + z\omega\sin(\omega t_n)]\cos(t_{n+1} - t_n) - z\omega\sin(\omega t_{n+1}) \end{aligned} \quad (12)$$

where, $x_{n+1} = x_n = g$. Thus the impact return map can be constructed from a three dimensions flow in (t, x, v) space onto a two dimension map (t,v) when x = g. Using Eq. 11 and 12, the Jacobian matrix for impact map can be derived[1,2],

$$DI = \begin{bmatrix} \frac{\partial t_{n+1}}{\partial t_n} & \frac{\partial t_{n+1}}{\partial v_n} \\ \frac{\partial v_{n+1}}{\partial t_n} & \frac{\partial v_{n+1}}{\partial v_n} \end{bmatrix} \quad (13)$$

The components in Eq. 13 are:

$$\begin{aligned} \frac{\partial t_{n+1}}{\partial t_n} &= \frac{1}{v_{n+1}}\frac{\partial x_{n+1}}{\partial t_n} \\ &= \frac{1}{v_{n+1}}\left[rv_n\cos(t_{n+1} - t_n) - A_n\sin(t_{n+1} - t_n)\right] \end{aligned} \quad (14)$$

$$\frac{\partial t_{n+1}}{\partial v_n} = \frac{1}{v_{n+1}}\frac{\partial x_{n+1}}{\partial v_n} = \frac{-r}{v_{n+1}}\left[\sin(t_{n+1} - t_n)\right] \quad (15)$$

$$\begin{aligned} \frac{\partial v_{n+1}}{\partial t_n} &= \frac{\partial v_{n+1}}{\partial t_n} + A_{n+1}\frac{\partial t_{n+1}}{\partial t_n} \\ &= -\left[A_n + \frac{rv_nA_{n+1}}{v_{n+1}}\right]\cos(t_{n+1} - t_n) \\ &\quad -\left[rv_n - \frac{A_nA_{n+1}}{v_{n+1}}\right]\sin(t_{n+1} - t_n) \end{aligned} \quad (16)$$

$$\frac{\partial v_{n+1}}{\partial v_n} = \frac{\partial v_{n+1}}{\partial v_n} + A_{n+1}\frac{\partial t_{n+1}}{\partial v_n} = -r\begin{bmatrix} \cos(t_{n+1} - t_n) - \\ \frac{A_{n+1}}{v_{n+1}}\sin(t_{n+1} - t_n) \end{bmatrix} \quad (17)$$

where,

$$A_n = \cos(\omega t_n) - x_n \text{ and } A_{n+1} = \cos(\omega t_{n+1}) - x_{n+1} \quad (18)$$

are the accelerations at the two impacts.

LYAPUNOV EXPONENTS

Using the Jacobian matrix of Eq. 13 and the definition of Eq. 3, two Lyapunov exponents of impact oscillator can be obtained by:

$$\lambda_{1,2} = \lim_{N\to\infty}\frac{1}{N}\ln(\mathrm{eig}\prod_{n=1}^{N}DI^{(n)}) \quad (19)$$

In computation of the eigenvalues of the matrix J, that is given by:

$$J = \prod_{n=1}^{N}DI^{(n)} \quad (20)$$

Only a few iterations, the matrix J becomes very large for chaotic case and null for the periodic case because the product of the matrix DI in Eq. 20. To overcome this problem, the QR-factorisation technique applied[15-17]. Given a matrix A, there is a unique factorization

$$A = QR \quad (21)$$

where, R is a square right upper triangular matrix and Q has orthogonal columns. Consider an arbitrary orthogonal matrix Q_0 and apply the orthogonalisation procedure to the impact map

$$Q_1R_0 = DI^{(1)}Q_0 \quad (22)$$

The columns Q_1 are now an orthogonal basis for the impact map at n = 1 and R_0 gives the relationship between this map and the image of the Q_0 under the Jacobian matrix $DI^{(1)}$. This process can be repeated recursively and gives:

$$\prod_{n=1}^{N}DI^{(n)}Q_0 = Q_nR_{n-1}R_{n-2}...R_1R_0 \quad (23)$$

Thus the calculation of the Lyapunov exponents in Eq. 19 is converted to the summation of the matrix $\ln(R_j)$ in Eq. 24.

$$\lambda_{1,2} = \lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1}\ln(R_j) \quad (24)$$

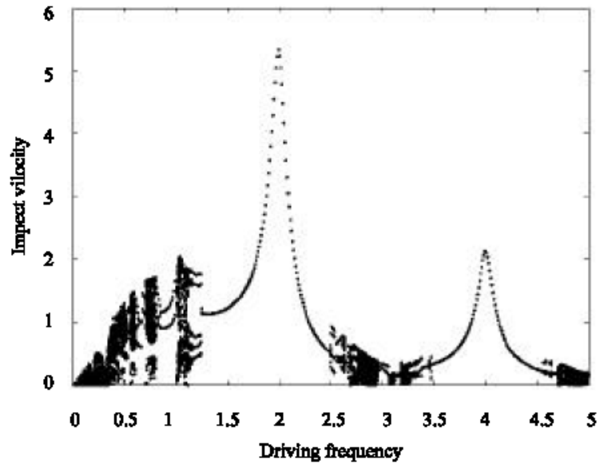


Fig. 2: Bifurcation diagram of the impact oscillator under the parameters m=1, c=0, k=1, p=1, r=0.8 and g=0

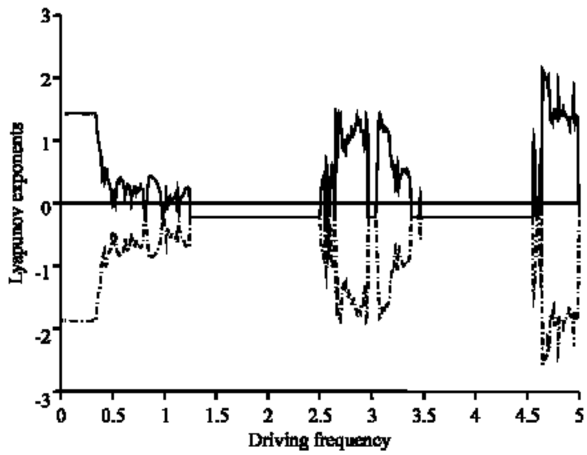
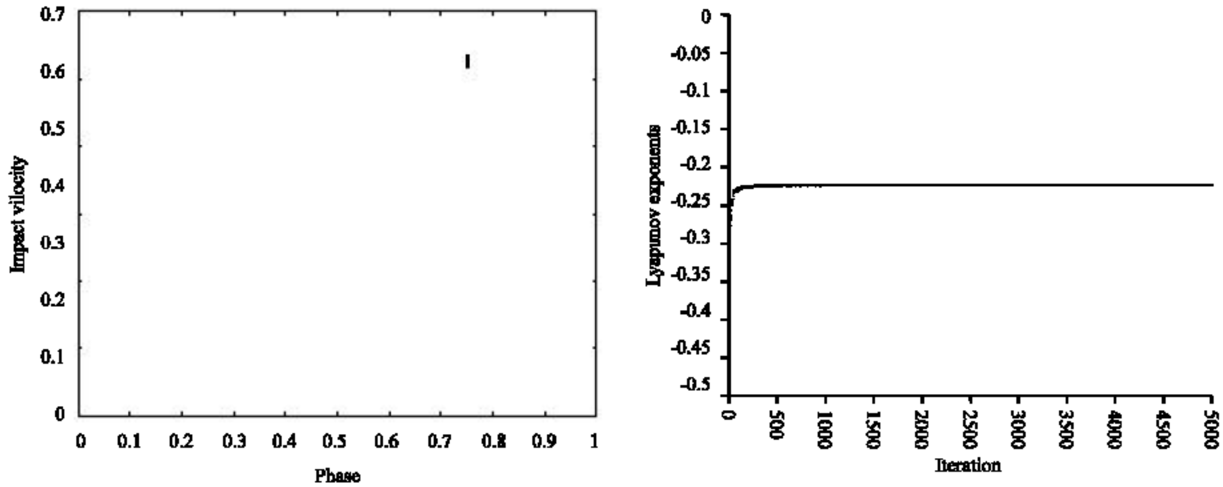


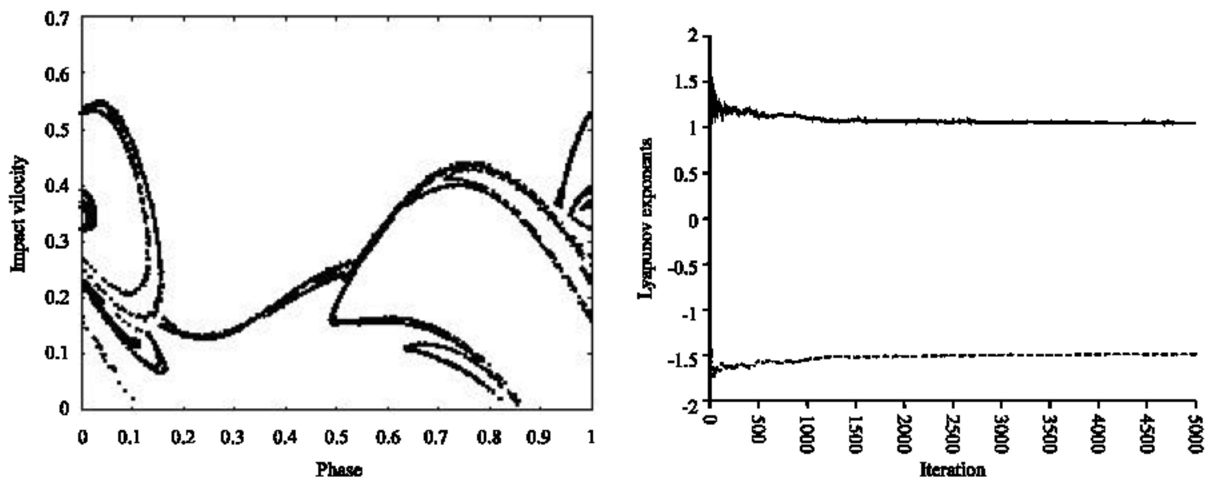
Fig. 3: Estimated Lyapunov exponents of the impact oscillator under the parameters m=1, c=0, k=1, p=1, r=0.8 and g=0

The result shown in Fig. 2 is the plot of bifurcation diagram for varying the driving frequency; the regions of the stable and unstable oscillator are demonstrated by points set in impact map. The corresponding results of the estimated Lyapunov exponents are shown in Fig. 3, where the positive values indicate the stable motions and the non-positive values indicate unstable motions. For example, the results in Fig. 4a show periodic points set in impact map. The corresponding results of the estimated Lyapunov exponents are shown in Fig. 4b, where both exponents converge to non-positive values after 5000 iterations. This indicates that the impact oscillator is stable at driving frequency $\omega = 2$. For the case of driving frequency at $\omega = 2.8$, the results show strange



(a) (b)

Fig. 4: Impact oscillator under the parameters m=1, c=0, k=1, p=1, r=0.8, g=0 and ω=2 (a) impact map (b) Lyapunov exponents



(a) (b)

Fig. 5: Impact oscillator under the parameters m=1, c=0, k=1, p=1, r=0.8, g=0 and ω=2.8 (a) impact map (b) Lyapunov exponents

points set in Fig. 5a and one of the estimated Lyapunov exponents converges to a positive value in Fig. 5b. This confirms that the impact oscillator is unstable.

CONCLUSIONS

The algorithm of computing Lyapunov exponents is successfully applied to the impact oscillator. The impact oscillator is considered as non-smooth dynamic system; therefore the Jacobian matrix for impact map is derived. The results show the plot of bifurcation diagram consists with the estimated Lyapunov exponents for varying the driving frequency.

REFERENCES

1. Shaw, S.W. and P.J. Holmes, 1983. A periodically forced linear piecewise linear oscillator. J. Sound Vib., 90: 129-155.
2. Budd, C., F. Dux and A. Cliffe, 1995. The effect of frequency and clearance variations on single-degree-of-freedom impact oscillators. J. Sound Vib., 184: 475-502.
3. Lee, J.-Y. and A.K. Nandi, 1998. Blind deconvolution of impacting signals using high order statistics. Mech. Sys. Sig. Process., 12: 357-371.

4. Luo, G. and J. Xie, 2001. Bifurcation and chaos in a system with impacts. Physica D., 148: 183-200.
5. Luo, C.J., 2002. An unsymmetrical motion in a horizontal impact oscillator. ASME J. Vib. Acoust., 124: 420-425.
6. Czolczynski, K. and T. Kapitaniak, 2003. Influence of the mass and stiffness ratio on a periodic motion of two impacting oscillators. Chaos Solitons Fractals, 17: 1-10.
7. Lee, J.Y., 2005. Motion behavior of impact oscillator. J. Marine Sci. Technol., 13: 86-92.
8. Darbyshire, A.G. and D.S. Broomhead, 1996. Robust estimation of tangent maps and *Lyapunov spectra*. Physica D., 89: 287-305.
9. Banbrook, M., G. Ushaw and S. McLaughlin, 1997. How to extract Lyapunov exponents from short and noisy time series. IEEE Signal Processing, 45: 1378-1382.
10. Guckenheimer, J. and P.J. Holmes, 1993. Nonlinear Oscillations. Dynamical Systems and Bifurcations of Vector Fields. Springer-Verlag.
11. Muller, P.C., 1995. Calculation of Lyapunov exponents for dynamic systems with discontinuities. Chaos Solitons Fractals, 5: 1671-1681.
12. Galvanetto, U., 2000. Numerical computation of Lyapunov exponents in discontinuous maps implicitly defined. Computer Physics Communications, 131: 1-9.
13. Stefanski, A., 2000. Estimation of the largest Lyapunov exponent in systems with impacts. Chaos Solitons Fractals, 11: 2443-2451.
14. Souza, S.L.T. and I.L. Caldas, 2004. Calculation of Lyapunov exponents in systems with impacts. Chaos Solitons Fractals, 19: 569-579.
15. Eckmann, J.P. and D. Rulle, 1985. Ergodic theory of chaos and strange attractors. Rev. Mod. Phys., 57: 617-657.
16. Johnson, R.A., K.J. Palmer and G.R. Sell, 1987. Ergodic properties of linear dynamical systems. SIAM J. Math. Anal., 18: 1-33.
17. Stewart, G.W., 1973. Introduction to Matrix Computations. Academic Press.