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Stability of Nonautonomous Ordinary Differential Equation via the Limiting Equation

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Abstract: The purpose of the present research is to investigate uniform asymptotic stability via limiting equations. Throughout the study we always dead with the asymptotic stability of the origin. The generalization to compact sets is straightforward. We also discuss the stability properties only on the positive half line and therefore use only positive limiting equations. By introducing negative limiting equations the stability on the whole line can be handle as well.

Key words: Stability, nonautonomous ordinary differential equation, limiting equation, liapunov function, Lasalle's invariance principle, uniform asymptotic stability, asymptotic stability

INTRODUCTION

The construction of Liapunov function is a most powerful tool for the study of the stability of nonautonomous systems. One find, however, that in many cases it is very complicated to construct the appropriate Liapunov function. Lasalle's invariance principle is another powerful tool.

The principle states that, if the positive limit sets of an autonomous ordinary differential equation have an invariance property, then Liapunove functions can be used to obtain information on the location of positive limit sets. The invariance principle enables us to handle a variety of equations for which the geometry of the law of motion is detectable, and then relatively simple Liapunov functions are sufficient. Results of Artstein (1977a-c, 1978) on the limiting equations of nonautonomous ordinary differential equations and the consequent invariance properties of the positive Limit sets of solutions of a broad class of nonautonomous systems, make possible a further extension of Lasalle's ideas. A major role in the new invariance properties is played by the limiting equations the equations which describe the limiting behavior of the original nonautonomous law of motion.

PRELIMINARIES

The basic differential equation is

$$x' = f(t,x) \qquad (1)$$

where, the $x \in R^n$, the n-dimensional Euclidean space and $f: R_+ \times R^n \rightarrow R^n$, where $R+ = [0, +\infty)$. We always assume that f is continuous in x, is measurable in t and satisfies the Caratheodory conditions locally, i.e., for x in bounded sets with $|f(t, x)| \le h(t)$ with h locally integrable.

Definition 1: An ordinary differential equation $x' = g(t, x)$ is a limiting equation of (1), if there is a sequence $t_k \rightarrow \infty$ such that whenever $\varphi_k: [a, b] \rightarrow R^n$ is a sequence of continuous functions which converge uniformly to $\varphi_k: [a, b] \rightarrow R^n$ then

$$\int_a^b g(s,\varphi(s))ds = \lim_{k\rightarrow\infty}\int_a^b f(t_k + s, \varphi_k(s))ds \qquad (2)$$

We denote by f^T the translation of f by the time T, i.e., $f^T(s, x) = f(T + s, x)$. If (2) holds we say that f^{tk} converges to g; compare with Artstein (1978).

Definition 2: Equation 1 is positively precompact in the restricted sense if for any sequence $t_j \rightarrow \infty$, there exist a subsequence t_k and a limiting equation $x' = g(t, x)$ of (1) such that f^{tk} converge to g.

Definition 3: Equation 1 is regular if for every limiting equation $x' = g(t, x)$ and every pair (t_0, x_0) the initial-value problem $x' = g(t, x)$, $x(t_0) = x_0$ has a unique solution.

Lemma 1: (Artstein, 1978) Let $x' = f(t, x)$ be positively precompact in the restricted sense and regular. Suppose that 0 is uniformly stable with respect to (1). Then 0 is uniformly asymptotically stable with respect to (1) if and only if there exist a neighborhood w of 0 such that w is a region of attraction of 0 with respect to every limiting equation of (1).

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Note that Theorem B of Artstein (1978) is stated for merely positively precompact equations, for computional purpose it will be convenient to have the precompactness in the restricted sense.

Lemma 2: Suppose that for every compact $K \subset R^n$ the function f satisfies the following conditions:

- $|F(t, x) - f(t, y)| \le m_k(t)\ |x-y|$ whenever $x, y \in K$, where m_k s locally integrable and such that

$$\int_t^{t+1} m_k(s)ds \le M$$

for a fixed $M < \infty$ and all t,

- $|f(t, x)| \le b_k(t)$ whenever $x \in K$, where b_k is locall integrable and

$$B(t) = \int_{t_0}^{t} b_k(s)ds$$

is uniformly continuous in t.

Then $x' = f(t, x)$ is positively precompact in the restricted sense and regular.

Lemma 3: Under the assumptions of Lemma 2 the convergence to the limiting equations is a metric convergence. Also, f^{tk} converge to g if and only if for every fixed interval $[t_0, t_1]$ and every fixed x the sequence of function $f^{tk}(t, x)$: $[t_0, t_1] \to R^n$ converge in the weak L_1-topology to g (t, x). Alternatively, for every fixed x the function g (t, x) is the almost everywhere derivative of

$$\lim_{k\to\infty}\int_0^t f(t_k + s, x)ds\ \cdot$$

MAIN RESULTS

Consider second order differential equation of the form

$$x'' + h(t,x,x')x' + f(x) + g(t,x,x') = 0 \quad (3)$$

Assume that

(H_1) h(t, x,y) is a continuous non-negative function on $R_+ \times R^2$, where the continuity in x, y is uniform for t on R_+. Also h(t, 0, 0) is bounded for $t \ge 0$.

(H_2) for each compact set $K \subset R^2$ the function h satisfies

$$|h(t,x_1,y_1) - h(t,x_2,y_2)| \le m_k(t)(|x_1 - x_2| + |y_1 - y_2|)$$

whenever $x, y \in k$, where m_k is locally integrable and such that

$$\int_t^{t+1} m_k(s)ds \le M \text{ for a fixed } M < \infty \text{ and for all t.}$$

(H_3) f (x) is locally Lipschitzian in x, $xf(x) > 0$ for all $x \ne 0$.

(H_4) g (t, x, y) is continuous and $yg(t, x, y) \ge 0$ for all $t > 0$ x, y in R.

(H_5) g^{tk} converges to 0.

The equation of the form (3) is the subject of a number scientific papers. Here we present the most famous results. Primary the simplest equation of the form (3)

$$x'' + h(t)x' + x = 0 \quad (4)$$

have been considered. Where $h(t) \ge 0$. Under the condition $0 < h_1 \le h(t) \le h_2$,

where h_1, h_2 are constants the origin is globally asymptotically stable (Matrosov, 1962). If h(t) is not bounded above, then the rest point $x' = x = 0$ is not necessarily asymptotically stable.

For instance the equation

$$x'' + (2 + e^t)x' + x = 0$$

has the solution $x = a(1+e^{-1})$ that does not tend to 0 as $t \to \infty$ (Lasalle, 1968). The equation of the same form has been studied Artstein and Infanter (1975). It is pointed that if $h(t) \ge 0$ and that the integral of h is uniformly continuous (i.e.,

$$\int_a^b h(s)ds \le \mu(b - a)$$

where μ is continuous at 0 and $\mu(0) = 0$), then the equilibrium point x' = x= 0 of (4) is uniformly asymptotically stable if and only if

$$\lim_{t\to\infty, T\to\infty} \inf \int_t^{t+T} h(s)ds > 0$$

Artstein and Infanter (1975), Ballien and Peiffer (1978) and Karsai (1987, 1984) are devoted to the investigation of the stability for the nonlinear equation of the following form

$$x'' + h(t,x,x')x' + f(x) = 0 \quad (5)$$

In Artstein and Infanter (1975) the global asymptotic stability of $x' = x = 0$ is shown in conditions

$$\int_0^x f(\xi)d\xi \to \infty \text{ as } |x| \to \infty,$$

$$h(t,x,x') \geq w(x,x') \geq 0\left(w(x,x') > 0(x' \neq 0)\right)$$

and

$$\frac{1}{T^2}\int_0^T h\left(t,x(t),y(t)\right)dt \leq K, \forall\left(x(t),y(t)\right): R_+ \to R^2 \quad (6)$$

for some K and all T > 0.

In Ballien and Peiffer (1978) this result is true under the conditions

$$0 \leq h_1(x) \leq h(t,x,x') \leq h_2(t)h_3(x,x')$$

$$\int_{-v}^{v} h_1(x)dx = m(v) > 0, (\forall v > 0), \int_0^\infty h_2^{-1}(s)ds = \infty \quad (7)$$

here $h_2(t)$ is the nondecreasing function, $h_2(t) > 0$ and $h_3(x, x') \geq 0$.

We note that the results obtained in Artstein and Infanter (1975) and Ballien and Peiffer (1978) are independed. Indeed, the result of Ballien and Peiffer (1978) does not follow from artstein and Infanter (1975) by taking $h(t) = t \ln t$. On the other hand, one can easily choose h(t), satisfying (6) but not (7). In Karsai (1987, 1984) is carried out the subsequent research in detail for the stability of the equilibrium position for a system of the form (5). Now we investigate the stability problem of the equilibrium $x' = x = 0$ of (3) by limiting methods. Under assumptions $(H_1) - (H_5)$.

We observe that condition (H_1) implies that h(t, x,y) is bounded on $R_+ \times B$ for every compact set B of R^2 and furthermore under the condition $(H_1) - (H_3)$ it follows that the Eq. 5 is positively precompact in the restricted sense and regular (Artstein, 1978).

Consider the function

$$V = \frac{x'^2}{2} + \int_0^x f(u)du$$

as a Liapunov function. The derivative of V with respect to Eq. 5 has the estimate

$$V'(t,x,x') = x'x'' + f(x)x' = -h(t,x,x')x'^2 \leq 0.$$

Hence, the equilibrium point of (5) $x' = x = 0$ is uniformly stable.

Lemma 4: All limiting Equations of (5) have the same form, namely,

$$x'' + P(t,x,x')x' + f(x) = 0 \quad (8)$$

where p satisfies

$$\int_0^t P\left(s,x(s),x'(s)\right)ds = \lim_{k\to\infty}\int_0^t h\left(t_k + s,x(s),x'(s)\right)ds.$$

Prove. Consider the equivalent system of (5)

$$x' = y,$$

$$y' = -h(t,x,y)y - f(x) \quad (9)$$

Assume that the limiting equations of (9) have the form

$$x' = g_1(t,x,y),$$

$$y' = g_2(t,x,y),$$

then for every fixed interval $[t_0, t_1]$ if there is a sequence $t_k \to \infty$ such that whenever φ_k: $[t_0, t_1] \to \infty$ is a sequence of continuous functions which converge uniformly to $\varphi(x, y)$ then

$$\lim_{k\to\infty}\int_{t_0}^{t_1} y_k(s)ds = \int_{t_0}^{t_1} g_1\left(s,x(s),y(s)\right)ds,$$

and

$$\lim_{k\to\infty}\int_{t_0}^{t_1}\left[-h\left(t_k + s,x_k(s),y_k(s)\right)y_k(s) - f\left(x_k(s)\right)\right] ds = \int_{t_o}^{t_1} g_2\left(s,x(s),y(s)\right)ds$$

In particular, for the case $x_k(t) \equiv x$, $y_k(t) \equiv y$ we have

$$\int_{t_0}^{t_1} yds = \int_{t_0}^{t_1} g_1(s,x,y)ds \quad (10)$$

and

$$\lim_{k\to\infty}\int_{t_0}^{t_1}\left[-h\left(t_k + s,x,y\right)y - f(x)\right]ds = \int_{t_0}^{t_1} g_2(s,x,y)ds \quad (11)$$

Notice that (10) and (11) remain hold when t_1 replaced with t for $\forall t \in (t_0, t_1)$. Therefore, it follows from (10) that the function $g_1(t, x,y)$ is the almost-everywhere (a. e. for short) derivative of

$$\int_{t_0}^{t} y ds, \text{ i.e.,}$$

$$g_1(t,x,y) = y, \text{ a. e.}$$

Also, it follows form (11) that the function g_2 (t, x, y) converges in weak L_1-topology to

$$-h(t, x,y)y - f(x), \quad \text{i.e.,}$$

$$g_2(t, x, y) = -h(t, x, y)y - f(x), \quad \text{a. e.}$$

Set

$$P(t,x,y) = \begin{cases} 0 & ,y = 0 \\ -\dfrac{g_2(t,x,y) + f(x)}{y} & ,y \neq 0 \end{cases}$$

then

$$g_2(t,x,y) = -P(t,x,y)y - f(x),$$

and

$$\lim_{k\to\infty} \int_{t_0}^{t} h(t_k + s,x,y)ds = \int_{t_0}^{t} P(t,x,y)ds$$

From this we deduce that all the limiting equations of (9) have the same form

$$\begin{cases} x' = y \\ y' = -P(t,x,y)y - f(x) \end{cases} \tag{12}$$

It is to say all the limiting equations of (5) have the form of (8). This completes the proof.

Theorem 1: Let hypotheses $(H_1) - (H_3)$ hold.

Then the equilibrium position of (5) $x' = x = 0$ is uniformly asymptotically stable if and only if the equation

$$x'' + f(x) = 0 \tag{13}$$

is not a limiting equation of (5).

Proof: The only if part follow from Lemma 2.4 by the fact that no nontrivial solution of (13) approach 0.

For proving the converse assume that the uniform asymptotic stability does not hold, then by lemma 2.4 a solution φ of a limiting equation, say (8), or rather its equivalent system (12), does not converge to 0 as $t \to \infty$. Since $V(t) = V(\varphi(t))$ is nonincreasing it follows that $\varphi(t)$ approaches a set

$$\Gamma = \left\{ (x,y) : \frac{y^2}{2} + \int_0^x f(\xi)d\xi = c \right\}$$

as $t \to \infty$ and $c > 0$ is constant. By the invariance principle stated by Artstein (1978), there exist a limiting equation of (12) and a solution ψ of the equation such that $\psi(t) \in \Gamma$ for every $t \geq 0$. This limiting equation is also a limiting equation of (9). The only possibility that such a function ψ is a solution of an equation of the form (12) is that $P(t, x, y) = 0$ for almost every t.

Therefore the limiting equation guaranteed by the invariance is $x' = y$, $y' = -f(x)$, or rather its equivalent equation $x'' + f(x) = 0$.

This completes the proof. We shall now state the necessary and sufficient conditions of the previous result in terms of coefficients h(t, x, y).

Theorem 2: Let hypotheses $(H_1) - (H_3)$ hold.

The equilibrium position $x' = x = 0$ of (5) is uniformly asymptotically stable if only if

$$\lim_{t\to\infty, T\to\infty} \inf \int_t^{t+T} h(s,x(s),y(s))ds > 0 \tag{14}$$

Proof: Equation 13 is a limiting equation of (5) if and only if there is a sequence $t_k \to \infty$ such that

$$\int_{t_k}^{t_k+T} h(s,x(s),y(s))ds \to 0$$

as $k \to \infty$ for every T and therefore if and only if (14) does not hold.

The result follows now from Theorem 1. Moreover it is pointed that this result is true for Eq. 3 under hypotheses $(H_1) - (H_5)$.

Let

$$V = \frac{x'^2}{2} + \int_0^x f(\xi)d\xi \cdot$$

The derivative of V with respect to Eq. 3 has the estimate

$$V' = -h(t,x,x')x'^2 - g(t,x,x')x' \leq 0$$

Therefore we easily get that the equilibrium point of (3) $x' = x = 0$ is uniform stable. Also, we compare Eq. 5 and 3 when the condition (H_5) hold. Clearly from the definition of limiting equations it follows that (5) and (3) share the same family of limiting equations and in particular (5) is positively precompact in the restricted sense or regular if and only if (3) is positively precompact in the restricted sense or regular, respectively. From the characterization of uniform asymptotic stability above we can deduce the following.

Theorem 3: Under the hypotheses (H_1) – (H_5) the equilibrium position of (3) $x' = x = 0$ is uniformly asymptotically stable if and only if the condition (14) hold.

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