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Particle in a Box with Dissipation

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Abstract: The canonical quantization of a system of free particle in a bounded volume in space, particle in a box, containing a dissipative medium, is carried out according to the Dirac method. A suitable Schrödinger equation is set up and solved for the Lagrangian representing this system. The viscous forces effects and hence the dissipation are represented clearly in the resultant wave function as will as in the energy eigenvalues.

Key words: Free particle in a box, Hamiltonian formulation, canonical quantization, fractional calculus, nonconservative systems, dissipation

INTRODUCTION

Dissipative forces include any and all types of such a nature that energy is dissipated from the system when motion takes place. The lost energy is usually accounted for by the formation of heat.

It frequently happens in practice that the magnitude of the force, F, on a particle may be closely represented, over certain ranges of velocity at least, by

$$F = av^n, \quad (1)$$

where v is the velocity of the particle, n is some number and α may be a constant or a function of coordinates and/or time.

- For a = 0 and a equals to the coefficient of friction times the normal force, Eq. 1 represents a frictional force. Frictional force required to slide one surface over another is assumed to be proportional to the normal force between surfaces, independent of the area of contact and independent of the speed, once motion is started.
- For n = 1, Eq. 1 represents viscous force. When the force on an object varies as the first power of its speed and is opposite in direction to its motion, it is said to be viscous. The drag on an object moving slowly through a fluid of any kind or the drag on a magnetic pole which is moving near a conducting sheet are examples of viscous forces.
- For n >1 and higher velocities, the drag on an object moving in a fluid is not a simple viscous force, instead of that and in certain cases, n may be considerably greater than one.

Most advanced methods of classical mechanics deal only with conservative systems, although all natural processes in the physical world are nonconservative. Classically or quantum-mechanically treated, macroscopically or microscopically viewed, the physical world shows different kinds of dissipation and irreversibility. Mostly ignored in analytical techniques, this dissipation appears in friction, Brownian motion, inelastic scattering, electrical resistance and many other processes in nature.

Many attempts have been made to incorporate nonconservative forces into Lagrangian and Hamiltonian formulations; but those attempts could not give a completely consistent physical interpretation of these forces. The Rayleigh dissipation function, invoked when the frictional force is proportional to the velocity, was the first to be used to describe frictional forces in the Lagrangian (Goldstein, 1980). However, in that case, another scalar function was needed, in addition to the Lagrangian, to specify the equations of motion. At the same time, this function does not appear in the Hamiltonian. Accordingly, the whole process is of no use when it is attempted to quantize nonconservative systems.

The most substantive study in this context was that of Riewe (1996, 1997) who used fractional derivatives to study nonconservative systems and was able to generalize the Lagrangian and other classical functions to take into account nonconservative effects.

Rabei *et al.* (2004) used Laplace transforms of fractional integrals and fractional derivatives to develop a general formula for the potential of any arbitrary force, conservative or nonconservative. This led directly to the consideration of the dissipative effects in Lagrangian and Hamiltonian formulations.

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In this study a method of quantizing nonconservative, or dissipative, systems is applied using fractional calculus (Appendix) (Rabei *et al.*, 2006a, b) to a system of free particle in a box, containing a dissipative medium.

QUANTIZATION OF NONCONSERVATIVE SYSTEMS

According to the most recent theory of the quantization of nonconservative systems (Rabei *et al.* 2006a, b), the starting point for quantizing the Hamiltonian is to change the coordinates and momenta, $q_{r,s(i)}$ and the $P_{r,s(i)}$, into operators satisfying commutation relations which correspond to the Poisson-bracket relations of the classical theory (Dirac, 1964). The first step was to connect the canonical conjugate variables, i.e., which of $p_{r,s(i)}$ and $q_{r,s(i)}$ are the canonical conjugate variables was determined.

This canonical conjugate relation was obtained directly from Hamilton's equation defined by Riewe (1996, 1997) as follows:

$$\frac{\partial H}{\partial p_{r,s(i)}} = q_{r,s(i+1)} \tag{2}$$

$$= \frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}} q_{r,s(i)}, \qquad 0 \le i \le N-1 \tag{3}$$

Rabie *et al.* (2006a, b)concluded that $p_{r,s(i)}$ is the canonical conjugate of $q_{r,s(i)}$. Accordingly, the Hamiltonian can be written in the following form:

$$\begin{aligned} H &= \sum_{i=0}^{N-1} \frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}} q_{r,s(i)} p_{r,s(i)} - L, \qquad 0 \le i \le N-1 \\ &= \sum_{i=0}^{N-1} q_{r,s(i+1)} p_{r,s(i)} - L \end{aligned} \tag{4}$$

The last result, Eq. 3, is equivalent to Riewe's Hamiltonian Riewe (1996, 1997) and it is applicable to higher-order Lagrangians with integer derivatives.

The most general classical Poisson bracket for any two functions was defined, F and G, in phase space (Rabei *et al.*, 2006a, b) as follows:

$$\{F, G\} = \sum_{k=0}^{N-1} \frac{\partial F}{\partial q_{r,s(k)}} \frac{\partial G}{\partial p_{r,s(k)}} - \frac{\partial F}{\partial p_{r,s(k)}} \frac{\partial G}{\partial q_{r,s(k)}} \tag{5}$$

The fundamental Poisson brackets then read (Rabei *et al.* 2006a, b):

$$\begin{aligned} \{q_{r,s(i)}, P_{l,s(j)}\} &= \sum_{i=0}^{N-1} \frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}} q_{r,s(i)} p_{r,s(i)} - L, \quad 0 \le i \le N-1 \\ &= \delta_{ij}\delta_{rl} \end{aligned} \tag{6}$$

Substituting integer derivatives, one recovers the well-known definition of Poisson brackets.

According to the definition of the Hamiltonian, Hamilton's equations of motion can be written in terms of Poisson brackets as (Rabei *et al.*, 2006a, b):

$$\frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}} q_{r,s(i)} = q_{r,s(i+1)} = \{q_{r,s(i)}, H\} \tag{7}$$

and

$$(-1)^{s(i+1)-s(i)} \frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}} p_{r,s(i)} = -\{p_{r,s(i)}, H\} \tag{8}$$

These two definitions are valid for higher-order Lagrangians with integer derivatives and lead to the same definitions given by Pimental and Teixeira (1997). These definitions are more generalized and are applicable for fractional as well as integer systems.

Connect the canonical conjugate variables quantum-mechanically by defining the momentum operator as (Rabei *et al.*, 2006a, b):

$$p_{s(i)} \equiv \frac{\hbar}{i} \frac{\partial}{\partial q_{s(i)}}, \qquad i = 0, 1, \ldots, N-1. \tag{9}$$

The correspondence between the quantum-mechanical operator bracket and the classical Poisson bracket is straightforward (Rabei *et al.*, 2006a, b):

$$[q_{r,s(i)}, p_{r,s(i)}]\Psi = [q_{r,s(i)} p_{r,s(i)} - p_{r,s(i)} q_{r,s(i)}]\Psi; \tag{10}$$

$$= i\hbar\Psi; \tag{11}$$

and the Schrödinger equation reads

$$H\Psi = i\hbar \frac{\partial}{\partial t}\Psi. \tag{12}$$

It follows that the commutators of the quantum-mechanical operators are proportional to the corresponding classical Poisson brackets (Rabei *et al.* 2006a, b):

$$\left[q_{r,s(i)}, p_{r,s(i)}\right] \leftrightarrow i\hbar\left\{q_{r,s(i)}, p_{r,s(i)}\right\}. \tag{13}$$

Further, Rabie *et al* (2006a, b)generalize Heisenberg's equation of motion for coordinate operators as follows:

$$\frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}}\left\langle \hat{q}_{r,s(i)} \right\rangle = \frac{1}{i\hbar}\left[\hat{q}_{r,s(i)}, \hat{H}\right]; \tag{14}$$

and for momentum operators:

$$(-1)^{s(i+1)-s(i)} \frac{d^{s(i+1)-s(i)}}{d(t-a)^{s(i+1)-s(i)}} \hat{p}_{r,s(i)} = -\frac{1}{i\hbar}\left[\hat{p}_{r,s(i)}, \hat{H}\right]. \tag{15}$$

Equation 14 and 15 are valid for integer-order derivatives as well as non-integer order.

PARTICLE IN A BOX CONTAINING DISSIPATIVE MEDIUM

Suppose a one dimensional box of width 2a and a potential

$$V(x) = \begin{cases} 0, & \text{if } -a < x < \alpha \\ \infty, & \text{otherwise} \end{cases} \tag{16}$$

A particle in this potential is completely free inside the box, except at the two ends ($x = -\alpha$ and $x = \alpha$), where an infinite force prevents it from escaping. This potential is awfully artificial, but it should be treated with respect. Despite its simplicity, or rather, precisely because of its simplicity, it serves as a wonderfully accessible test case for many real cases in quantum mechanics (Griffiths, 1995). A classical model would be a cart on a frictionless horizontal air track, with perfectly elastic bumpers--it just keeps bouncing back and forth forever. Outside the well the probability of finding the particle there is zero). Inside the box, where V = 0, the time-independent Schrödinger equation reads (Griffiths, 1995; Merzbacher, 1970).

$$E\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi, \tag{17}$$

which has the general solution

$$\Psi = A \sin kx + B \cos kx. \tag{18}$$

Where A and B are arbitrary constants fixed by the boundary conditions of the problem. Ordinarily, the appropriate boundary conditions leads to the normalized solution (Griffiths, 1995; Merzbacher, 1970).

$$\Psi = \sqrt{\frac{2}{\alpha}} \sin(\frac{n\pi}{\alpha}x), \tag{19}$$

with the possible values of energy

$$E = \frac{n^2\pi^2\hbar^2}{2m\alpha^2}. \tag{20}$$

In the following we are going to treat the same problem with a box containing viscous material. Consider a free particle moving in a dissipative medium, the particle is moving in a box containing a fluid, the viscous forces on an object, varies as the first power of its speed, n = 1 in Eq. 1, i.e.,

$$f = -\gamma q_1, \tag{21}$$

γ being a positive constant. Using the formula given by Rabei *et al.* (2004).

$$U = (-1)^{-(\alpha+1)} \int \left[L^{-1}\left\{ \frac{1}{S^\alpha} L\left(F(q_\beta)\right) \right\} \right] dq_\alpha, \tag{22}$$

one can derive the potential of a nonconservative force.

The potential corresponding to this dissipation is

$$U = \frac{i\gamma}{2} q_{1/2}^2. \tag{23}$$

The Lagrangian is

$$L = \frac{1}{2} m q_1^2 - \frac{i\gamma}{2} q_{1/2}^2, \tag{24}$$

where

$$q_0 = x, \quad q_1 = \frac{dx}{dt}, \quad q_{1/2} = \frac{d^{1/2}x}{d(t-\alpha)^{1/2}}; \tag{25}$$

and

$$s(0) = 0, s(1) = ½, s2) = 1. \tag{26}$$

The generalized Euler-Lagrange equation for this problem reads

$$\frac{\partial L}{\partial q_0} + (-1)^{1/2} \frac{d^{1/2}}{d(t-\alpha)^{1/2}} \frac{\partial L}{\partial q_{1/2)}} - \frac{d}{dt}\frac{\partial L}{\partial q_1} = 0 \tag{27}$$

Substituting the Lagrangian given by Eq. 24, we obtain the equation of motion

$$m\ddot{q} + \gamma\dot{q} = 0. \tag{28}$$

The canonical momenta are

$$\begin{aligned} p_0 &= \frac{\partial L}{\partial q_{1/2}} + i\frac{d^{1/2}}{d(t-a)^{1/2}}\frac{\partial L}{\partial q_1} \\ &= i\gamma\, q_{1/2} + imq_{3/2}; \end{aligned} \tag{29}$$

and

$$p_{1/2} = \frac{\partial L}{\partial q_1} \qquad = mq_1 \tag{30}$$

Making use of Eq. 4, we have for the Hamiltonian

$$\begin{aligned} H &= \frac{d^{1/2}}{d(t-a)^{1/2}}(q_0)p_0 + \frac{d^{1/2}}{d(t-a)^{1/2}}\left(q_{1/2}\right)p_{1/2} - L \\ &= \frac{(p_{1/2})^2}{2m} + q_{1/2}p_0 + \frac{\gamma}{2i}q_{1/2}^2 \end{aligned} \tag{31}$$

Here p_0 and $p_{1/2}$ are the canonical conjugate momenta to q_0 and $q_{1/2}$, respectively.

The fundamental Poisson brackets are calculated as

$$\{q_0, p_0\} = \frac{\partial q_0}{\partial q_0}\frac{\partial p_0}{\partial p_0} - \frac{\partial p_0}{\partial q_0}\frac{\partial q_0}{\partial p_0} + \frac{\partial q_0}{\partial q_{1/2}}\frac{\partial p_0}{\partial p_{1/2}} - \frac{\partial p_0}{\partial q_{1/2}}\frac{\partial q_0}{\partial p_{1/2}} = 1 \tag{32}$$

and

$$\left\{q_{1/2}, p_{1/2}\right\} = \frac{\partial q_{1/2}}{\partial q_0}\frac{\partial p_{1/2}}{\partial p_0} - \frac{\partial p_{1/2}}{\partial q_0}\frac{\partial q_{1/2}}{\partial p_0} + \frac{\partial q_{1/2}}{\partial q_{1/2}}\frac{\partial p_{1/2}}{\partial p_{1/2}} - \frac{\partial p_{1/2}}{\partial q_{1/2}}\frac{\partial q_{1/2}}{\partial p_{1/2}} = 1. \tag{33}$$

It is clear that other brackets vanish, i.e.,

$$\begin{aligned} &\left\{q_0, p_{1/2}\right\} = \left\{q_0, q_{1/2}\right\} = \left\{q_{1/2}, q_{1/2}\right\} = \left\{p_{1/2}, p_{1/2}\right\} = \\ &\left\{p_0, p_{1/2}\right\} = \left\{q_{1/2}, p_0\right\} = \{q_0, q_0\} = \{p_0, p_0\} = 0. \end{aligned} \tag{34}$$

From Eq. 7, 8 we obtain Hamilton's equations of motion as

$$\frac{d^{1/2}}{d(t-a)^{1/2}}q_0 = q_{1/2} = \{q_0, H\} = \sum_{i=0}^{1}\frac{\partial q_0}{\partial q_{s(i)}}\frac{\partial H}{\partial p_{s(i)}} - \frac{\partial H}{\partial q_{s(i)}}\frac{\partial q_0}{\partial p_{s(i)}} = \frac{\partial H}{\partial p_0}; \tag{35}$$

and

$$\begin{aligned} \frac{d^{1/2}}{d(t-a)^{1/2}}q_{1/2} &= q_1 = \left\{q_{1/2}, H\right\} \\ &= \sum_{i=0}^{1}\frac{\partial q_{1/2}}{\partial q_{s(i)}}\frac{\partial H}{\partial p_{s(i)}} - \frac{\partial H}{\partial q_{s(i)}}\frac{\partial q_{1/2}}{\partial p_{s(i)}} = \frac{\partial H}{\partial p_{1/2}}. \end{aligned} \tag{36}$$

On the other hand,

$$\begin{aligned} i\frac{d^{1/2}}{d(t-a)^{1/2}}p_0 &= -\{p_0, H\} = -\sum_{i=0}^{1}\frac{\partial p_0}{\partial q_{s(i)}}\frac{\partial H}{\partial p_{s(i)}} - \frac{\partial H}{\partial q_{s(i)}}\frac{\partial p_0}{\partial p_{s(i)}} \\ &= \frac{\partial H}{\partial q_0}. \end{aligned} \tag{37}$$

and

$$\begin{aligned} i\frac{d^{1/2}}{d(t-a)^{1/2}}p_{1/2} &= -\left\{p_{1/2}, H\right\} = -\sum_{i=0}^{1}\frac{\partial p_{1/2}}{\partial q_{s(i)}}\frac{\partial H}{\partial p_{s(i)}} - \frac{\partial H}{\partial q_{s(i)}}\frac{\partial p_{1/2}}{\partial p_{s(i)}} \\ &= \frac{\partial H}{\partial q_{1/2}}. \end{aligned} \tag{38}$$

Equation 35 is an identity; Eq. 36 gives the definition of the momentum introduced in Eq. 30; while Eq. 37 represents the q_0-independence of the Hamiltonian. Further, Eq. 38 can be written in the form

$$i\frac{d^{1/2}}{d(t-a)^{1/2}}(mq_1) = p_0 - i\gamma q_{1/2} \tag{39}$$

which is equivalent to Eq. 29 and yields the equation of motion, Eq. 28, if the semi-derivation of both sides is taken.

The quantum-mechanical brackets are obtained directly from Eq. 10 as

$$[q_0, p_0]\Psi = i\hbar\Psi; \tag{40}$$

and

$$[q_{1/2}, p_{1/2}]\Psi = i\hbar\Psi \tag{41}$$

Equations 40, 41 are compatible with Eq. 32, 33.

With Eq. 9, 12 and 31, Schrödinger's equation reads

$$i\hbar\frac{\partial}{\partial t}\Psi = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_{1/2}^2} + \frac{\hbar}{i}q_{1/2}\frac{\partial}{\partial q_0} + \frac{1}{2i}\gamma q_{1/2}^2\right]\Psi \tag{42}$$

Heisenberg's equation of motion for the damped free particle system can be obtained using Eq. 14, 15:

$$\frac{d^{1/2}}{d(t-a)^{1/2}}\hat{q}_0 = \frac{1}{i\hbar}\left[\hat{q}_0, \hat{H}\right] = \frac{1}{i\hbar}\left[\hat{q}_0, \hat{q}_{1/2}\hat{p}_0\right] = q_{1/2} \tag{43}$$

Also

$$\begin{aligned}\frac{d^{1/2}}{d(t-a)^{1/2}}\hat{q}_{1/2} &= \frac{1}{i\hbar}\left[\hat{q}_{1/2}, \hat{H}\right]\\ &= \frac{1}{i\hbar}\left[\hat{q}_{1/2}, \frac{(p_{1/2})^2}{2m}\right]\\ &= \frac{1}{i\hbar}\frac{1}{2m}\left[\hat{q}_{1/2}, p_{1/2}\right]p_{1/2} + p_{1/2}\left[\hat{q}_{1/2}, p_{1/2}\right]\\ &= \frac{p_{1/2}}{m} = q_1\end{aligned} \tag{44}$$

On the other hand,

$$i\frac{d^{1/2}}{d(t-a)^{1/2}}\hat{p}_0 = -\frac{1}{i\hbar}\left[\hat{p}_0, \hat{H}\right] = 0; \tag{45}$$

and

$$i\frac{d^{1/2}}{d(t-a)^{1/2}}\hat{p}_{1/2} = -\frac{1}{i\hbar}\left[\hat{p}_{1/2}, \hat{H}\right] = p_0 - i\gamma q_{1/2} \tag{46}$$

Equation 43-46 are compatible with Eq. 35-38.

Substituting Eq. 29 into Eq. 45, we get

$$-\gamma\frac{d^{1/2}}{d(t-\alpha)^{1/2}}\hat{q}_{1/2} - m\frac{d^{1/2}}{d(t-\alpha)^{1/2}}\hat{q}_{3/2} = 0, \tag{47}$$

which, after semi-derivation, may be written as

$$m\frac{d^2}{dt^2}q_0 + \gamma\frac{d}{dt}q_0 = 0. \tag{48}$$

This equation is equivalent to the well-known equation for a free particle passing through a dissipative medium.

In other words, the results obtained from Heisenberg's equation of motion are compatible with those obtained classically.

Invoking Eq. 42, which is Schrödinger's equation for a free particle in a box containing a dissipative medium and defining Ψ as

$$\Psi = F\,(q_{1/2}, q_0)\,T(t), \tag{49}$$

then, using the method of separation of variables, we obtain the following:
the time-dependent part:

$$i\hbar\frac{d}{dt}T = E_0 T, \tag{50}$$

which has the solution

$$T = T_0 \exp\frac{-iE_0 t}{\hbar}; \tag{51}$$

and the other, time-independent, part:

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2i}\gamma y^2 + \frac{\hbar}{i}y\frac{\partial}{\partial x}\right]F = E_0 F, \tag{52}$$

where $q_0 = x$ and $q_{1/2} = y$. Letting $x = uy$ and substituting into Eq. 52, we have

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2i}\gamma y^2 + \frac{\hbar}{i}\frac{\partial}{\partial u}\right]F = E_0 F. \tag{53}$$

The y-part reads

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{1}{2i}\gamma y^2\right]Y = E_y Y. \tag{54}$$

or

$$\left[\frac{d^2}{dy^2} - \left(\frac{m\gamma}{i\hbar^2}\right)y^2\right]Y = -2E_y\left(\frac{m}{\hbar^2}\right)Y. \tag{55}$$

This has the solution (Dass and Sharma, 1998; Arfken, 1985)

$$Y_n = Y_0^{H_n}\left[\left(\frac{m\gamma}{i\hbar^2}\right)^{1/4} y\right]\exp\frac{-\left(\frac{m\gamma}{i\hbar^2}\right)^{1/2} y^2}{2} \tag{56}$$

where H_n are Hermite polynomials
The u-part of Eq. 53 reads

$$\left[\frac{\hbar}{i}\frac{d}{du} - E_x\right]U = 0 \tag{57}$$

which has the solution

$$U = U_0 \exp\left(\frac{iE_x u}{\hbar}\right) \tag{58}$$

or, by considering the boundary conditions $\Psi = 0$ at x = -a and x = a,

$$U = U_0 \cos\left(\frac{E_x u}{\hbar}\right) \tag{59}$$

The energy eigenvalues

$$E_{yn} = \frac{1}{2}(2n+1)\hbar\left(\frac{i\gamma}{m}\right) \text{ and } E_{xn} = \frac{n\pi\hbar}{2a} \tag{60}$$

Thus,

$$\Psi_n = AHn\left[\left(\frac{m\gamma}{i\hbar^2}\right)^{1/4} y\right]\exp\frac{-\left(\frac{m\gamma}{i\hbar^2}\right)^{1/2} y^2}{2}\cos\left(\frac{E_x}{\hbar}\right)\frac{x}{y}\exp\frac{-i}{\hbar}E_0 t. \tag{61}$$

In terms of $q_{s(i)}$,

$$\Psi_n = AHn\left[\left(\frac{m\gamma}{i\hbar^2}\right)^{1/4} q_{1/2}\right]\exp\frac{-\left(\frac{m\gamma}{i\hbar^2}\right)^{1/2} q^2_{1/2}}{2}\cos\left(\frac{Eq_0}{\hbar q_{1/2}}\right)q_0 \exp\left(\frac{-i}{\hbar}E_0 t\right). \tag{62}$$

The canonical coordinates are q_0 and $q_{1/2}$. The wave function Ψ depends on these. The total energy eigenvalues

$$E_n = E_{yn} + E_{xn} = \frac{1}{2}(2n+1)\hbar\left(\frac{i\gamma}{m}\right) + \frac{n\pi\hbar}{2a} + n = 1,2,.. \tag{63}$$

the drag force effects are represented clearly in the wave function and in the energy eigenvalues.

CONCLUSION

The quantization of a nonconservative particle in a box system has been carried out according to the theory have been proposed recently. A potential corresponding to the viscous force and a Hamiltonian are constructed. The results obtained using Heisenberg's equation of motion are compatible with those obtained classically. The relevant Schrödinger's equation has then been set up and solved. The viscous force effects and hence the dissipation are represented clearly in the resultant wave function as will as in the energy eigenvalues.

APPENDIX

Fractional calculus

The fractional integral of a function f (t) is defined as (Oldham and Spanier, 1974)

$$J^\alpha f(t) \equiv \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau\,, \qquad \alpha \in R^+ \tag{A-1}$$

where J^α represents the fractional integral operator of order α and R^+ represents the set of positive real numbers.

If we introduce the positive integer m such that $m-1<\alpha \leq m$, the fractional derivative of order $\alpha > 0$ may be defined as

$$D^\alpha f(t) = D^m J^{m-\alpha} f(t), \tag{A-2}$$

D^α being the fractional deferential operator of order α. Equation (A-2) may be rewritten using Eq. (A-1) as follows:

$$D^\alpha f(t) \equiv \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau\right], \qquad m-1 < \alpha \leq m \tag{A-3}$$

Here, we formulate the problem in terms of the left fractional derivative the left Riemann-Liouville fractional derivatives, which are defined in Eq. (A-1, A-2). Most of the left fractional operations also hold for the right ones. For the left operations f (t) must vanish for $t < \alpha$; while f (t) = 0 fort > b for the right operation. Thus, the left operations are causal. Conversely, the right operations are anti-causal (Dreisigmeyer and Young, 2003). From the physical point of view, when we differentiate with respect to time, the right differentiation represents an operation performed on the future state of the process f (t) (Agrawal, 2001).

Fractional integral and differential operators have the following properties (Oldham and Spanier, 1974):

For I, the identity operator:

$$D^n J^n = 1 \tag{A-4}$$

but the inverse application of the two operators is not necessarily true.

For n > 0, J^n and D^n are linear operators, i.e.

$$J^n[f_1(x)+f_2(x)] = J^n f_1(x)+J^n f_2 (x): \qquad (A\text{-}5)$$

$$D^n[f_1(x)+f_2(x)] = D^n f_1(x)+D^n f_2 (x). \qquad (A\text{-}6)$$

For a constant c, J^n and D^n are homogeneous operators, i.e.,

$$J^n[cf (x)] = cJ^n f(x) \qquad (A\text{-}7)$$

$$D^n[cf (x)] = cD^n f(x) \qquad (A\text{-}8)$$

For α, β>0, J^n obeys the additive index law, but not necessarily D^n, i.e.,

$$J^\alpha J^\beta[f (x)] = J^{\alpha+\beta} f (x); \qquad (A\text{-}9)$$

$$D^\alpha D^\beta[f (x)] \neq D^{\alpha+\beta} f (x). \qquad (A\text{-}10)$$

Of special importance are the fractional integrals and fractional derivatives of the function $(t\text{-}\alpha)^\beta$, which are given by

$$\frac{d^\alpha}{d(t-\alpha)^\alpha}[t-\alpha]^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}[t-\alpha]^{\beta-\alpha} \qquad (A\text{-}11)$$

For α = ½ this equation is called semi-derivative; for $(t\text{-}\alpha)^\beta$ it is called semi-integral.

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