Journal of Applied Sciences

ISSN 1812-5654

Journal of Applied Sciences 7 (9): 1271-1278, 2007
ISSN 1812-5654

Analytical Solution of Simplified Phan-Thien Tanner Fluid between Nearly Parallel Plates of a Small Inclination

S.H. Hashemabadi and S.M. Mirnajafizadeh
Computational Fluid Dynamics Research Laboratory, Department of Chemical Engineering,
Iran University of Science and Technology, Narmak, Tehran, 16846, Iran

Abstract: The steady state laminar flow of non-linear viscoelastic fluid with Simplified Phan-Thien Tanner (SPTT) model between two plates with an angle is studied analytically for the case of continuous motion of the lower plate. Linear form of stress coefficient is used in the constitutive equation. For the linear stress coefficient the dimensionless pressure gradient and velocity profile are obtained for a wide range of upper plate slope, Deborah number and elongational parameter. The results indicate the strong effects of the viscoelastic parameters on velocity profile and the dimensionless pressure. While upper plate slope decreases for constant dimensionless viscoelastic group εDe^2, maximum pressure reduces and location of maximum shifts to entrance of channel, as limiting case for two parallel plates the dimensionless pressure gradient approaches to zero. Increasing of dimensionless viscoelastic group εDe^2 in constant upper plate slope, decreases maximum pressure but mean dimensionless pressure promotes and pressure profile changes to more uniform distribution.

Key words: Nonlinear viscoelastic fluid, lubrication flow, simplified Phan-Thien Tanner model, analytical solution

INTRODUCTION

Lubrication flows are flows between nearly parallel walls of a small inclination with respect to each other, as well as thin film flows under nearly planar interfaces. Important operational flows such as journal-bearing and piston-ring lubrication of engines and processing flows such as application of thin films wire or roll coating and multilayer extrusion can be analyzed as lubrication flows. Lubrication of journal-bearing and piston-ring is performed to reduce the friction of two bodies in near contact and is usually accomplished by viscous fluids moving through the narrow but variable distance between two bodies. This thin viscous film should create to somehow extremely large pressure difference in order to prevent the contact of two bodies. Hence the properties of lubricants are usually modified by additives to provide various requirements of machinery systems. The effects of elastic fluids on velocity and pressure field studied for viscoelastic fluids obeying Upper Convected Maxwell model (UCM) (Tichy, 1996) and second order model (Sawyer and Tichy, 1998). Advantages of Phan-Thien-Tanner (PTT) model (Phan-Thien and Tanner, 1977; Phan-Thien, 1978) encouraged many authors to use this model to investigate the problems. Alves *et al.* (2001) presented an analytical solution of fluids following single mode PTT model for steady flows through pipe and parallel plates. Hashemabadi *et al.* (2003) also provided an analytical solution for dynamic pressurization of viscoelastic fluids obeying SPTT between two parallel plates for simulation in single screw extruder. Mirzazadeh *et al.* (2005) presented an analytical solution for purely tangential flow of Phan-Thien-Tanner (PTT) viscoelastic fluid model in a concentric annulus with relative rotation of the inner and outer cylinders. Oliviera and Pinho (1999) have published a series of analytical solutions for PTT model for flows through ducts.

The SPTT non-Newtonian model for lubrication flows has not yet been considered. Therefore, the objective of the present investigation is to solve analytically the lubrication flows of a PTT viscoelastic fluid flowing between two plates with small inclination.

PROBLEM DESCRIPTION AND MATHEMATICAL FORMULATION

Figure 1 illustrates two plates, the stationary upper one has a small slope respect to the other and the lower one is horizontal and moving with a constant velocity U. The flow is assumed to be laminar, low Reynolds number, steady state, isothermal and incompressible. The plates are subjected to no slip condition and the gravitational forces for the flow domain are negligible. The necessary conditions of lubrication flow in X direction are:

Corresponding Author: S.H. Hashemabadi, Computational Fluid Dynamics Research Laboratory, Department of Chemical Engineering, Iran University of Science and Technology, Narmak, Tehran, 16846, Iran Tel: +98-217-391-2714 Fax: +98-217-724-0495

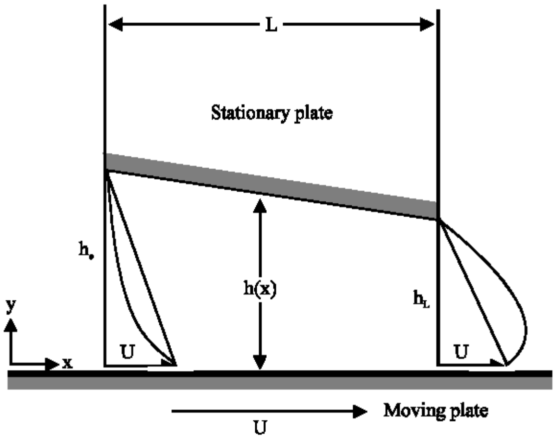

Fig. 1: Schematic diagram of flow domain

$$v << u, \quad \frac{\partial}{\partial x} << \frac{\partial}{\partial y}, \quad u = u(x, y)$$

The continuity and momentum equation in lubrication approximation can be developed respectively by an order of magnitude of full two dimensional Navier-Stokes equations as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$-\frac{\partial p}{\partial x} = \frac{\partial \tau_{yx}}{\partial y} \tag{2}$$

The PTT constitutive equation is given by Bird *et al.* (1987):

$$Z(tr\tau)\tau + \lambda\tau_{(1)} + \frac{\xi}{2}\lambda(\dot{\gamma}.\tau + \tau.\dot{\gamma}) = -\eta\dot{\gamma} \tag{3}$$

Where Z usually is suggested in exponential form and reported as follows:

$$Z(tr\tau) = \exp\left(-\frac{\varepsilon\lambda tr(\tau)}{\eta}\right) \tag{4}$$

As mentioned previously, Reynolds number is assumed to be low, so small molecular deformation occurs and therefore considering the linearized form of Eq. 4 as follows is acceptable (Tanner, 2000):

$$Z(tr\tau) = 1 - \varepsilon\lambda\frac{tr\tau}{\eta} \tag{5}$$

where ε is related to the elongational behavior, η is the viscosity coefficient of the model, λ is the relaxation time and trτ is the trace of stress tensor τ.ξ is a constant parameter of PTT model. It is related to the slip velocity between the continuum medium and molecular network. For weak flows where the rate of deformation of fluid elements is not noticeable, the constant ξ is equal to zero (Alves *et al.*, 2001) and the model is called simplified Phan-Thien Tanner (SPTT). $\tau_{(1)}$ is the convected time derivative of stress tensor and is defined by:

$$\tau_{(1)} = \frac{D\tau}{Dt} - \{(\nabla\mu)^{\dagger}.\tau + \tau.(\nabla\mu)\} \tag{6}$$

Where velocity gradient tensor ∇μ, for the problem depicted above, is simplified as:

$$\nabla u = \begin{pmatrix} 0 & \dot{\gamma}_{yx} \\ 0 & 0 \end{pmatrix} \tag{7}$$

By substituting Eq. 6 into Eq. 3, finally, it reduces to

$$Z\tau_{xx} - 2\lambda\dot{\gamma}_{yx}\tau_{yx} = 0 \tag{8}$$

$$Z\tau_{yy} = 0 \tag{9}$$

$$Z\tau_{yx} - \lambda\dot{\gamma}_{yx}\tau_{yy} = -\eta\dot{\gamma}_{yx} \tag{10}$$

It was found from Eq. 9 that $\tau_{yy} = 0$, hence the trace of stress tensor will be equal to τ_{xx}. The shear stress (τ_{yx}) can be obtained by integrating Eq. 2:

$$\tau_{yx} = \tau_0 - \left(\frac{\partial p}{\partial x}\right)y \tag{11}$$

Where τ_0 indicates the shear stress at the moving wall. By dividing Eq. 8 by Eq. 10 and using Eq. 11, we get

$$\tau_{xx} = -\frac{2\lambda}{\eta}\left[\tau_0 - \left(\frac{\partial p}{\partial x}\right)y\right]^2 \tag{12}$$

By substituting Eq. 11 into Eq. 10, the following equation is found for the velocity gradient:

$$\dot{\gamma}_{yx} = \frac{\partial u}{\partial y} = -\frac{Z(tr\tau)}{\eta}\left(\tau_0 - \frac{\partial p}{\partial x}\right)y \tag{13}$$

By using the linear form of stress coefficient and exerting the dimensionless terms into Eq. 13, the dimensionless velocity gradient yields:

$$\dot{\gamma}^*_{yx} = \frac{\partial u^*}{\partial y^*} = \left[1+2\varepsilon De^2\left(Gy^*-\tau^*_0\right)^2\right]\left(Gy^*-\tau^*_0\right) \quad (14)$$

Where the dimensionless terms are defined as

$$u^* = \frac{u}{U},\ y^* = \frac{y}{h_e},\ x^* = \frac{x}{h_e},\ De = \frac{\lambda U}{h_e},$$
$$G = \frac{dp^*}{dx^*},\ p^* = \frac{ph_e}{\eta U},\ \tau^*_0 = \frac{\tau_0 h_e}{\eta U}$$

Where the dimensionless group De is the Deborah number that is a measure for elasticity of the fluid. The term εDe^2 is referred to as the viscoelastic dimensionless group. G is a dimensionless group for the pressure drop and finally, τ^*_0 is the dimensionless shear stress at the moving wall. Non-dimensionalized boundary conditions are written as

$$y^* = 0 \qquad u^* = 1 \quad (15)$$

$$y^* = h^*(x)^* \qquad u^* = 0 \quad (16)$$

Where $h^*(x^*)$ indicates the dimensionless height profile at any point in x-direction for upper plate.

RESULTS AND DISCUSSION

By integrating Eq. 14 and using boundary condition (15), the dimensionless velocity profile is obtained:

$$u^* = \frac{1}{2},\ y^*\left(Gy^*-2\tau_0\right)+\frac{1}{4}\varepsilon De^2 y^*\left(Gy^*-2\tau^*_0\right)\left[\left(Gy^*\right)^2+\left(Gy^*-2\tau^*_0\right)^2\right]+1 \quad (17)$$

When εDe^2 approaches to zero, Eq. 17 converts to Newtonian velocity profile for flow between two plates depicted in Fig. 1 (White, 1991). To obtain the unknown τ^*_0 in Eq. 17, we should substitute boundary condition (16) into Eq. 17, finally, a cubic equation in the term of dimensionless shear stress at the moving wall (τ^*_0) yields. Solution of this equation leads to one real root which is acceptable and two complex roots which are not acceptable:

$$\tau^*_0 = \frac{\alpha}{6\varepsilon De^2 h^*} - \frac{1}{2\alpha}\left(\varepsilon De^2 G^2 h^{*2}+2\right)h^* + \frac{1}{2}Gh^* \quad (18)$$

Where

$$\alpha = \left[54+\left(3\varepsilon De^2 h^*\right)^{\frac{3}{2}}\left[\frac{8h^{*2}+12h^{*4}\varepsilon De^2 G^2+6h^{*6}}{\left(\varepsilon De^2\right)^3 G^6+108\varepsilon De^2}\right]^{\frac{1}{2}}\right]^{\frac{1}{3}}$$

From Eq. 12 can be concluded $\tau^*_{xx}\big|_{y^*=0} = -De\tau^{*2}_0$, it means the dimensionless shear stress on lower plate is depend to elastic effects of viscoelastic fluid. In Eq. 18, G, the dimensionless pressure gradient is unknown. The correct pressure gradient must satisfy the continuity equation everywhere between two plates. By integrating of Eq. 1, we get:

$$\int_0^{h^*}\left(\frac{\partial u^*}{\partial x^*}\right)dy^* = -\int_0^{h^*}\left(\frac{\partial v^*}{\partial y^*}\right)dy^* = v^*(0) - v^*(h^*) \quad (19)$$

Where in this particular case we are assuming that the y-component of velocities $v^*(0)$ and $v^*(h^*)$ are zero at both walls. By Leibniz formula, Eq. 19 can be rearranged as follows:

$$\frac{\partial}{\partial x^*}\left(\int_0^{h^*} u^* dy^*\right) = 0 \quad (20)$$

Substitution of dimensionless x-velocity, Eq. 17 and integration of Eq. 20, obtains a nonlinear second order differential equation for the dimensionless pressure:

$$a\frac{dG}{dx^*} + a_2 G^2 + a_1 G + a_0 = 0 \quad (21)$$

Where

$$a = \frac{1}{6}h^* + \varepsilon De^2\left[\tau^*_0 h^*\left(\tau^*_0 - Gh^*\right) + \frac{3}{10}G^2 h^{*3}\right],\ a_1 = 2h^*\varepsilon De^2\tau^*_0\frac{d\tau^*_0}{dx^*},$$
$$a_2 = -\frac{1}{2}\varepsilon De^2 h^{*2}\frac{d\tau^*_0}{dx^*},\quad a_0 = -\frac{1}{2}\left(1+6\varepsilon De^2\tau^{*2}_0\right)\frac{d\tau^*_0}{dx^*},$$

If we assume the pressure out of the channels is equal to gage pressure, $p^* = 0$, Therefore the appropriate boundary conditions are:

$$p^*(0) = p^*(1) = 0 \quad (22)$$

If the dimensionless distance between two plates depends upon position in the linear form, then we have:

$$h^* = 1+\beta x^* \quad (23)$$

Where

$$\beta = \left(\frac{h_L}{h_e} - 1\right)\frac{h_e}{L}$$

For Newtonian fluids, while εDe^2 approaches to zero, Eq. 21 can be simplified as follows:

$$\frac{1}{6}(1+\beta x^*)\frac{d^2p^*}{dx^{*2}} + \frac{1}{2}\beta\frac{dp^*}{dx^*} - \frac{\beta}{(1+\beta x^*)^2} = 0 \qquad (24)$$

With using of boundary conditions, Eq. 22, the dimensionless pressure distribution for Newtonian fluids between the plates can be derived as (White, 1991):

$$p^*(x^*) = \frac{6\beta x^*(x^*-1)}{(2+\beta)(1+\beta x^*)^2} \qquad (25)$$

Figure 2a illustrates the dimensionless pressure p*, along the channel for various values of inclination β for the case of Newtonian fluid. For small inclination β, the pressure distribution is nearly symmetric and the maximum dimensionless pressure occurs at $x^* \approx 0.5$. As the slope of upper plate promotes, p^*_{max} increases and moves toward the exit plane.

Equation 21 is a highly nonlinear ordinary differential equation and cannot be solved analytically, however it is instructive to study limiting case while dimensionless pressure gradient G, approaches to zero. We can presume the dimensionless pressure gradient is negligible while the values of β is less or the dimensionless viscoelastic group εDe^2 is high, therefore the differential Eq. 21 is simplified as follows:

$$\frac{dG}{dx^*} = \frac{3}{h^*}\frac{d\tau_0^*}{dx^*} \qquad (26)$$

And also Eq. 18 is simplified as follows

$$\tau_0^* = \frac{\alpha}{6\varepsilon De^2 h^*} - \frac{h^*}{\alpha} \qquad (27)$$

Where

$$\alpha = \left(54 + 3(\varepsilon De^2 h^*)^{\frac{3}{2}}\sqrt{24h^{*2} + 324\varepsilon De^2}\right)^{\frac{1}{3}}$$

Two times of integrating of Eq. 26 and using of boundary conditions Eq. 22, leads to an expression for dimensionless pressure:

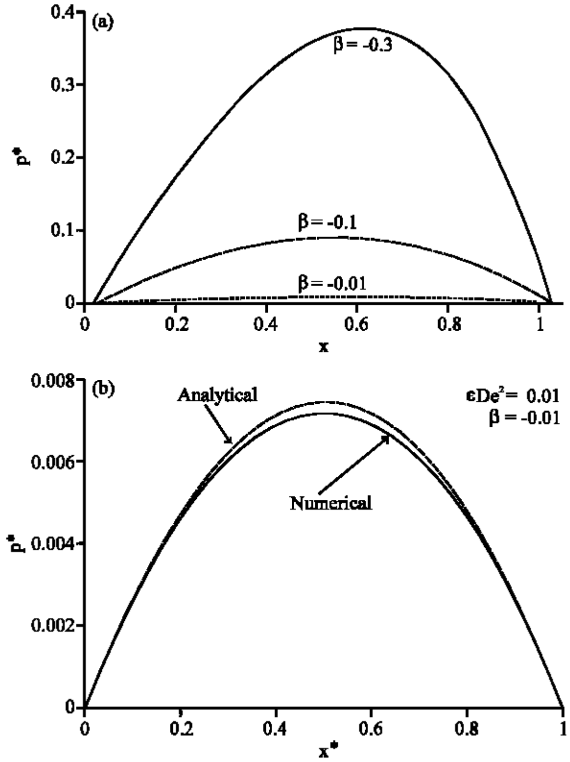

Fig. 2: (a) Dimensionless pressure distribution along the channel for Newtonian fluid ($\varepsilon De^2 = 0$), (b) Differences of numerical and analytical results on dimensionless pressure

$$p^*(x^*) = \int_0^{x^*}\int_0^{X}\frac{3}{h^*(x)}\left(\frac{d\tau_0^*(X)}{dx}\right) dXdx^* - \left[\int_0^1\int_0^X \frac{3}{h^*(x)}\left(\frac{d\tau_0^*(X)}{dx}\right)dXdx^*\right]x^* \qquad (28)$$

For small values of β and εDe^2, the results show good agreement between numerically solution of Eq. 21 and approximate solutions of Eq. 28, (Fig. 2b).

The dimensionless pressure distribution derived from numerical solution of Eq. 21 has been shown in Fig. 3 for various values of β and εDe^2 as effective parameters. From the dimensionless pressure profiles (Fig. 3) we anticipate two flow regions, a region where the pressure gradient is positive ($0 < x^* < x^*_{max}$) and a region where the pressure gradient is negative ($x^*_{max} < x^* < 1$). For very small dimensionless viscoelastic group εDe^2, the pressure distribution is the same as Newtonian fluids (White, 1991). As a limiting case, while β approaches to zero, the resulting solution converts to the solution reported for two parallel plates according to SPTT fluid flow (Hashemabadi *et al.*, 2003). Totally, as the degree of

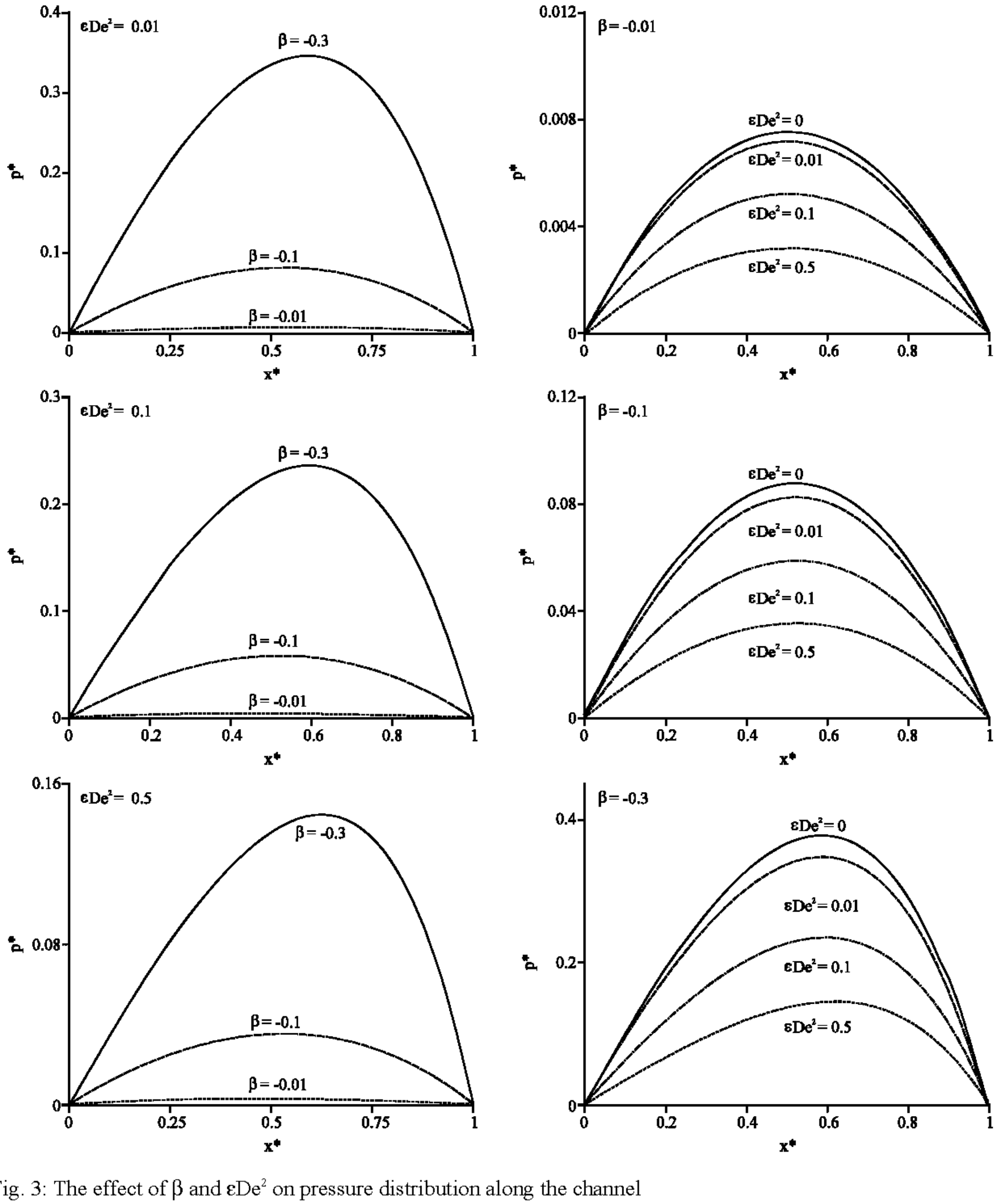

Fig. 3: The effect of β and εDe^2 on pressure distribution along the channel

contraction (β) increases, p^*_{max} increases and moves toward the end of channel. It must be noted that the increasing of dimensionless pressure from β = -0.1 to β = -0.3 is larger than the increasing of pressure while the inclination of upper plate changes from β = -0.01 to β = -0.1.

For constant angle of upper plate, the results show while εDe^2 increases, the maximum pressure decreases. In other words, existence of elasticity effects and normal stresses could change the pressure distribution between two plates how to be more uniform respect to Newtonian fluids. The dimensionless apparent viscosity is defined as the ratio of dimensionless shear stress to dimensionless shear rate:

$$\eta^* = \frac{\tau^*_{yx}}{\dot{\gamma}^*} = \frac{1}{1+2\varepsilon De^2[G(x^*)h^*(x^*) - \tau^*_0(x^*)]^2} \quad (29)$$

Figure 4 shows the dimensionless viscosity profile at two cross sections x* = 0.25 and 0.75. As it's shown, the average viscosity at the entrance of the channel is higher than the average viscosity at the end region.

The typical velocity profiles have been illustrated at three cross sections of channel in Fig. 5. It can be concluded the velocity magnitude decreases relative to the Couette flow at entrance region, increases at exit region and near to location of maximum pressure, the flow is almost Couette and the velocity profiles are linear at cross section. The combined effects of Couette and Poissule flow in exit region causes lower apparent viscosity (Fig. 4) and this create higher velocity gradient at two walls, meanwhile there is no changes in velocity gradient at lower plate in entrance region. On the other hand, the dimensionless pressure gradient decreases with increasing of dimensionless viscoelastic group, so the velocity profile approaches to the velocity profile of Couette flow for this situation (Fig. 5).

The skin friction coefficient is defined as (Shah and London, 1978):

$$C_{f,x}^* = \frac{\bar{\tau}_w(x^*)}{\frac{1}{2}\rho\bar{u}^2} \quad (30)$$

By using the dimensionless groups, the product of skin friction factor and Reynolds number becomes as follows:

$$C_{f,x}^*\,Re = \frac{4\bar{\tau}^*_w(x^*)h^*(x^*)}{\bar{u}^*} \quad (31)$$

Where $\bar{\tau}^*_w$, the dimensionless average wall shear stress can be obtained by

$$\bar{\tau}^*_w = \frac{1}{1+\sqrt{1+\beta^2}}\left[\tau^*_0 + (\tau^*_0 - Gh^*)\sqrt{1+\beta^2}\right] \quad (32)$$

The dimensionless average velocity can be obtained by the following equation:

$$\bar{u}^* = \frac{1}{h^*(x^*)}\int_0^{h^*} u^*\,dy^* \quad (33)$$

From substitution of Eq. 17, the non dimensional average velocity can be obtained:

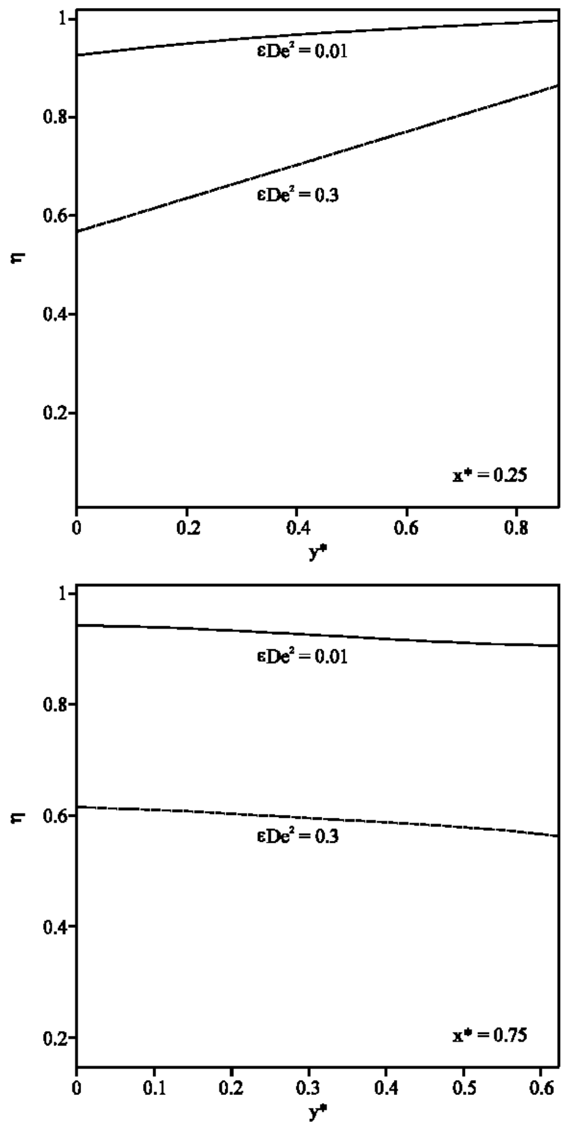

Fig. 4: Variation of apparent viscosity across the channel for β = -0.5 at x* = 0.25 and x* = 0.75

$$\bar{u}^* = h^*\left[\frac{1}{6}(Gh^* - 3\tau^*_0) + \varepsilon De^2\left(\frac{1}{10}G^3h^{*3} + \tau^{*2}_0\,Gh^* - \frac{1}{2}\tau^*_0\,G^2h^{*2} - \tau^{*3}_0\right)\right] + 1 \quad (34)$$

For Newtonian fluids, the dimensionless average velocity equals to

$$\frac{1}{2} - \frac{1}{12}Gh^{*2}$$

that for parallel plate the solution converts to the previous solution (Hashemabadi *et al*., 2003). Figure 6 shows variation of production of skin friction factor $C_{f,x}$* and

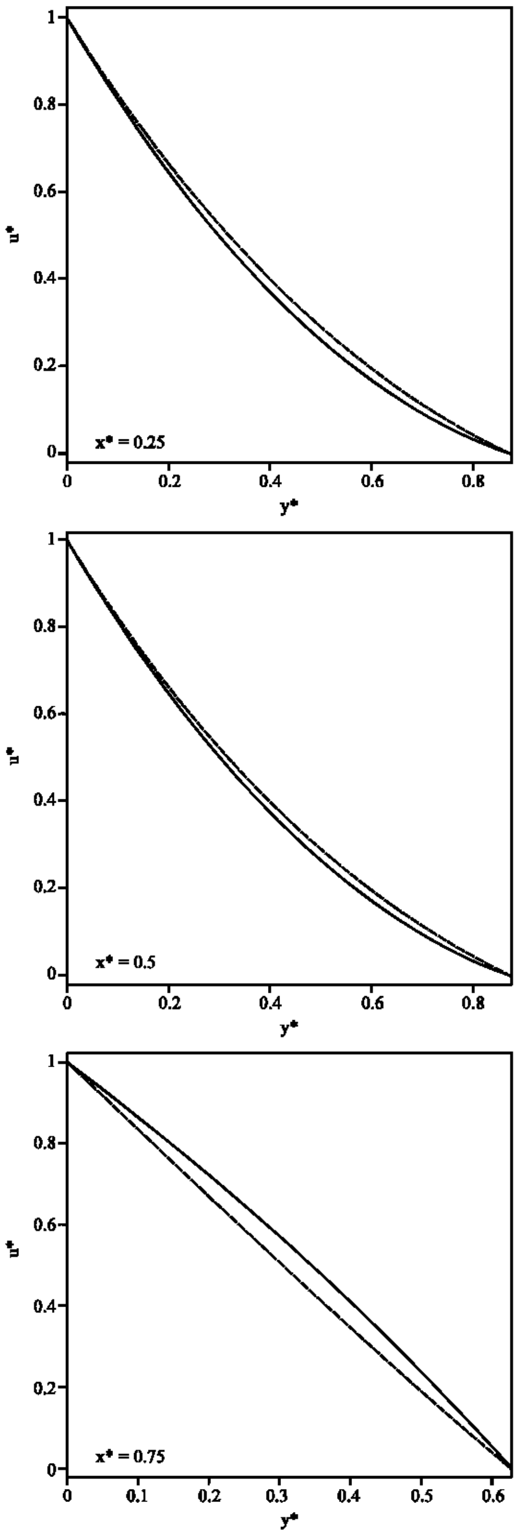

Fig. 5: Velocity profile at three cross sections in the duct β = -0.5, εDe^2 =0.01 (solid line), εDe^2 = 0.5 (dashed line)

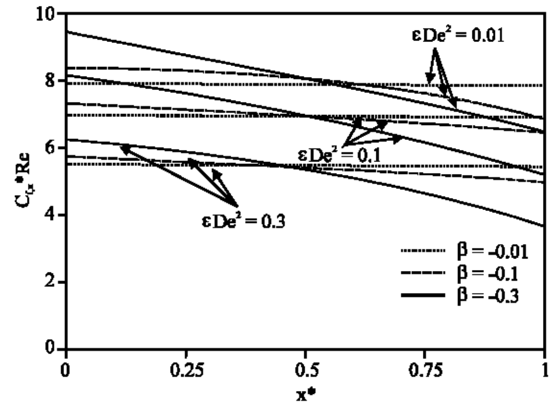

Fig. 6: Effects of viscoelastic group εDe^2, on C_{fx}* Re for various values of β

Reynolds number for different values of β and εDe^2. The results illustrate while the inclination is small, the magnitude of $C_{f\,x}$ * Re becomes nearly constant and when the inclination increases, the $C_{f\,x}$* Re reduces. Also the results show when the viscoelastic group increases, due to shear thinning effects of fluid the value of $C_{f\,x}$* Re decreases. For Newtonian fluids

when $\left|G\right| > \frac{2}{h^{*2}}$:

$$C_{f,x}*Re=\frac{24}{\left(1+\sqrt{1+\beta^2}\right)}\left\{\frac{2\left(1-\sqrt{1+\beta^2}\right)-Gh^{*2}\left(1-\sqrt{1+\beta^2}\right)}{6-Gh^{*2}}\right\} \tag{35}$$

When $\left|G\right| < \frac{2}{h^{*2}}$:

$$C_{f,x}*Re=\frac{24}{\left(1+\sqrt{1+\beta^2}\right)}\left\{\frac{2\left(1+\sqrt{1+\beta^2}\right)+Gh^{*2}\left(1-\sqrt{1+\beta^2}\right)}{6-Gh^{*2}}\right\} \tag{36}$$

When the plates are parallel and G approaches to negative infinity, Eq. 35 approaches to 24 which is equivalent to the purely Poiseuille flow. Also when the plates are parallel and G approaches to zero, then Eq. 36 approaches to 8 which is equivalent to the purely Couette flow.

CONCLUSIONS

The analytical solution was suggested for lubrication flow of nonlinear viscoelastic fluid between two plates that make an angle with respect to each other. It was assumed that fluid obeys the simplified Phan-Thien Tanner (SPTT) model. Effects of elasticity and elongational behavior of fluid on pressure distribution through the channel and velocity profile were investigated. The results show with increasing of the viscoelastic dimensionless group (εDe^2), the velocity gradient at moving wall decreases and as a result, the friction factor of the fluid at this wall decreases and the amount of maximum pressure relative to Newtonian fluids reduces and the pressure distribution through the channel becomes more uniform for a given upper plate inclination β. When β decreases, the pressure distribution becomes symmetric and for two parallel plate dimensionless pressure gradient approaches to zero. Effect of inclination of upper plate (β) on pressure distribution for higher viscoelastic group is more considerable. Influence of viscoelasticity on skin friction coefficient is noticeable and results show when the contraction is slight, magnitude of C_{fx}* Re becomes nearly constant and when the contraction is steeper, the changes of C_{fx}* Re take the falling form. Also the results show when elasticity of the fluid increases, the values of C_{fx}* Re decrease.

Nomenclature

a, a_0, a_1 and a_2	Parameters of Eq. 21
D_h	Hydraulic diameter
D	Deborah number ($\lambda U/h_0$)
G	Dimensionless pressure gradient
h	Gap between two plates through the channel
P	Pressure
Re	Reynolds number ($\rho u D_h/\eta$)
U	Velocity of moving plate
u	Velocity profile
x	Axial coordinate
y	Lateral coordinate
Z	Stress coefficient function

Greek symbols

α	Parameter of Eq. 18
β	Parameter of Eq. 23
ε	Elongational parameter of PTT model
η	Viscosity coefficient of PTT model
$\dot{\gamma}$	Shear rate tensor
λ	Relaxation time in PTT model
ξ	Slip parameter
τ	Stress tensor

Subscripts

e	Values at entrance of channel
L	Values at end of channel
0	Values at the moving wall
w	At the wall

Superscripts

*	Refers to dimensionless quantities
-	Refers to average quantities
†	Transpose of tensor

REFERENCES

Alves, M.A., F.T. Pinho and P.J. Oliviera, 2001. Study of steady pipe and channel flows of single mode phan-thien-tanner fluid. J. Non-Newtonian Fluid Mech., 101: 55-76.

Bird, R.B., R.C. Amstrong and O. Hassager, 1978. Dynamics of polymeric liquids. Vol. 1, Fluid mechanics. 2nd Edn., John Wiley, New York.

Hashemabadi, S.H., S.Gh. Etemad, J. Thibault and M.R. Golkar Naranji, 2003. Analytical Solution for Dynamic Pressurization of Viscoelastic Fluid. Int. J. Heat Fluid Flow, 24: 137-144.

Mirzazadeh, M., M.P. Escudier, F. Rashidi and S.H. Hashemabadi, 2005. Purely tangential flow of a PTT-viscoelastic fluid within a concentric annulus. J. Non-Newtonian Fluid Mech., 129: 88-97.

Oliviera, P.J. and F.T. Pinho, 1999. Analytical solution for fully developed channel and pipe flow of phan-thien-tanner fluid. J. Fluid Mech., 387: 271-280.

Phan-Thien, N. and R.I. Tanner, 1977. A new constitutive equation derived from network theory. J. Non-Newtonian Fluid Mech., 2: 353-365.

Phan-Thien, N., 1978. A non-linear network viscoelastic model. J. Rheo., 22: 259-283.

Sawyer, W.G. and J.A. Tichy, 1998. Non-newtonian lubrication with the second-order fluid. ASME J. Tribol., 120: 622-628.

Shah, R.K. and A.L. London, 1978. Laminar Flow Forced Convection in Ducts, Academic Press Inc., New York.

Tanner, R.I., 2000. Engineering Rheology, Clarendon Press, Oxford.

Tichy, J.A., 1996. Non-newtonian lubrication with the convective max-well model. ASME J. Tribol., 118: 344-349.

White, F.M., 1991. Viscous Fluid Flow, McGraw-Hill, Inc.