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LQ-moments: Application to the Extreme Value Type I Distribution

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Abstract: The objective of this study is to develop improved LQ-moments that do not impose restrictions on the value of p and α such as the median, trimean or the Gastwirth but we explore an extended class of LQMOM with consideration combinations of p and α values in the range 0 and 0.5. The popular quantile estimator namely the Weighted Kernel Quantile (WKQ) estimator will be proposed to estimate the quantile function. The performances of the proposed estimators of the Extreme Values Type 1 (EV1) distribution were compared with the estimators based on conventional LMOM, MOM (method of moments), ML (method of maximum likelihood) and the LQ-moments based on LIQ (linear interpolation quantile) for various sample sizes and return periods.

Key words: The weighted kernel quantile, linear interpolation quantile, LQ-moments, L-moments, quick estimator

INTRODUCTION

The L-moments, certain linear functions of the expectations of order statistics, were introduced and comprehensively reviewed by Hosking (1990). L-moments have found wide applications in such fields of applied research as civil engineering, meteorology and hydrology. The method of L-moments has become a standard procedure in hydrology for estimating the parameters of certain statistical distributions. The L-moments are defined as:

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E(X_{r-k:r}), \quad r = 1,2,.... \tag{1}$$

Mudolkar and Hutson (1998) extended LMOM to new moment like entitles called LQ moments (LQMOM). They found LQMOM always exists, are often easier to compute than LMOM and in general behave similarly to the LMOM. The LQ-moments are defined as:

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\tau_{p,\alpha}(X_{r-k:r}), \quad r = 1,2,.... \tag{2}$$

Where, $0 \le \alpha \le 1/2$, $0 \le p \le 1/2$ and

$$\tau_{p,\alpha}(X_{r-k:r}) = pQ_{X_{r-k:r}}(\alpha) + (1-2p)Q_{X_{r-k:r}}(1/2) + pQ_{X_{r-k:r}}(1-\alpha) \tag{3}$$

is the linear combination and defined as a quick measures of the location of the sampling distribution of the order statistic $X_{r-k:r}$ and $Q_X(u)$ is the quantile function. Mudolkar and Hutson (1998) discussed a robust modification in which the mean of the distribution of $X_{r-k:r}$ in (1) is replaced by the common quick estimators $\hat{\tau}_{p,\alpha}(X_{r-k:r})$ using the median ($p = 0$, $\alpha = 1$), the trimean ($p = 1/4$, $\alpha = 1/4$) and Gastwirth ($p = 0.3$, $\alpha = 1/3$) for some symmetric and asymmetric distributions.

The objective of this research is to develop improved LQ-moments that does not impose restrictions on the value of the quick estimators parameters p and α such as $p = 0$, $\alpha = 0.5$ for the median, $p = 1/4$, $\alpha = 1/4$ for the trimean and $p = 0.3$, $\alpha = 1/3$ for the Gastwirth but we explore an extended class of LQ-moments with consideration combinations of p and α values in the range 0 and 0.5. Rather, we seek to determine optimal combination of p and α values of LQ-moments, assuming that the underlying distribution is correctly specified. More specifically, we develop the method of LQ-moments for the Extreme Values Type 1 (EV1) distribution, which is often employed in statistical analyses of hydrological data. The popular quantile estimator namely the Weighted Kernel Quantile (WKQ) estimator will be proposed to estimate the quantile function. The performances of the proposed estimators of the EV1 distribution were compared with the estimators based on conventional LMOM, MOM (method of moments), ML (method of maximum likelihood) and LIQ (linear interpolation quantile) estimator for various sample sizes and return periods.

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DEFINITION AND PROPERTIES OF LQ-MOMENTS ESTIMATORS

Let X_1, X_2,...X_n be a random sample from a continuous distribution function F(.) with quantile function $Q(u) = F^{-1}(u)$ and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ denote the corresponding order statistics (Mudolkar and Hutson, 1998). Then the rth LQ-moments ξ_r is given by:

$$\xi_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} \tau_{p,\alpha}(X_{r-k:r}), \quad r = 1, 2, \ldots \tag{4}$$

Where, $0 \le \alpha \le 1/2$, $0 \le p \le 1/2$,

$$\begin{aligned}\tau_{p,\alpha}(X_{r-k:r}) &= pQ_{X_{r-k:r}}(\alpha) + (1-2p)Q_{X_{r-k:r}}(1/2) \\ &\quad + pQ_{X_{r-k:r}}(1-\alpha) \\ &= pQ[B^{-1}_{r-k:r}(\alpha)] + (1-2p)Q[B^{-1}_{r-k:r}(1/2)] \\ &\quad + pQ[B^{-1}_{r-k:r}(1-\alpha)]\end{aligned} \tag{5}$$

is the quick estimator of location and $B^{-1}_{r-k:r}(\alpha)$ is the quantile of a beta random variable with parameter r–k and k+1 and Q(.) denotes the quantile estimator. The first four LQ-moments of the random variable X are defined as:

$$\xi_1 = \tau_{p,\alpha}(X), \tag{6}$$

$$\xi_2 = \tfrac{1}{2}[\tau_{p,\alpha}(X_{2:2}) - \tau_{p,\alpha}(X_{1:2})], \tag{7}$$

$$\xi_3 = \tfrac{1}{3}[\tau_{p,\alpha}(X_{3:3}) - 2\tau_{p,\alpha}(X_{2:3}) + \tau_{p,\alpha}(X_{1:3})], \tag{8}$$

$$\xi_4 = \tfrac{1}{4}[\tau_{p,\alpha}(X_{4:4}) - 3\tau_{p,\alpha}(X_{3:4}) + 3\tau_{p,\alpha}(X_{2:4}) - \tau_{p,\alpha}(X_{1:4})] \tag{9}$$

The quantile estimators: The sample quantiles estimators of the values of the population quantile Q(.), are used widely in a variety of applications such as a Q-Q plots and a box plot in the exploratory data analysis, non-parametric estimators involving statistics such as the quartiles and their ranges, to theoretical topics such as density function estimation.

Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the corresponding order statistics. The population quantiles estimators of a distribution is defined as:

$$Q(u) = F^{-1}(u) = \inf\{x : F(x) \ge u\}, \quad 0 < u < 1 \tag{10}$$

Where, F(x) is the distribution function (Hydnman and Fan, 1996).

The linear interpolation quantile estimator: The linear interpolation (LIQ) quantile estimator is used commonly in statistical packages such as MINITAB, SAS, IMSL and S-PLUS. The LIQ estimator is given by:

$$\hat{Q}(u) = (1-\varepsilon)X_{[n'u]:n} + \varepsilon X_{[n'u]+1:n} \tag{11}$$

Where, $\varepsilon = n'u - [n'u]$, $n' = n+1$ and [nu] denotes the integral part of nu (Mudolkar and Hutson, 1998).

The weighted kernel quantile estimator: A popular class of L quantile estimators is called kernel quantile estimators has been widely applied (Sheather and Marron, 1990). The L quantile estimators is given by:

$$\hat{Q}(u) = \sum_{i=1}^{n}\left[\int_{(i-1)/n}^{i/n} K_h(t-u)dt\right]X_{i:n} \tag{12}$$

Where, K is a density function symmetric about 0 and

$$K_h(\bullet) = (1/h)K(\bullet/h)$$

In this study, the approximation of the L quantile estimator is called as the weighted kernel quantile estimator (WKQ) proposed by Huang and Brill (1999). The WKQ is given by:

$$\hat{Q}(u) = \sum_{i=1}^{n}\left[n^{-1}K_h\left(\sum_{j=1}^{i} w_{j,n} - u\right)\right]X_{i:n}, \quad 0 < u < 1 \tag{13}$$

And the data point weights are:

$$w_{i,n} = \begin{cases}\dfrac{1}{2}\left(1 - \dfrac{n-2}{\sqrt{n(n-1)}}\right), & i = 1, n \\ \dfrac{1}{\sqrt{n(n-1)}}, & i = 2, 3, \ldots, n-1\end{cases} \tag{14}$$

Where, $K(t) = (2\pi)^{-1/2}\exp(-t^2/2)$ is the Gaussian Kernel, $h = [uv/n]^{1/2}$ and $v = 1-u$ is an optimal bandwidth proposed by Sheather and Marron (1990).

THE EV1 DISTRIBUTION

The EV1 distribution or Gumbel distribution, named after Gumbel (1958), has been extensively used in various fields including hydrology for modeling extreme events. The EV1 CDF has Cumulative Distribution Function (CDF)

$$F(x) = \exp\{-\exp[-(x-\mu)/\sigma]\} \quad -\infty < x < \infty \tag{15}$$

Where, μ and σ are location and scale parameters, respectively. Quantiles function of EV1 distribution is given by:

$$Q(u) = \mu - \sigma\log(-\log(u)) \tag{16}$$

METHOD OF LQ-MOMENTS

The LQ-moments estimators for the EV1 distribution behave similarly to the LMOM. From Eq. 6, 7 and 16, the first two LQ-moment of the EV1 distribution can be written as:

$$\xi_1 = \mu + \sigma[\tau_{p,\alpha}(X_{11})] \tag{17}$$

$$\xi_2 = \tfrac{1}{2}\sigma[\tau_{p,\alpha}(X_{22}) - \tau_{p,\alpha}(X_{12})] \tag{18}$$

The LQMOM estimators μ and σ of the parameters are the solution of (17) and (18). $\hat{\sigma}$ and $\hat{\mu}$ can be estimated successively from Eq.(18) and (17) as:

$$\hat{\sigma} = \frac{2\hat{\xi}_2}{\hat{\tau}_{p,\alpha}(X_{22}) - \hat{\tau}_{p,\alpha}(X_{12})} \tag{19}$$

$$\hat{\mu} = \hat{\xi}_1 - \hat{\sigma}[\hat{\tau}_{p,\alpha}(X_{11})] \tag{20}$$

OTHERS METHODS OF PARAMETER ESTIMATION

Several methods can be used to estimate the parameters of the EV1 distribution. The methods of L-moments (LMOM), ordinary product moments (MOM) and maximum likelihood (ML) are commonly used to estimate the parameters of the EV1 distribution. The method of ML is known to be asymptotically unbiased and optimal for the EV1 distribution. However, there is no guarantee that the ML method is the best in small samples. The method of probability weighted moments (PWM) or L-moments method has a become a standard procedure in hydrology for estimating the parameters of certain statistical distributions.

Landwehr *et al.* (1979) was developed the method of probability weighted moments (PWM) or L-moments (LMOM) method. The LMOM method was compared to the MOM and the ML method. The results show that LMOM estimates were comparable to other estimators.

A number of methods of fitting the EV1 distribution to sample data were compared by Jain and Singh (1987). The MOM method was found to be most accurate, next to the ML method. The MOM method also found to be virtually unbiased and the simplest to apply.

Raynal and Salas (1986) analyzed six different methods of parameter estimation and preferred PWM for large samples. Phien (1987) compared the MOM, ML and ME (maximum entropy) and LMOM methods for the EV1 distribution. The LMOM method was found to be best in terms of bias and the ML method was found to be best in terms of mean square error.

Methods of moments: This is one of the most popular methods of estimating the parameters of EV1 distribution. The MOM estimators of the EV1 parameters are given by:

$$\hat{\sigma} = \frac{s}{\pi}\sqrt{6} \tag{21}$$

$$\hat{\mu} = \bar{x} - 0.450041s \tag{22}$$

Where, $\bar{x}$ and s are the mean and standard deviation of a sample of size n.

Method of maximum likelihood: The ML estimators of the EV1 are given by:

$$\hat{\sigma} = \bar{x} - \frac{\sum_{i=1}^{n} x_i \exp(-x_i/\hat{\sigma})}{\sum_{i=1}^{n} \exp(-x_i/\hat{\sigma})} \tag{23}$$

$$\hat{\mu} = -\hat{\sigma}\log\left\{\frac{1}{n}\sum_{i=1}^{n}\exp(-x_i/\hat{\sigma})\right\} \tag{24}$$

Method of L-moments: The LMOM estimators for the EV1 distributions are given by:

$$\hat{\sigma} = \frac{L_2}{\log 2} \tag{25}$$

$$\hat{\mu} = b_0 - \hat{\sigma}\gamma \tag{26}$$

$$L_2 = 2b_1 - b_0$$

Where,

$$b_r = \sum_{i=1}^{n} \frac{(i-1)(i-2)(i-3)\ldots(i-r)}{n(n-1)(n-2)\ldots(n-r)} x_{i:n}$$

And γ = 0.577 is Euler's constant.

SIMULATION STUDY

A number of simulation experiments were conduct to investigate the properties of LQ-moments estimators for the EV1 distribution. A set of 10 000 random samples of sizes varying from 10 to 100 were generated from EV1 distribution. The location and scale parameters (μ, σ) were set 0 and 1, respectively. From each generated sample of a given size n the root mean square error (RMSE) for F = 0.9, 0.95, 0.99 and 0.999 corresponding to return periods T = 10, 20, 100 and 1000 years, respectively were

Table 1: Combination α and p that produces of Root Mean Square Error (RMSE) for the LQ-Moments method based on the WKQ and LIQ estimator of a 100-year EV1 quantile from 50 observations

WKQ estimator (α)					LIQ estimator (α)				
p	0.20	0.22	0.24	0.26	p	0.26	0.28	0.30	0.32
0.10	0.6334	0.6350	0.6394	0.6463	0.10	0.7578	0.7751	0.7951	0.8176
0.12	0.6298	0.6281	0.6288	0.6318	0.12	0.7379	0.7497	0.7641	0.7811
0.14	0.6330	0.6291	0.6273*	0.6275	0.14	0.7217	0.7284	0.7376	0.7493
0.16	0.6400	0.6347	0.6311	0.6292	0.16	0.7158	0.7188	0.7242	0.7318
0.18	0.6490	0.6427	0.6379	0.6345	0.18	0.7153	0.7156	0.7180	0.7225
0.20	0.6591	0.6523	0.6467	0.6422	0.20	0.7168	0.7148	0.7147*	0.7163
0.22	0.6699	0.6629	0.6568	0.6516	0.22	0.7251	0.7218	0.7202	0.7202
0.24	0.6812	0.6742	0.6680	0.6624	0.24	0.7285	0.7233	0.7195	0.7170
0.26	0.6928	0.6861	0.6799	0.6743	0.26	0.7410	0.7355	0.7312	0.7280

*Indicates the smallest RMSE obtained

Table 2: Values of the quick estimator parameters (α, p) leading to minimum RMSE of LQ-moments estimators for (a) WKQ and (b) linear interpolation quantiles

	T = 10			T = 100			T = 1000		
n	α	p	RMSE	α	p	RMSE	α	p	RMSE
Weighted kernel quantiles									
10	0.26	0.26	0.780	0.26	0.26	1.423	0.26	0.28	2.110
25	0.18	0.24	0.492	0.16	0.24	0.894	0.16	0.26	1.302
50	0.14	0.24	0.346	0.14	0.24	0.627	0.14	0.26	0.913
75	0.14	0.24	0.283	0.14	0.24	0.513	0.12	0.24	0.747
100	0.14	0.22	0.248	0.12	0.24	0.449	0.12	0.24	0.654
Linear interpolation quantiles									
10	0.14	0.26	0.800	0.14	0.28	1.445	0.14	0.28	2.110
25	0.16	0.26	0.550	0.16	0.26	1.081	0.16	0.28	1.559
50	0.18	0.24	0.386	0.16	0.24	0.726	0.16	0.24	1.071
75	0.18	0.24	0.307	0.16	0.24	0.575	0.16	0.24	0.847
100	0.16	0.26	0.263	0.16	0.26	0.496	0.16	0.26	0.730

Table 3: Relative Performance of the method of WKQ over the method of LIQ, LMOM, MOM and ML as measured by the ratio of RMSE of quantile estimators

n	Methods	T = 10	T = 100	T = 1000
10	LIQ	0.973	0.985	0.984
	LMOM	0.972	0.972	0.972
	MOM	0.962	0.953	0.949
	ML	1.044	1.072	1.081
25	LIQ	0.890	0.855	0.843
	LMOM	0.978	0.979	0.981
	MOM	0.939	0.915	0.908
	ML	1.016	1.038	1.046
50	LIQ	0.910	0.881	0.865
	LMOM	0.981	0.986	0.982
	MOM	0.927	0.903	0.888
	ML	1.011	1.038	1.042
75	LIQ	0.921	0.892	0.881
	LMOM	0.984	0.987	0.991
	MOM	0.924	0.896	0.887
	ML	1.022	1.049	1.061
100	LIQ	0.929	0.905	0.896
	LMOM	0.996	1.002	1.003
	MOM	0.939	0.913	0.902
	ML	1.069	1.109	1.124

Values<1 indicate that WKQ are superior

computed. The presentation of our results will focus on the properties of quantile estimators because they are more direct practical interest. Initially, parameters were estimated using combinations of the quick estimators parameters (α and p) values in the ranges 0 to 0.5. In the computer simulations the values of αand p were chosen in small steps by adding 0.02 and all possible combination of αand p were examined in order to find the best combination in term of RMSE.

All possible for combinations of αand p that produces of RMSE of a 100-year quantile estimator computed from 50 observations for the LQ-moments method based on the WKQ an LIQ estimator were examined and only limited result presented as shown in Table 1. The minimum value of RMSE lies at α is 0.24 and p is 0.14 for WKQ estimator and α is 0.30 and p is 0.20 for LIQ estimator. The results show that the choice α and p of LQ-moments based on the median, trimeans and Gaswirth is not optimal.

The Table 2 shows that the minimum value of RMSE decreases as the sample size increases. The optimal values of α lies in the range (0.14, 0.26) and p in the range (0.22, 0.26) for WKQ and α value fall in the range (0.14, 0.20) and p in the range (0.24, 0.28) for the LIQ estimator. The WKQ has consistently performed better than the LIQ quantile estimators for any combinations of return period and sample size.

COMPARISON OF WKQ, LIQ, LMOM, MOM AND ML METHODS

The RMSE of 10-year, 100-year and 1000-year quantiles estimated by conventional LIQ, LMOM, MOM and ML, relative to the new WKQ estimator for different combinations of sample size, are compared and shown in Table 3. Values less than one suggest superiority of the new method. For any values of sample size, n, the ML estimator generally has the lowest RMSE in comparison to the other estimators. The WKQ perform next followed by LMOM, MOM and LIQ.

The results show that the new estimator performs better than LIQ, MOM and LMOM. It is however, slightly inferior to the ML method in terms of RMSE.

CONCLUSIONS

The LQ-moments are constructed by using functional defining the quick estimators, such as the median, trimean or Gastwirth, in places of expectations in L-moments have are re-examined. The quick estimators based on three-mean of quantiles using weighted kernel estimators are introduced for fitting the data and for characterizing the upper part of distributions in a sample. This study has compared the LQMOM based on LIQ estimator, the traditional method of MOM, LMOM and ML commonly used in hydrological frequency analysis with the LQMOM based on the WKQ estimator in which the quick estimators parameters α and p are not restricted, such as the median, trimean or Gastwirth.

Results from fitting the EV1 distribution function to generated EV1 samples for the WKQ estimator shows that the optimal values of α lies in the range (0.14, 0.26) and p in the range (0.22, 0.26) for different combinations of return period and sample size. In many cases of practical interest, the method of WKQ outperforms the method of ML over the entire sample size n considered but always perform better than the LMOM, LIQ and MOM methods in term of RMSE.

This study has demonstrated that the conventional L moment is not optimal for the estimation of the EV1 distribution. The new method of estimation, denoted the LQ-moments based on WKQ method, in many cases represents higher efficiency in the quantile estimation compared the L moments method. The simplicity and generally good performance of this method make it an attractive option for estimating quantiles in the EV1 distribution.

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