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The Bounded Solutions of Lienard Equation

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Abstract: In this study by using qualitative method, we obtain sufficient conditions under which we can guarantee that bounded of lienard equation $x'' + f(x)x' + g(x) = p(t)$.

Key words: Bounded solution, lienard equation, qualitative method, liapunove function

INTRODUCTION

Consider the system

$$x' = f(t,x), \quad (1)$$

where f is continuous on I × Q, I = [0, ∞) and Q is an open set in R^n.

Yoshizawa (1975) studied bounded of solution of system (1) and gave the following result.

Theorem 1: Suppose that there exists a Liapunov function v(t, x) defined on $I \times K$, $K = \{x \mid |x| \geq k,$ for some $k > 0, x \in R^n\}$, which satisfies the following conditions:

(i) $a(|x|) \leq v(t,x) \leq b(|x|)$, where a(r), b(r) are continuous and a(r) → ∞ as r → ∞,

(ii) $v'_{(1)}(t,x) \leq 0$.

Then the solutions of (1) are uniformly bounded.

MAIN RESULTS

However sometimes it may be difficult to find a Liapunov function which satisfies all of the conditions of theorem 1. In such cases it is often useful to prove the bounded of solutions by using qualitative method. We shall demonstrate this by means of Lienard equation,

$$x'' + f(x)x' + g(x) = p(t) \quad (2)$$

Theorem 2: Assume that:

(a) f and g are continuous for all x ∈ R,

(b) $G(x) = \int_0^x g(u)du > 0$,

(c) $f(x) > 0$ for all $x \in R$ and $\lim_{|x|\to\infty} \int_0^x f(u)du = \infty$,

(d) p(t) is continuous on I and $\int_0^\infty |p(t)|dt < \infty$.

Then every solution x(t) of (2) satisfies $|x(t)| < M$, $|x'(t)| < M$, where M may depend on the solution.

Proof: Letting $x_1 = x, x_2 = x'$, (2) is equivalent to the system of equations,

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -f(x_1)x_2 - g(x_1) + p(t) \end{aligned} \quad (3)$$

We choose a Liapunove function

$$v(t,x_1,x_2) = \sqrt{x_2^2 + 2G(x_1)} - \int_0^t |p(s)|ds$$

then along the solution of system (3) we have,

$$v'_{(3)}(t,x_1,x_2) = \frac{x_2x_2' + g(x_1)x_1'}{\sqrt{x_2^2 + 2G(x_1)}} - |p(t)|$$
$$= \frac{1}{\sqrt{x_2^2 + 2G(x_1)}}\left[-f(x_1)x_2^2 - g(x_1)x_2 + x_2p(t) + g(x_1)x_2\right] - |p(t)| \leq 0.$$

That there not exists a(r) satisfies condition (i) in Theorem 1 so that we cannot apply Theorem 1. In such case, it is often useful to prove the bounded of solution by using qualitative method.

To this end, let l and m be arbitrary given positive numbers and consider the region U defined by the inequalities

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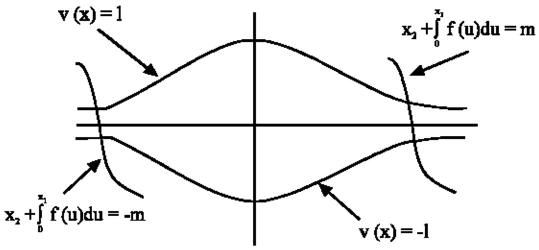

Fig. 1: Pairs of numbers (l, m) in bounded region

$$v(x) < 1 \text{ and } \left[x_2 + \int_0^{x_1} f(u)du\right]^2 < m^2$$

For each pair of numbers (l, m), U is a bounded region as shown, e.g., in Fig. 1 Miller and Michel (1982).

Now let $x_0^T = (x_{10}, x_{20}) = (x_1(0), x_2(0))$ be any point in R^2. If we choose (l, m) properly, x_0 will be in the interior of U. Now let $\phi(t, x_0)$ be a solution of (3) such that $\phi(0, x_0) = x_0$. We shall show that cannot leave the bounded region U. This in turn will show that all solutions of (3) are bounded, since $\phi(t, x_0)$ is arbitrary.

In order to leave U, the solution $\phi(t, x_0)$ must either cross the locus of points determined by v(x) = 1 or one of the loci determined by

$$x_2 + \int_0^{x_1} f(u)du = \pm m.$$

Here we choose, without loss of generality, the positive number

$$m > \sup_{t\in I} \int_0^t p(s)ds \text{ and } -m < \inf_{t\in I} \int_0^t p(s)ds$$

so large that the part of the curve determined by

$$x_2 + \int_0^{x_1} f(u)du = m$$

that is also the boundary to U corresponds to $x_1 > 0$ and the part of the curve determined by,

$$x_2 + \int_0^{x_1} f(u)du = -m$$

corresponds to $x_1 < 0$. Now since $v'(\phi(t, x_0)) \le 0$, the solution $\phi(t, x_0)$ cannot cross the curve determined by v(x) = 1.

To show that it does not cross either of the curves determined by,

$$x_2 + \int_0^{x_1} f(u)du = \pm m$$

we consider the function

$$w(t) = \left[\phi_2(t, x_0) + \int_0^{\phi_1(t,x_0)} f(u)du - \int_0^t p(s)ds\right]^2$$

where $\phi(t, x_0)^T = [\phi_1(t, x_0), \phi_2(t, x_0)]$. Then

$$w'(t) = 2\left[\phi_2(t, x_0) + \int_0^{\phi_1(t,x_0)} f(u)du - \int_0^t p(s)ds\right]$$
$$\left[\phi_2'(t, x_0) + f(\phi_1(t, x_0))\phi_1'(t, x_0) - p(t)\right]$$
$$= -2\left[\phi_2(t, x_0) + \int_0^{\phi_1(t,x_0)} f(u)du - \int_0^t p(s)ds\right] g(\phi_1(t, x_0)).$$

Now suppose that $\phi(t, x_0)$ reaches the boundary determined by the equation;

$$x_2 + \int_0^{x_1} f(u)du = m,\ x_1 > 0.$$

Then along this part of the boundary $w'(t) = -2[m - \int_0^t p(s)ds] g(\phi_1(t, x_0)) < 0$ because $x_1 > 0$ and $m - \int_0^t p(s)ds > 0$.

Therefore, the solution $\phi(t, x_0)$ cannot cross outside of U through that part of the boundary determined by

$$x_2 + \int_0^{x_1} f(u)du = m.$$

We apply the same argument to the part of the boundary determined by

$$x_2 + \int_0^{x_1} f(u)du = -m.$$

Therefore, every solution of (3) is bounded.

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