Journal of Applied Sciences

ISSN 1812-5654

Journal of Applied Sciences 6 (4): 843-853, 2006
ISSN 1812-5654

Constitutive Equations for an Elastic Media with Micro-voids

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Abstract: Continuum Damage Mechanics deals with elastic or inelastic materials which undergo structural weakening as a result of micro-crack formation or micro-voids growth. In this study, a relevant mathematical model is developed in the context of Continuum Damage Mechanics. This mathematical model represents mechanical behavior of an elastic media which have micro-voids and which is subjected to a mechanical loading. As a result, constitutive equations for the stress and strain-energy density release rate are presented for the media under consideration.

Key words: Continuum media, damage, helmholtz free energy, stress, strain-energy density release rate, constitutive equations

INTRODUCTION

Continuum Damage Mechanics (CDM) is a research field rendering necessary a combined application of large number of mechanics branches, engineering mathematics and material science. Damage mechanics examines mechanisms addressing situations like failure, deterioration or becoming useless that are covered by the general concept of damage in materials affected by various loads. The literature states that the first theories related with CDM, especially with creep and fatigue have been put forward by Kachanov in 1958 and by Rabotnov in 1969 (Chaboche, 1981; Murakami, 1988).

Later on, Kachanov (1986) discussed subjects like types of damage, damage variables, isotropic damage, kinetic law of damage, damaged elastic materials, creep and crack under uniaxial stress, crack growth and fatigue damage (Kachanov, 1986). Having clearly defined borders and relationships between the Classical Crack Mechanics and CDM, Lemaitre (1996) has presented a summary of various damage variables and constitutive equations obtained according to these variables. Subjects like basic concepts and definitions related with CDM, selection of damage parameters, brittle and ductile deformation processes and constitutive equations of damaged materials have been researched by Krajcinovic (1983), Krajcinovic and Mier (2000). Examining the damage mechanics in the most systematic way by separating it into subdivisions of micromechanical models and continuous media models, Krajcinovic (2003) refers to about 900 sources related with the subject in his publication.

As is known, considering micro scales, when the distance between atoms is at a critical value, the energy between these atoms reaches its maximum level. As a result of external loads and consequent change in this distance between the atoms, a decrease occurs in the interaction energy between the atoms. This leads to a weakening in atomic bonds and a decrease in cohesion powers. This situation creates micro voids and discontinuity surfaces in a material. Thus, it can be stated that a micro level damage takes a start in the material (Woo and Li, 1993).

In damage mechanics, scales are considered on three different levels: micro, meso and macro. A micro level damage is the accumulation of dislocations and micro stresses close around defects and interfaces as well as bond tear-off. On the meso level, beginning of a crack forms due to unified motion and growth of micro cracks or micro voids inside Representative Volume Element (RVE). In the macro level the cracks and voids grow. The first two scales can be examined by using Continuum Mechanics (CM) damage variables while macro scale is subject to the crack mechanics (Lemaitre, 1996; Krajcinovic, 2003). A systematic study was done by Kattan and Voyiadjis (2002) that considers damage models using the finite element method based on the ground principles of CM and the effective stress concept. In a work there general expressions pertaining to constitutive equations of isotropic elastic damaged materials was derived from the basic law of thermodynamics of irreversible processes, Helmholtz free energy was selected as a free energy constitutive functional for a damaged material and opened in Taylor

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series with two conditional variables. The classical damage constitutive equation developed in relation with the strain equivalence principle is expressed to be a simplified form of the general expression stated in this study (Song *et al*., 2001). Studying the electro-mechanic behavior of damage in ferroelectric materials, Bassiouny has developed a phenomenological model setting off from the principles of thermodynamics. Considering effective values he studied the effects of different type electromechanical couplings (Bassiouny, 2005).

In this study in the scope of CDM, considering RVE, the mechanical representation of damage was expressed with two interior conditional variables showing the properties of a second degree symmetrical tensor. In scope of CM, balance equations have been summarized and, without diving into too much detail, the combined form was stated for the energy equation and entropy inequality. Considering necessary constitutive axioms, after determining the arguments affecting the stress potential it was further proceeded to the formulation of the constitutive theory and a model has been formed for the damaged elastic isotropic medium. Considering fully orthogonal transformations for the material coordinate system, after determining the common invariants affecting the stress potential, constitutive equations have been obtained related with the stress and the strain energy density release rate.

Mechanical representation of damage: To obtain constitutive equations and thus determines the mechanical behavior it is first necessary to determine damage variables, consider the free energy as a function of a damage variable and determine scale invariants. For this purpose we have to determine the structure of the damage variables. Regarded in the beginning as linear, homogenous and isotropic, the material becomes anisotropic under the effect of damage. For example, in case of a uniaxial stress the damage parameter shown by D can be defined as follows (Fig. 1):

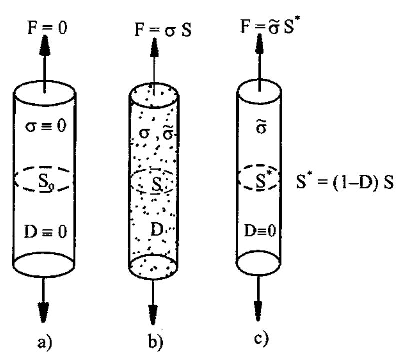

Fig. 1: Definition of damage parameter of a bar under uniaxial tension (Murakami, 1988)

Due to the existence in the material of dispersed microscopic cracks and voids representing damage, the effective area carrying the load is less than the S_o area that existed in the beginning. In this situation, S^* representing the remaining effective area after the micro voids have been subtracted should be given as follows. DS here represents the value of area that turned into voids due to the damage and DS<S. Thus, for D<1, the expression

$$S^* \equiv S - DS \quad \Rightarrow \quad D = 1 - \frac{S^*}{S} \tag{1}$$

can be written. S^* here represents the cross sectional area not affected by the damage. Considering the definition provided above the following limitations can be written.

$$0 \le D \le 1$$
$$D = 0 \text{ (Initial undamaged state)} \tag{2}$$

$$D = 1 \text{ (Final ruptured state)} \tag{3}$$

The decrease in the total area carrying the load determines the distribution of the stress σ created by the external force F. Referring to definition (1), for the effective stress $\tilde{\sigma}$ the following equality can be written.

$$\tilde{\sigma} = \frac{F}{S^*} = \frac{F}{S(1-D)} = \frac{\sigma}{(1-D)} \tag{4}$$

Just in a similar way to the effective stress principle, a suitable definition has been made for the effective strain by some researchers (Lemaitre, 1996; Ibijola, 2002). Equivalence strain for the isotropic damage condition can be expressed as follows:

$$\tilde{\varepsilon} = (1-D)\varepsilon$$

Using effective stress principle the relationship between a damaged and an undamaged material can be expressed as below:

$$\varepsilon = \frac{\tilde{\sigma}}{E} = \frac{\sigma}{(1-D)E} = \frac{\sigma}{\tilde{E}} \Rightarrow \tilde{E} = E\,(1-D) \tag{5}$$

Here, DE expressed the decrease in the elasticity module occurring due to damage whereas $\tilde{E}$ stands for the Young's modulus of damaged material.

Now, to define the fictitious undamaged state a cross sectional area S^* mechanically equivalent to a real damaged state and an imaginary element under affected by the load F applied can be considered, naming this an undamaged situation (Fig. 1c). If, in the two mechanically equivalent states, the relationship between the areas S

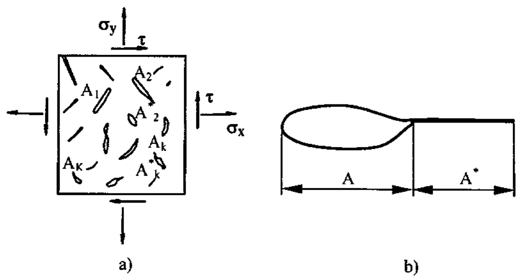

Fig. 2: a) A sketch of a representative volume element containing K micro cracks (Weitsman, 1988a), b) Open and closed surfaces of a micro crack

and S^* can be somehow defined then the damage variable D can be assigned using Eq. 1 (Murakami, 1988; Lemaitre, 1996).

In some researches, in order to be able to define the damage variable, a RVE has been considered that has a K number of micro cracks. While the open or active part of any k^{th} micro crack has been shown by $A^{(k)}$, its closed or passive surface has been shown by $A^{*(k)}$. Active or passive surfaces of a crack can switch positions among each other depending on stress, temperature and humidity percentage. Despite that, Weitsman states that these open and closed surfaces can be selected as independent variables characterizing the state of a material at a certain time range (Weitsman, 1988a, b).

Stress and strain at the macro level are average values over the RVE volume. Infinitesimal deformations can also be considered among these macro values. To fully consider the behaviors of RVE it is necessary to deal with a K number of crack parameters representing $A^{(k)}$ and $A^{*(k)}$, (no sum on k, k=1,...,K) surfaces. Because the real shape of these surfaces is unknown on the meso scale, assuming them to be equivalent plane surfaces, Weitsman represented mthem by vectors $\mathbf{A}^{(k)} = A^{(k)} n^{(k)}$ and $\mathbf{A}^{*(k)} = A^{*(k)} n^{(k)}$. Here, $n^{(k)}$ stands for a unit normal vector of a micro crack surface (Weitsman, 1988a). Let us consider two micro cracks inside a material with different convexities around a material point. Depending on the load applied on the material the cracks having different convexities can demonstrate different types of behavior depending on their crack surfaces. Different infinitesimal crack surfaces with very big curvature radii can be accepted topologically and mechanically equivalent. In this case topologic representation of the crack surface can be expressed independent of the direction of that surface. Mathematically that representation can be shown by using the symmetric tensor, which is a dyadic product of two vectors. Boldface letters are used to describe vectors and tensors. Thus, any micro crack can be defined using symmetric dyads as follows.

$$\mathbf{H}^{(k)} = \mathbf{A}^{(k)} \otimes \mathbf{A}^{(k)} \text{ and } \mathbf{H}^{*(k)} = \mathbf{A}^{*(k)} \otimes \mathbf{A}^{*(k)}, \quad H_{ij}^{(k)} = A_i^{(k)} A_j^{(k)} \tag{6}$$

Because detailed information about the value and location of surfaces $A^{(k)}$ and $A^{*(k)}$ can only be found statistically on the micro scale, on the meso scale where Continuum Mechanics is used, we can show the combined effects of the tensor expressions stated in (6) by the sum of the dyadic products given below. This operation represents homogenization when moving from the micro to the meso scale.

$$\mathbf{H} = \sum_{k=1}^{K} \mathbf{A}^{(k)} \otimes \mathbf{A}^{(k)} \text{ and } \mathbf{H}^* = \sum_{k=1}^{K} \mathbf{A}^{*(k)} \otimes \mathbf{A}^{*(k)} \tag{7}$$

Thus, the effect of damage on the meso scale can be expressed with two interior conditional variables, the variables bearing second degree symmetric tensor characteristics, as stipulated by their definitions. Dealing with infinitesimal deformations does not mean that tensors representing damage bear separate infinitesimal characteristics (Weitsman, 1988a). Therefore, while power series is being used for representative strains is may not be useable for the damage tensors. As the constitutive variable, in this study we are going to deal with only one damage tensor taking into consideration only the effect of open micro surfaces.

FORMULATION-BALANCE EQUATIONS

In this part, in order not to keep the size of our study compact, we provide the summary of the local form of Continuum Mechanics equilibrium equations. Researches wishing to obtain detailed information on this subject are advised to refer to basic sources related with Continuum Mechanics (Eringen, 1967; Şuhubi, 1993; Spencer, 1972).

$$\dot{\rho} + \rho \nabla \cdot v = 0 \quad \text{(Conservation of mass)} \tag{8}$$

$$\nabla \cdot t + \rho \left(f - a \right) = 0 \quad \text{(Balance of linear momentum)} \tag{9}$$

$$t = t^T \text{ or } t_{ij} = t_{ji} \quad \text{(Balance of angular momentum)} \tag{10}$$

$$\rho\dot{\varepsilon} - t : d + \nabla \cdot q - \rho h = 0 \quad \text{(Conservation of energy} \tag{11}$$

$$\rho\dot{\eta} - \rho \frac{h}{\theta} + \nabla \cdot \left(\frac{q}{\theta} \right) \geq 0 \quad \text{(Entropy inequality)} \tag{12}$$

Here, ρ, v, t, f, a, d, q, h, η and θ respectively represent, mass density per unit volume, velocity vector, stress tensor, body force density per unit volume, acceleration vector, deformation rate tensor $\left(d_{kl} = \frac{1}{2}\left(v_{k,l} + v_{l,k}\right)\right)$, heat flux vector, heat source per unit mass, entropy density and absolute temperature. Dots over the symbols show derivatives of the related values following the motion (e.g. $\dot{\rho} \equiv \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho$).

After the expressions (11) and (12) stated below are suitably combined, we get the following inequality

$$-\frac{\rho}{\theta}\left(\dot{\varepsilon} - \theta\,\dot{\eta}\right) + \frac{1}{\theta}\,t : d - \frac{1}{\theta^2}\,q \cdot \nabla\theta \geq 0 \tag{13}$$

Because the material derivative of the entropy density cannot be controlled inside a thermodynamic process, Helmholtz free energy should be defined by a Legendre transformation as follows

$$\psi \equiv \varepsilon - \theta\,\eta \tag{14}$$

in order to get rid of this derivative in the inequality (13). ι here is called Helmholtz free energy and represents the thermodynamically usable portion of energy. By separating ϵ out of the definition (14) stated above, taking its material derivative and using it back in the inequality (13), the entropy inequality can be written in terms of controllable independent variables as follows

$$-\rho\left(\dot{\Psi} + \eta\,\dot{\theta}\right) + t_{kl}\,v_{(l,k)} - \frac{1}{\theta}\,q_k\,\theta_{,k} \geq 0 \tag{15}$$

Here,

$$v_{(l,k)} = d_{kl} = \frac{1}{2}\,\dot{C}_{KL}\;X_{K,k}\;X_{L,l},$$
$$C_{KL} \equiv x_{k,K}\;x_{k,L}\;,\;\dot{C}_{KL} \equiv \frac{DC_{KL}}{Dt} \tag{16}$$

C_{KL} is known as Green deformation tensor whereas d_{kl} is the deformation rate tensor. Dots over the symbols show the derivatives of the related values following the motion. Using the expression of mass conservation on material coordinates

$$\rho = \rho_0\; j^{-1}\;\;(\text{Conservation of mass}),$$
$$j \equiv \det\left[\,x_{k,K}\,\right] = \det \underline{\underline{F}} \tag{17}$$

and expressions (15) and (16),

$$-\rho_0\left(\dot{\Psi} + \eta\,\dot{\theta}\right) + j\;t_{kl}\;d_{lk} - \frac{j}{\theta}\,q_k\,\theta_{,k} \geq 0 \tag{18}$$

can be written. Here, ρ_0 shows the object's initial mass density. By defining the stress potential as below

$$\Sigma \equiv \rho_0\;\Psi \tag{19}$$

We obtain the expression provided below:

$$-\left(\dot{\Sigma} + \rho_0\;\eta\;\dot{\theta}\right) + j\;t_{kl}\;d_{lk} - \frac{j}{\theta}\,q_k\;\theta_{,k} \geq 0 \tag{20}$$

Because all arguments of Σ in expression (20) depend on spatial coordinates, let us express the other terms in expression (20) objectively, as required by the Objectivity axiom. For this purpose, if we write the definitions below:

$$T_{KL} = j\,X_{K,k}\;X_{L,l}\;t_{kl},\;t_{kl} = j^{-1}\;x_{k,K}\;x_{l,L}\;T_{KL} \tag{21}$$

$$Q_K = j\;X_{K,k}\;q_k,\;q_k = j^{-1}\,x_{k,K}\;Q_K \tag{22}$$

$$\Sigma \;=\; \rho_0\;\Psi = \rho_0\left(\varepsilon - \theta\,\eta\right) \tag{23}$$

$$\frac{1}{2}\,\dot{C}_{KL} \;=\; d_{kl}\;x_{k,K}\;x_{l,L},\,\theta_{,K} \;=\; \theta_{,k}\;x_{k,K} \tag{24}$$

integrating them into the entropy inequality given by Eq. 20 the entropy inequality in terms of components of vector and tensor values on material coordinates should be written as follows (Usal, 1994).

$$-\left(\dot{\Sigma} + \rho_0\;\eta\;\dot{\theta}\right) + \frac{1}{2}\,T_{KL}\;\dot{C}_{KL} - \frac{1}{\theta}\,Q_K\;\theta_{,K} \geq 0 \tag{25}$$

Now that this stage is over, it should be clearly expressed on what arguments the stress potential depends, using the necessary constitutive axioms. These operations are to be completed in the next part.

DEFINITION OF STRESS POTENTIAL

According to the causality and determinism axioms (Eringen, 1967; Şuhubi, 1993), Σ (stress potential of a material point X at point t in time) depends on the history of motion and temperature of all material points comprising the object. According to this:

$$\Sigma(X,t) \;=\; \Sigma\!\left[x(X',t'),\,\theta\,(X',t'),X\,\right],$$
$$X' \in B,\, -\infty < t' < t \tag{26}$$

Here, t is any time, present or past while t′ is time in the past. According to the situation discussed here, presuming that the material does not have any memory, this expression is converted to be as follows:

$$\Sigma(X,t) \;=\; \Sigma\!\left[x(X',t),\,\theta\,(X',t),X\,\right] \tag{27}$$

On the other hand, the objectivity axiom determines that Σ depends on the difference of motions X' and X, rather than on individual motion of material points of X'. In the meanwhile, the axiom of neighborhood expressed that dependence of Σ on its arguments will damp fast as the distance between X' and X decreases. To realize these axioms the stress potential should be specified as follows.

$$\Sigma(X,t) = \Sigma\left[C_{KL}(X,t), \theta(X,t), X, L_K\right] \quad (28)$$

L_K here is called the material description vector and is used to determine anisotropy of the medium. Furthermore, the thermodynamic processes in the studied medium have been accepted to run under isothermal conditions.

The expression (28) stated above constitutes a prototype in the aspect of ability to take into account the other physical interactions on non-linear media (Usal, 1994). In this study, due to the existence of micro voids in the material is it assumed that the material has gained directed medium characteristics, i.e. that an anisotropic structure has appeared due to the damage. We assume that initially the material was isotropic and that the anisotropy is only caused by the dispersion of micro voids. For a medium like that the role of material description vectors will be played by the vector representing the mean values in RVE and the vector $\dot{A}(X,t)$ representing the change in time of the preceding vector (Korkmaz, 2001). We believe that, by dividing these vectors by the area of any characteristic surface pertaining to RVE, we render them dimensionless. Therefore, arguments in the stress potential of an elastic medium undergoing a mechanical load application, bearing voids and where these voids are believed to be changing with time, can be expressed as follows:

$$\Sigma(X,t) = \Sigma\left[C_{KL}(X,t), \theta(X,t), A(X,t), \dot{A}(X,t), X\right] \quad (29)$$

On the other hand, because the material will not be able to detect the positive and negative sides of micro void surfaces, we had previously specified that the dependence on vectors $A(X,t)$ and $\dot{A}(X,t)$ can be expressed by a product of tensors.

$$H \equiv A \otimes A, \dot{H} \equiv \dot{A} \otimes A + A \otimes \dot{A} \quad (30)$$

We can specify it as follows in the index form:

$$H_{KL} \equiv A_K A_L, \dot{H}_{KL} \equiv \dot{A}_K A_L + A_K \dot{A}_L \quad (31)$$

In this case the stress potential will be as follows:

$$\Sigma(X,t) = \Sigma\left(C_{KL}, H_{KL}, \dot{H}_{KL}, \theta\right) \quad (32)$$

Thus, assuming that the material is homogeneous (isotropic damage), its direct dependence on X is out of question. At this stage, the arguments determining the stress potential of the micro void medium studied by us have been found out. However, the flow of this determination process will be explained during later stages.

FORMULATION OF CONSTITUTIVE THEORY

From the Eq. 32 the material derivative of Σ can be written as follows:

$$\dot{\Sigma} = \frac{\partial \Sigma}{\partial C_{KL}} \dot{C}_{KL} + \frac{\partial \Sigma}{\partial H_{KL}} \dot{H}_{KL} + \frac{\partial \Sigma}{\partial \dot{H}_{KL}} \ddot{H}_{KL} + \frac{\partial \Sigma}{\partial \theta} \dot{\theta} \quad (33)$$

Substituting this expression into the entropy inequality (29) and arranging the inequality afterwards, we obtain the following expression:

$$\left(\frac{1}{2} T_{KL} - \frac{\partial \Sigma}{\partial C_{KL}}\right) \dot{C}_{KL} - \frac{\partial \Sigma}{\partial H_{KL}} \dot{H}_{KL} - \frac{\partial \Sigma}{\partial \dot{H}_{KL}} \ddot{H}_{KL}$$
$$-\rho_0 \left(\eta + \frac{1}{\rho_0} \frac{\partial \Sigma}{\partial \theta}\right) \dot{\theta} - \frac{1}{\theta} Q_K \theta_{,K} \geq 0 \quad (34)$$

Because in the inequality (34) stated above, starting from the right, we can change the arguments to in the θ as $\dot{\theta}$ and $\theta_{,K}$ consecutive order, as well as arbitrarily represent the derivative of $\dot{H}$ as $\ddot{H}$ and H as $\dot{H}$ for the expression (34) to be valid for any thermodynamic process, coefficients of $\dot{\theta}$, $\theta_{,K}$, $\ddot{H}_{KL}$ and $\dot{C}_{KL}$ have to be equal to zero. Coefficient of $\dot{H}_{KL}$ cannot be equal to zero because, due to the presence of $\dot{H}_{KL}$ in the arguments of Σ, $\dot{H}_{KL}$, cannot be arbitrarily changed. Therefore, is Y_{KL}, called as the strain energy density release rate, is assigned as the coefficient of $\dot{H}_{KL}$, Y_{KL} can be defined as follows (Lemaitre and Chaboche, 2000; Simo and Ju, 1987):

$$\overline{Y}_{KL} \equiv -\frac{\partial \Sigma}{\partial H_{KL}} \quad (35)$$

Furthermore, using the definition $Y_{KL} \equiv -\overline{Y}_{KL}$ in order to deal with a positive value, the strain energy density release rate can be re-written as follows:

$$Y_{KL} \equiv \frac{\partial \Sigma}{\partial H_{KL}} \quad (36)$$

By making coefficients $\dot{C}_{KL}$, $\ddot{H}_{KL}$, $\dot{\theta}$ and $\theta_{,K}$ equal to zero in the inequality (34), the following expressions are obtained:

$$T_{KL} = 2\frac{\partial \Sigma}{\partial C_{KL}}, \quad \frac{\partial \Sigma}{\partial \dot{H}_{KL}} = 0, \quad \eta = -\frac{1}{\rho_0}\frac{\partial \Sigma}{\partial \theta}, \quad Q_K = 0$$
$$\bar{Y}_{KL}\,\dot{H}_{KL} \geq 0, \quad -Y_{KL}\,\dot{H}_{KL} \geq 0, \quad Y_{KL}\,\dot{H}_{KL} \leq 0$$
$$Y_{KL} > 0, \quad D_{KL} > 0, \quad \dot{D}_{KL} > 0 \quad (37)$$

From the expressions (37) stated above it is clear that there is no heat transfer in the medium and that the free energy density does not depend on the material change rate of the damage, as we assumed in the beginning. Therefore, arguments determining the free energy density and the internal energy can be expressed as follows:

$$\Sigma = \Sigma\left(C_{KL}, H_{KL}, \theta\right) \quad (38)$$

$$\varepsilon = \frac{1}{\rho_0}\left(\Sigma + \rho_0\,\theta\,\eta\right) \quad (39)$$

In sources related with CM, Cauchy stress tensor is expressed as follows:

$$t_{kl} = \frac{\rho}{\rho_0}\frac{\partial \Sigma}{\partial x_{k,K}}\,x_{l,K} \quad (40)$$

If the medium is incompressible, the condition $j^2 = \det C = 1$ or $III = 1$ should be justified (Eringen, 1967; Şuhubi, 1993). Therefore in the Eq. 40, the function below that is equivalent to Σ but that contains the said limitation, can be used in lieu of Σ.

$$\Sigma - p\left(x,t\right)\left(j-1\right) \quad (41)$$

p in the expression (41) is called Lagrange multiplier. If the derivative of the function in this expression is taken according to $x_{k,K}$ and then substituted to the Eq. 40, the Equation

$$t_{kl} = -p\,\delta_{kl} + 2\,x_{k,K}\,x_{l,L}\frac{\partial \Sigma}{\partial C_{KL}} \quad (42)$$

is obtained. Form of this expression on material coordinates is found to be (Usal, 2001).

$$T_{KL} = -p\,C^{-1}_{KL} + 2\frac{\partial \Sigma}{\partial C_{KL}} \quad (43)$$

Here, $C^{-1}_{KL} = X_{K,k}\,X_{L,k}$ is known as Piola deformation tensor.

The strain energy density release rate has been previously defined to be $Y_{KL} \equiv \frac{\partial \Sigma}{\partial H_{KL}}$. Because the derivative of Σ according to the deformation gradient does not exist in this expression the Lagrange coefficient will be equal to zero. In this case the constitutive equations that we should obtain, depending on the assumption made, are T_{KL} and Y_{KL}, their free energy function depending on Σ, as clearly shown by Eq. 36 and 43. Therefore, the first thing to do is to find out the open form of Σ.

CONSTITUTIVE MODEL FOR DAMAGED-ELASTIC-ISOTROPIC MEDIA

In this study, a micro void elastic medium exposed to a mechanic load has been assumed to be isotropic. Therefore, for concrete determination of arguments of Σ, findings of the invariant theory have been used. The medium owes its anisotropy only to micro cracks or micro voids. Accordingly, form of the Σ stress potential should remain invariant under the fully orthogonal transformation group of material coordinate system (Spencer, 1971). To express it mathematically, Σ should justify the limitation below:

$$\Sigma\left(C,H,\theta\right) = \Sigma\left(MCM^T, MHM^T, \theta\right) \quad (44)$$

M here is an orthogonal matrix showing the fully orthogonal transformations of material coordinate systems that justifies the condition $M^{-1} = M^T$. As we know from the invariant theory, Σ being a scalar function of these arguments, it has to be dependent on these arguments through common invariants. Thus, we can show that symmetric matrices C and H have 8 common invariants that are not interdependent.

$$I_1 = \mathrm{tr}C, \quad I_2 = \mathrm{tr}C^2, \quad I_3 = \mathrm{tr}C^3, \quad I_4 = \mathrm{tr}H,$$
$$I_5 = \mathrm{tr}H^2, \quad I_6 = \mathrm{tr}H^3, \quad I_7 = \mathrm{tr}C\,H,$$
$$I_8 = \mathrm{tr}C^2H, \quad I_9 = \mathrm{tr}C\,H^2, \quad I_{10} = \mathrm{tr}C^2H^2 \quad (45)$$

Because we have selected the damage tensor H so that it is the tensor product of vector A with itself $(30)_1$ and because $I_9 = I_4\,I_7$ and $I_{10} = I_4\,I_8$ we can ignore the invariants I_9 and I_{10} from the list of invariants, depending on this. Relying on the fact that invariants must be a symmetric function of units comprising them, the assumption that we made above for the damage tensor does not prejudice the generality. Thus, our free energy function can be written as follows as a function of the arguments defined above:

$$\Sigma = \Sigma\left(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8 \right) \tag{46}$$

A second degree tensor, the Green deformation tensor is in the form of its principal invariants.

$$I = I_1,\ \ II = \frac{1}{2}\left(I_1^2 - I_2 \right),\ \ III = \frac{1}{6}\left(I_1^3 - 3\, I_1 I_2 + 2\, I_3 \right) \tag{47}$$

Taking this into account, the principal invariants given in the Eq. 47 can be used in lieu of invariants (I_1, I_2, I_3) in the expression (45). Because the medium is assumed to be incompressible, III = 1. In this case, Σ in terms of owns invariants can be stated as follows using the expressions (45) and (47):

$$\Sigma = \Sigma\left(I, II, I_4, I_5, I_6, I_7, I_8 \right) \tag{48}$$

Because the medium is incompressible the constitutive equation of stress and strain energy density release rate has been defined as follows:

$$T_{PQ} = -p\, C_{PQ}^{-1} + 2\frac{\partial \Sigma}{\partial C_{PQ}} \tag{49}$$

$$Y_{PQ} = \frac{\partial \Sigma}{\partial H_{PQ}} \tag{50}$$

Partial derivatives in expressions (49) and (50) can be written as follows from the expression (48):

$$\frac{\partial \Sigma}{\partial C_{PQ}} = \frac{\partial \Sigma}{\partial I}\frac{\partial I}{\partial C_{PQ}} + \frac{\partial \Sigma}{\partial II}\frac{\partial II}{\partial C_{PQ}} + \frac{\partial \Sigma}{\partial I_7}\frac{\partial I_7}{\partial C_{PQ}} + \frac{\partial \Sigma}{\partial I_8}\frac{\partial I_8}{\partial C_{PQ}} \tag{51}$$

$$\frac{\partial \Sigma}{\partial H_{PQ}} = \frac{\partial \Sigma}{\partial I_4}\frac{\partial I_4}{\partial H_{PQ}} + \frac{\partial \Sigma}{\partial I_5}\frac{\partial I_5}{\partial H_{PQ}} + \frac{\partial \Sigma}{\partial I_6}\frac{\partial I_6}{\partial H_{PQ}} + \frac{\partial \Sigma}{\partial I_7}\frac{\partial I_7}{\partial H_{PQ}} + \frac{\partial \Sigma}{\partial I_8}\frac{\partial I_8}{\partial H_{PQ}} \tag{52}$$

After the derivatives of invariants, stated in the expressions (45) and (47) taken according to C_{PQ}, that are not equal to zero, are substituted at the Eq. 51 and integrated in the Eq. 49 and after those derivatives of the invariants taken according to H_{PQ}, that are not equal to zero, are substituted at the Eq. 52 and integrated in the Eq. 50, the constitutive stress equation and the constitutive equation of the strain energy density release rate in an elastic medium exposed to a mechanical load, assumed to have micro voids and to be incompressible in terms of components on material coordinates can be obtained as follows:

$$T_{PQ} = -p C_{PQ}^{-1} + 2\begin{bmatrix} \frac{\partial \Sigma}{\partial I}\delta_{PQ} + \frac{\partial \Sigma}{\partial II}\left(\delta_{PQ} C_{RR} - C_{PQ} \right) + \frac{\partial \Sigma}{\partial I_7} H_{PQ} \\ + \frac{\partial \Sigma}{\partial I_8}\left(C_{PR} H_{RQ} + H_{PR} C_{RQ} \right) \end{bmatrix} \tag{53}$$

$$Y_{PQ} = \frac{\partial \Sigma}{\partial I_4}\delta_{PQ} + 2\frac{\partial \Sigma}{\partial I_5} H_{PQ} + 3\frac{\partial \Sigma}{\partial I_6} H_{PM} H_{MQ} + \frac{\partial \Sigma}{\partial I_7} C_{PQ} + \frac{\partial \Sigma}{\partial I_8} C_{PL}\, C_{LQ} \tag{54}$$

Expressions (53) and (54) can be further rearranged and expressed in the matrix form as follows:

$$T = -p\, C^{-1} + 2\begin{bmatrix} \frac{\partial \Sigma}{\partial I} I + \frac{\partial \Sigma}{\partial II}\left(\mathrm{tr}C\, I - C \right) + \\ \frac{\partial \Sigma}{\partial I_7} H + \frac{\partial \Sigma}{\partial I_8}\left(C\,H + H\,C \right) \end{bmatrix} \tag{55}$$

$$Y = \frac{\partial \Sigma}{\partial I_4} I + 2\frac{\partial \Sigma}{\partial I_5} H + 3\frac{\partial \Sigma}{\partial I_6} H^2 + \frac{\partial \Sigma}{\partial I_7} C + \frac{\partial \Sigma}{\partial I_8} C^2 \tag{56}$$

The Lagrange multiplier p observed in the Eq. 55 is known as hydrostatic pressure and can be defined by area equations and limit conditions. To obtain more the Eq. 55 and 56 in more concrete forms, derivatives of Σ according to its invariants in these equations should be used.

The invariants determining Σ in Eq. 53 and 54 have been previously shown in the expression (48). However, we have not yet defined how Σ depends on these invariants. If Σ is an analytic function of these invariants, it can be represented by a power series. However, the degree of the said power series and the number of terms to be taken into account, to put it differently, the grade of the polynomial to represent Σ, depends on the size of the deformation invariant and on their interaction proportion in the event, shortly on the grade of non-linearity.

On the other hand, because the internal energy has a positive definition, this polynomial must also be defined positively. Besides, for the order of invariant multiplication not to affect Σ, the polynomial must have symmetric coefficients, i.e., must be in the quadratic form. If a polynomial approach is accordingly selected, the following expression can be written for the free energy function Σ in terms of existing invariants.

$$\Sigma = \sum_{i,j} a_{ij}\, I_i\, I_j \qquad (i, j = 1,2,4,5,6,,7,8), \quad a_{ij} = a_{ji} \tag{57}$$

All a_{ij} coefficients in this expression depend on the X particle and on the medium temperature θ. Derivatives of Σ according to its invariants in the expressions (55) and (56) can be found using the polynomial expansion in (57). By taking derivatives of Σ according to invariants in the expression (57) and substituting them in the Eq. 55, considering the terms of deformation tensor C and damage tensor H up to the second degree, the equation

$$\begin{aligned}
T_{PQ} = & -pC^{-1}_{PQ} + 2\{2a_{11} C_{KK} \delta_{PQ} + a_{12} C_{KK} C_{LL} \delta_{PQ} - a_{12} C_{KL} C_{LK} \delta_{PQ} \\
& + 2a_{14} H_{KK} \delta_{PQ} + 2a_{15} H_{KL} H_{LK} \delta_{PQ} + 2\,a_{17} C_{KL}\, H_{LK} \delta_{PQ} \\
& + 2a_{18} C_{KL}\, C_{LM} H_{MK} \delta_{PQ} + 2a_{12} C_{KK} \delta_{PQ} C_{RR} - 2\,a_{12}\, C_{KK}\, C_{PQ} \\
& + 2\,a_{24}\, H_{KK}\, \delta_{PQ}\, C_{RR} - 2\,a_{24}\, H_{KK}\, C_{PQ} + 2\,a_{25}\, H_{KL}\, H_{LK}\, \delta_{PQ}\, C_{RR} \\
& - 2a_{25} H_{KL} H_{LK} C_{PQ} + 2\,a_{27} C_{KL} H_{LK} \delta_{PQ} C_{RR} - 2\,a_{27}\, C_{KL}\, H_{LK} C_{PQ} \\
& + 2\,a_{17} C_{KK} H_{PQ} + a_{27}\, C_{KK}\, C_{LL}\, H_{PQ} - a_{27}\, C_{KL}\, C_{LK}\, H_{PQ} \\
& + 2\,a_{47}\, H_{KK}\, H_{PQ} + 2\,a_{77}\, C_{KL}\, H_{LK}\, H_{PQ} + 2a_{78}\, C_{KL} C_{LM} H_{MK} H_{PQ} \\
& + 2a_{18} C_{KK} C_{PR} H_{RQ} + 2a_{18} C_{KK} H_{PR} C_{RQ} + 2a_{48} H_{KK} C_{PR} H_{RQ} \\
& + 2\,a_{48}\, H_{KK}\, H_{PR}\, C_{RQ} + 2\,a_{78}\, C_{KL}\, H_{LK}\, C_{PR}\, H_{RQ} \\
& + 2\,a_{78}\, C_{KL}\, H_{LK}\, H_{PR}\, C_{RQ}\}
\end{aligned} \tag{58}$$

can be obtained. To re-define coefficients in the Eq. 58 using coefficients such as α_i (i = 1, 2, 3, ..., 13), the following expressions can be written:

$$\begin{aligned}
& \alpha_1 \equiv 4a_{11}\,,\ \alpha_2 \equiv 2a_{12}\,,\ \alpha_3 \equiv 4a_{14}\,,\ \alpha_4 \equiv 4a_{15}\,,\ \alpha_5 \equiv 4a_{17}\,,\ \alpha_6 \equiv 4a_{18}\,,\ \alpha_7 \equiv 4a_{24} \\
& \alpha_8 \equiv 4a_{25}\,,\ \alpha_9 \equiv 2a_{27}\,,\ \alpha_{10} \equiv 4a_{47}\,,\ \alpha_{11} \equiv 4a_{77}\,,\ \alpha_{12} \equiv 4a_{78}\,,\ \alpha_{13} \equiv 4a_{48}
\end{aligned} \tag{59}$$

By using these coefficients the constitutive stress equation can be re-written as follows:

$$\begin{aligned}
T_{PQ} = & -pC^{-1}_{PQ} + \alpha_1 C_{KK} \delta_{PQ} + \alpha_2\, C_{KK}\, C_{LL} \delta_{PQ} - \alpha_2 C_{KL} C_{LK}\, \delta_{PQ} + \alpha_3 H_{KK} \delta_{PQ} \\
& + \alpha_4 H_{KL} H_{LK} \delta_{PQ} + \alpha_5 C_{KL} H_{LK} \delta_{PQ} + \alpha_6 C_{KL}\, C_{LM} H_{MK} \delta_{PQ} \\
& + 2\alpha_2 C_{KK} \delta_{PQ} C_{RR} - 2\alpha_2 C_{KK} C_{PQ} + \alpha_7 H_{KK} \delta_{PQ} C_{RR} - \alpha_7 H_{KK} C_{PQ} \\
& + \alpha_8 H_{KL} H_{LK} \delta_{PQ} C_{RR} - \alpha_8 H_{KL} H_{LK} C_{PQ} + 2\alpha_9 C_{KL} H_{LK} \delta_{PQ} C_{RR} \\
& - 2\alpha_9 C_{KL} H_{LK} C_{PQ} + \alpha_5 C_{KK} H_{PQ} + \alpha_9 C_{KK} C_{LL} H_{PQ} - \alpha_9 C_{KL} C_{LK} H_{PQ} \\
& + \alpha_{10} H_{KK} H_{PQ} + \alpha_{11} C_{KL} H_{LK} H_{PQ} + \alpha_{12} C_{KL}\, C_{LM} H_{MK} H_{PQ} \\
& + \alpha_6 C_{KK} C_{PR} H_{RQ} + \alpha_6 C_{KK} H_{PR} C_{RQ} + \alpha_{13} H_{KK} C_{PR} H_{RQ} \\
& + \alpha_{13} H_{KK} H_{PR} C_{RQ} + \alpha_{12} C_{KL} H_{LK} C_{PR} H_{RQ} + \alpha_{12} C_{KL} H_{LK} H_{PR} C_{RQ}
\end{aligned} \tag{60}$$

Taking partial derivative of Σ according to invariants in the expression (57) and substituting them in the Eq. 56, considering terms of deformation tensor C and damage tensor H up to the second degree, we can reach the following expression:

$$
\begin{aligned}
Y_{PQ} = {} & 2a_{14} C_{KK} \delta_{PQ} + a_{24} C_{KK} C_{LL} \delta_{PQ} - a_{24} C_{KL} C_{LK} \delta_{PQ} + 2a_{44} H_{KK} \delta_{PQ} \\
& + 2a_{45} H_{KL} H_{LK} \delta_{PQ} + 2a_{47} C_{KL} H_{LK} \delta_{PQ} + 2a_{48} C_{KL} C_{LM} H_{MK} \delta_{PQ} \\
& + 4a_{15} C_{KK} H_{PQ} + 2a_{25} C_{KK} C_{LL} H_{PQ} - 2a_{25} C_{KL} C_{LK} H_{PQ} + 4a_{45} H_{KK} H_{PQ} \\
& + 4a_{57} C_{KL} H_{LK} H_{PQ} + 4a_{58} C_{KL} C_{LM} H_{MK} H_{PQ} + 6a_{16} C_{KK} H_{PM} H_{MQ} \\
& + 3a_{26} C_{KK} C_{LL} H_{PM} H_{MQ} - 3a_{26} C_{KL} C_{LK} H_{PM} H_{MQ} + 2a_{17} C_{KK} C_{PQ} \\
& + 2a_{47} H_{KK} C_{PQ} + 2a_{57} H_{KL} H_{LK} C_{PQ} + 2a_{77} C_{KL} H_{LK} C_{PQ} \\
& + 2a_{48} H_{KK} C_{PL} C_{LQ} + 2a_{58} H_{KN} H_{NK} C_{PL} C_{LQ}
\end{aligned}
\tag{61}
$$

To re-define coefficients in the Eq. 61 using coefficients such as β_i = (i = 1, 2, 3, ...,4), the following expressions can be written:

$$
\begin{aligned}
& \beta_1 \equiv 2a_{14},\ \beta_2 \equiv a_{24},\ \beta_3 \equiv 2a_{44},\ \beta_4 \equiv 2a_{45},\ \beta_5 \equiv 2a_{47},\ \beta_6 \equiv 2a_{48},\ \beta_7 \equiv 4a_{15}, \\
& \beta_8 \equiv 2a_{25},\ \beta_9 \equiv 2a_{57},\ \beta_{10} \equiv 2a_{58},\ \beta_{11} \equiv 6a_{16},\ \beta_{12} \equiv 3a_{26},\ \beta_{13} \equiv 2a_{17},\ \beta_{14} \equiv 2a_{77}
\end{aligned}
\tag{62}
$$

By using these coefficients, constitutive equation of the strain energy density release rate can be written as follows in terms of components on material coordinates:

$$
\begin{aligned}
Y_{PQ} = {} & \beta_1 C_{KK} \delta_{PQ} + \beta_2 C_{KK} C_{LL} \delta_{PQ} - \beta_2 C_{KL} C_{LK} \delta_{PQ} + \beta_3 H_{KK} \delta_{PQ} \\
& + \beta_4 H_{KL} H_{LK} \delta_{PQ} + \beta_5 C_{KL} H_{LK} \delta_{PQ} + \beta_6 C_{KL} C_{LM} H_{MK} \delta_{PQ} + \beta_7 C_{KK} H_{PQ} \\
& + \beta_8 C_{KK} C_{LL} H_{PQ} - \beta_8 C_{KL} C_{LK} H_{PQ} + 2\beta_4 H_{KK} H_{PQ} + 2\beta_9 C_{KL} H_{LK} H_{PQ} \\
& + 2\beta_{10} C_{KL} C_{LM} H_{MK} H_{PQ} + \beta_{11} C_{KK} H_{PM} H_{MQ} + \beta_{12} C_{KK} C_{LL} H_{PM} H_{MQ} \\
& - \beta_{12} C_{KL} C_{LK} H_{PM} H_{MQ} + \beta_{13} C_{KK} C_{PQ} + \beta_5 H_{KK} C_{PQ} + \beta_9 H_{KL} H_{LK} C_{PQ} \\
& + \beta_{14} C_{KL} H_{LK} C_{PQ} + \beta_6 H_{KK} C_{PL} C_{LQ} + \beta_{10} H_{KN} H_{NK} C_{PL} C_{LQ}
\end{aligned}
\tag{63}
$$

The Eq. 63 is a constitutive equation obtained under the above-mentioned assumptions for strain energy density release rate that we were trying to find out in this study and should be restricted so as to justify the inequality $\bar{Y}_{PQ}\,\dot{H}_{PQ} \geq 0$.

CONCLUSIONS

A path has been followed in scope of the modern Continuum Mechanics in hope of providing an opportunity to model the non-linear behavior of an elastic medium with micro voids exposed to a mechanical load. Doing this modeling, first and second law of thermodynamics (Clausius-Duhem inequality), constitutive theories in general and especially axioms of objectivity and material symmetry from among them, concepts related with a material's symmetry group, findings of the invariant theory for determining to structure functionals and concrete determination of arguments have been taken as theoretic basis in modeling the non-linear behavior of the material in question.

Constitutive equation as well as stress potential with Green deformation tensor and damage tensor as its arguments has been determined for the material in question. Due to this constitutive, functional the constitutive equations pertaining to stress and strain energy density release rate in the material under mechanical load have been obtained in terms of components on material coordinates. The studied material being structurally isotropic, it was believed that, due to the existence of micro voids, i.e., due to the damage that occurred, the material gained anisotropic characteristics. Therefore an isotropic material was picked up and constitutive equations of stress and strain energy density release rate were determined in a non-linear form with Eq. 54 by using the findings of the invariant theory (53). Because derivatives of Σ must be known according to its determining invariants in order to solidify the said constitutive equations, the stress potential Σ was represented by a second degree polynomial and its derivatives according to its invariants have been calculated. During these operations in order to determine the degree of non-linear behavior, effects of the deformation tensor C and the damage tensor H have been taken into account up to the second degree. In this case constitutive equations of stress and strain energy density release rate have been presented in terms of components on material coordinates by expressions (60) and (63). Of these equations, the expression (60) that gives constitutive equations of stress, can be simplified by considering the linear contribution of the deformation tensor and taking the damage tensor into account linearly or its terms up to the second grade.

The Eq. 60 is the expression yielding a constitutive equation for stress in an incompressible elastic medium with micro voids exposed to a mechanical load, where mechanical interactions are accepted to be non-linear. In this expression the first term is produced by the assumption of incompressibility. While the second term is produced by the linear effect of the deformation tensor, the 5th term is caused by the linear effect of the damage tensor, terms 3, 4, 9 and 10 are caused by the non-linear effect of the deformation tensor, terms 6 and 20 are caused by the non-linear effect of the damage tensor, terms 7, 11, 12 and 17 are caused by the linear interaction between deformation and damage tensors, terms 8, 15, 16, 18, 19, 23 and 24 are caused by non-linear interaction of the deformation tensor and linear interaction of the damage tensor, terms 13, 14, 21, 25 and 26 are caused by the linear interaction of the deformation tensor and non-linear interaction of the damage tensor and terms 22, 27 and 28 are caused by non-linear interaction of the deformation and the damage tensors.

Equation 63 is an expression yielding the constitutive equation of the strain energy density release rate in an incompressible elastic medium with micro voids, which is exposed to a mechanical load. Terms of the equation and their causes are stated as follows: first term-linear effect of the deformation tensor, 4th term-linear effect of the damage tensor, terms 6, 8 and 18-linear interaction of the deformation and damage tensors, terms 2, 3 and 17-non-linear effect of the deformation tensor, terms 5 and 11-non-linear effect of the damage tensor, terms 7, 9, 10, 20 and 21-non-linear interaction of the deformation tensor and linear interaction of the damage tensor, terms 12, 14 and 19-non-linear interaction of the damage tensor and linear interaction of the deformation tensor, terms 13, 15, 16 and 22-non-linear interaction of the deformation and the damage tensors.

As a continuation of this study, material and spatial forms of the damage tensor can be suitably interconnected and components of the obtained constitutive equations in spatial coordinates-determined. Besides, considering multiple damage tensors representing damage new models can be developed. Furthermore, damage interactions in plastic areas can be considered and constitutive equations can be developed containing different interactions resulting from incorporation of a damage tensor into independent variables of visco-elastic, thermo-elastic and pieso-electric materials.

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