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HAN New Plate Bending Macro-Element

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Abstract: Macro-elements are one of the powerful means in reducing number of equations to be solved in finite element analysis. This is because a single macro-element will represent many finite elements. In this study a rectangular plate bending macro-element was developed. This macro-element is based on equivalent energy theory. The developed macro-element was tested and the results were compared with the results of the conventional plate bending finite element solutions. Excellent results were achieved with substantional reduction in number of equations required for the solutions. This reduction in number of equations will save computer storage and time especially for large structures.

Key words: Rectangular finite element, macro-element

INTRODUCTION

The analysis of large structural systems using the conventional finite element method is impractical. This is because of the necessity to use relatively fine mesh to obtain an accurate model. This will lead to a large number of equations to be solved. Therefore, it is advantageous to seek for approaches that reduce the total number of degrees of freedom needed to successfully model large systems. One of these methods is to use macro-elements.

In this study a new rectangular plate bending macro-element were developed and called Hamid, Armanios and Negm (HAN) macro-element.

This macro-element is based on transformation of many structural finite elements into single equivalent macro-element. This is done by preserving the same potential energies of the structure modeled by finite elements and the same structure modeled by macro-elements.

The macro-element is based on the improved rectangular finite element of Armanios and Negm (AN) (1983).

The A.N. finite element is an interesting one this because the variation of the normal deflection over the element in terms of the various nodal displacements is expressed in simple parametric form scanning the space between Adini-Clough-Melosh (ACM) finite element and the Papenfuss finite element. That is, the parametric shape functions are chosen in such a way as to create a family of elements with varying ability to produce basic bending modes and satisfy inter element compatibility requirements.

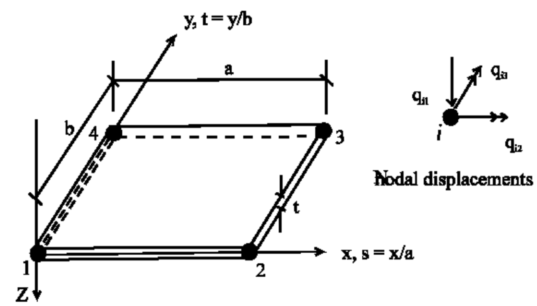

Fig. 1: The Armanios and Negm rectangular finite element

PLATE BENDING FINITE ELMENT USED IN THE FORMULATION OF THE MACRO-ELEMENT

The A.N. finite element has four nodes with three degree of freedom per node (Fig. 1). It is Kirchhoff type plate bending finite element.

The displacement vector is:

$$\lfloor q_i \rfloor = [\ W_i\text{-}W_{i,y}\ \ W_{i,x}\ ]$$

Where,
I = 1, 2, 3, 4

The nodal forces vector corresponding to the displacement vector is

$$\lfloor F_i \rfloor = [\ P_{zi}\ M_{xi}\ W_{yi}\ ]$$

FORMULATION OF MACRO-ELEMENTS

The stiffness matrix of a macro-element is formulated by equating the strain energy of the original structure modeled by finite-elements and that of the equivalent macro-element model as follows:

$$U_o = U_m \qquad (1)$$

Where,

U_o = The strain energy of the original structure modeled by many finite elements that constitute one macro-element.

U_m = The strain energy of the macro-element.

$$\tfrac{1}{2} \lfloor q_o \rfloor [Sk_o] \{ q_o \} = \tfrac{1}{2} \lfloor q_m \rfloor [K_m] \{ q_m \} \qquad (2)$$

Where,

q_o = Displacement vector of the structure modeled by many finite elements that constitute one macro-element.

q_m = Displacement vector of one macro-element.

$[Sk_o]$ = The assembled stiffness matrix of all stiffness matrices of the finite elements constituting one macro-element.

$[K_m]$ = The stiffness matrix of the macro-element.

Let the displacement vector of the original structure, (which constitute one macro-element) $\{q_o\}$ be related to that of the macro-element $\{q_m\}$ as:

$$\{q_o\} = [T] \{q_m\} \qquad (3)$$

Where: [T] is the transformation matrix for the macro-element. Substituting Eq. 3 into Eq. 2 gives:

$$\lfloor q_m \rfloor [T]^T [SKo] [T] \{q_m\} = \lfloor q_m \rfloor [K_m] \{q_m\}$$
$$[T]^T [SK_o] [T] = [K_m] \qquad (4)$$

In the solution, matrix $[SK_o]$ is not needed, only $[K_o]$, the stiffness matrix of a single finite element bounded by the macro-element is needed. To explain this let.

n = The number of finite elements comprising the macro-element.

$[T_e]$ = The finite-element transformation matrix.

Every time $[T_e]$ carries a partition of the transformation matrix [T] that corresponds to the degrees of freedom of the finite-element under consideration. The transformed stiffness matrix for each finite-element is placed in its proper place in the structural stiffness matrix of the equivalent model, which is the place of $[K_m]$, as:

$$\sum_{e=1}^{n} [T_e]^T [K_o] [T_e] = [K_m] \qquad (5)$$

The transformation matrix [T] is simply the evaluation of the shape functions of the macro-element at the nodes of the finite-element. This evaluation is based on local coordinates for the nodal points of the finite-elements with respect to the macro-element nodes.

To form a general transformation matrix $[T_i]$ corresponding to an arbitrary nodal point i of a certain finite element within a certain macro-element, consider the notation N_{kI} which means that shape function k of node I of this macro-element is evaluated at point i using its local coordinates within the macro-element.

The transformation matrix For the Armanios and Negm rectangular element will be as :

$$\begin{Bmatrix} q_{i1} \\ q_{i2} \\ q_{i3} \end{Bmatrix} = [T_i]_{3*12} \quad \{q_m\}_{12*1} \qquad (6)$$
$$[\; Ti \;] = [T_{i1} \quad T_{i2} \quad T_{i3} \quad T_{i4}]$$

Where,

T_{iI} = The participation of (T_I) that correspond to node I of the macro-element under consideration. Where: I = 1, 2, 3 and 4 and:

$$[T_{iI}] = \begin{bmatrix} N_{1I} & N_{2I} & N_{3I} \\ -N_{1I},y & -N_{2I},y & -N_{3I},y \\ N_{1I},x & N_{2I},x & N_{3I},x \end{bmatrix}$$

The shape function derivatives N_{kI},y and N_{kI}, x (k = 1, 2, 3) are evaluated using the chain rule.

The shape functions (N_i) and the stiffness matrix of this element are stated in (Armanios and Negm, 1983).

Macro-element load vector: The externals loading are applied at known nodes of the finite element model. However, these nodes may not necessarily coincide with the macro-elements nodes. It is required to calculate the equivalent consistent nodal load vector of each macro-element.

In general, all forms of loading other than concentrated loads subjected to the original structure nodes must be first reduced to equivalent nodal forces acting on the original structure, as with the conventional finite element method. The nodal load vector of the original structure can then be transformed to equivalent macro element structural load vector by equating the external work done on the original structure modeled by finite-elements and that of the macro-element model as follows:

$$W_o = W_m \quad (7)$$

Where,

W_o = The external study done on the original structure that constitute one macro-element.

W_m = The external study done on the macro-element.

$$\lfloor q_o \rfloor \{F_o\} = \lfloor q_m \rfloor \{F_m\} \quad (8)$$

Where,

$\{F_o\}$ = The assembled nodal load vector of the finite-elements constituting one macro-element.

$\{F_m\}$ = The equivalent nodal load vector of the macro-element.

Substituting Eq. 3 into Eq. 8 gives:

$$\lfloor q_m \rfloor [T]^T \{F_o\} = \lfloor q_m \rfloor \{F_m\}$$
$$[T]^T \{F_o\} = \{F_m\} \quad (9)$$

Where, [T] is the same transformation matrix used in deriving $[K_m]$.

The assembly of all the macro-element stiffness matrices into a structural stiffness matrix and also the construction of the macro-element structural load vector and solution of the structure equation are the same as that of conventional finite element method.

APPLICATIONS

Various problems of plate bending analysis are solved and presented below in order to demonstrate the efficiency of the macro-elements developed.

The accuracy of the macro-elements are checked by using the conventional finite elements method and, if available, the exact solution.

The moments and stresses are generally calculated at the Gauss points of the macro-elements in the problems presented here unless it is stated differently.

Problem No. 1: The analysis of thin rectangular orthotropic plate simply supported along two opposite sides and free along the others and under two concentrated loads as shown in Fig. 2.

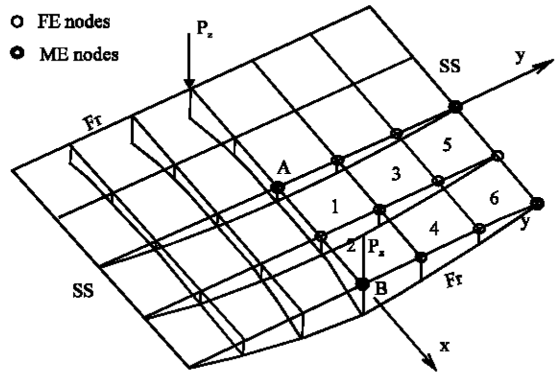

Fig. 2: The plate for problem No. 1

Table 1: Matrices D in (MN.m) for the finite elements of plate for problem No. 1

FE	D11	D12	D13	D22	D23	D33
1 and 2	5.0	0.4	0.0	1.0	0.0	0.80
3 and 4	4.0	0.3	0.0	1.0	0.0	0.65
5 and 6	3.0	0.2	0.0	1.0	0.0	0.50

The following data given for this problem:

Plate dimension = 4*6 m
FE Dimension = 1*1 m
P_z = 0.01 M N

The basic finite elements in the mesh have different orthotropies as shown in Table 1. Due to symmetry, only one quarter of the plate is analyzed. The analysis is done using a special case of the Armanios and Negm element. Table 2 shows that the total number of degree of freedom is reduced to (33.33%) with the macro-element. In the formulation of the elements stiffness matrices and calculation for stresses a (3*3) Gaussian integration order is used. The results for deflections along x- and y-axis are shown in Fig. 3 and 4. Table 3 shows the errors for deflection at points A and B Fig. 2. These errors are measured from the (2*3) conventional FE analyses.

The results for moments M_x and M_y are shown in Fig. 5 and 6. These moments are at the nearest line of Gauss points of the original 2*3 FE modeling beside the x- and y-axis, respectively, i.e., section A-A and B-B as shown in Fig. 7. Extrapolations for moments are done where is required.

Figure 5 and 6 represent two types of stress calculations for the macro-element analysis. The ME_1 represents the moments at the Gauss points of the sub-elements forming the ME Fig. 7. The ME_2 is achieved after calculating the nodal displacements of the sub-elements from the ME nodal displacement vector and using the transformation matrix (T).

Table 2: Details for problem No. 1

Mesh	No. of nodes	Total degree of freedom
2*3 conventional FE	12	36
1*1 equivalent ME	4	12
1*3 conventional FE	8	24

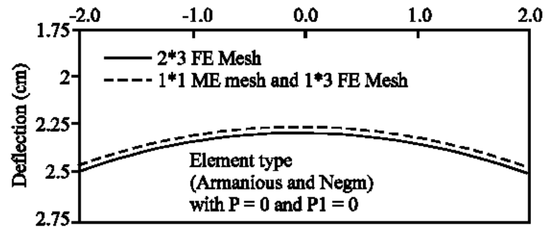

Fig. 3: X-axis deflection for problem No. 1

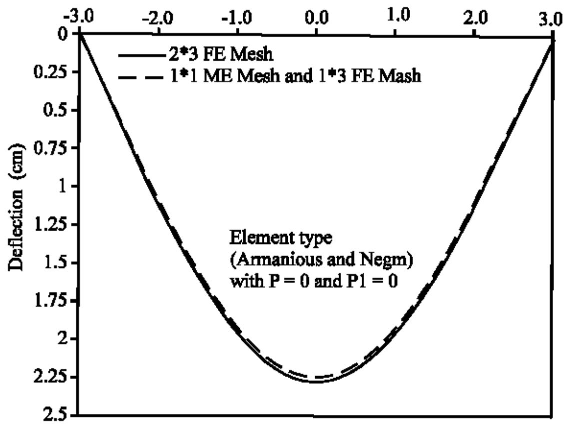

Fig. 4 : Y-axis deflection for problem No. 1

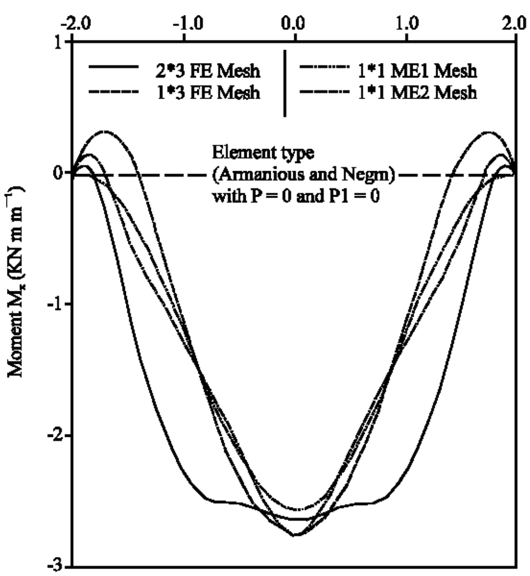

Fig. 5: M_x-Diagram along section A-A for problem No. 1

Table 3: Deflections and their corresponding errors at selected points on the plate of problem No. 1

Mesh	Deflection at point A (cm)	Error (%)	Deflection at point B (cm)	Error (%)
2*3 conventional FE	2.275906	-	2.489945	-
1*1 equivalent ME	2.248874	1.19	2.447316	1.71
1*3 conventional FE	2.248826	1.19	2.443514	1.86

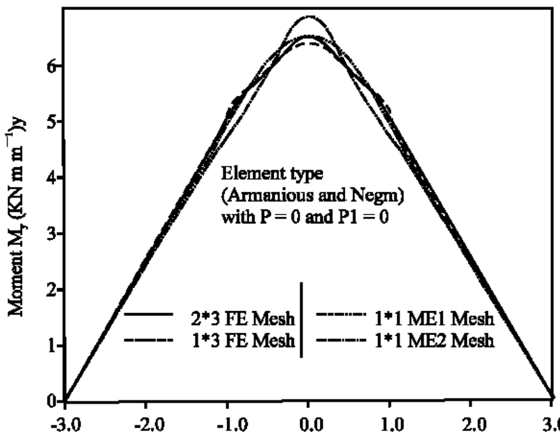

Fig. 6: M_y-diagram along section B-B for problem No. 1

Problem No. 2: The analysis of thin square clamped isotropic plate with sloped bottom surface under a uniformly distributed load as shown in Fig. 8.

The following data are given to this problem :

L = 7.2 m.

t = Variable from 0.105 m at center of plate to 0.225 m at the edges of the plate, i.e., bottom slope is 3.33% in x or y direction.

E = $25 * 10^6$ KN m^{-2}.

G_{xy} = $10.5932 * 10^6$ KN m^{-2}.

μ (nu) = 0.18

Q_z = 9.126 KN m^{-2}.

Due to symmetry only one quarter of the plate is analyzed.

The analysis is done using the Armanios and Negm element in its optimum case for isotropic plate i.e., P = 1 and P1 = 1.

Figure 9 and 10 shows the central deflection values and central moments (M_x or M_y) values obtained from different conventional finite element meshes. The corresponding percentages of error are also shown in the Fig. 9 and 10.

These errors are measured from the (12 * 12) conventional FE analysis of the problem.

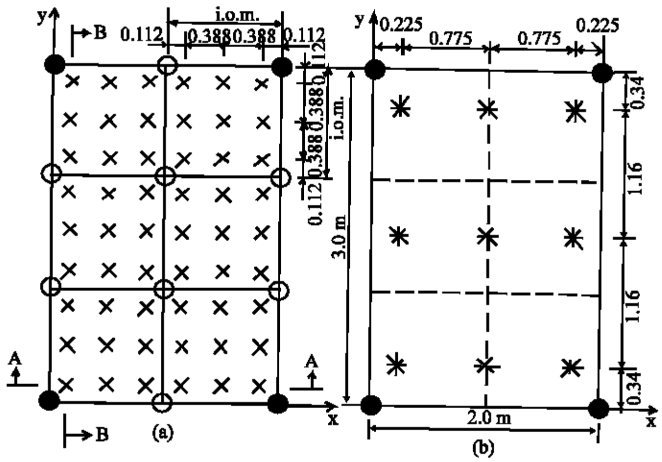

Fig. 7: Gauss points locations for problem No. 1 of the original FE X of the ME

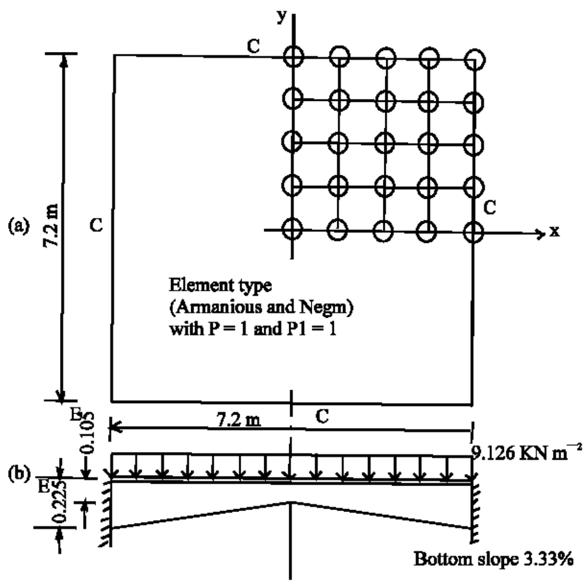

Fig. 8: The plate for problem No. 2

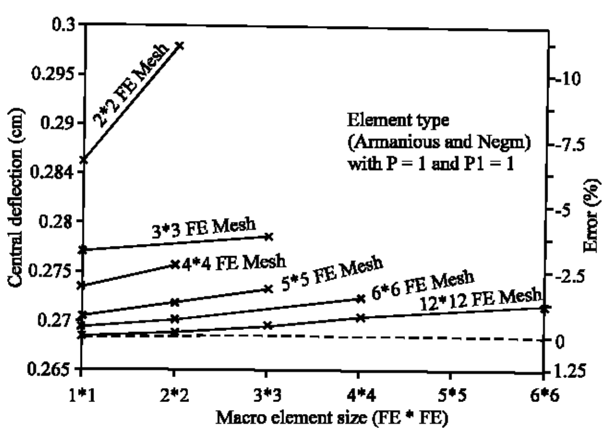

Fig. 9: Central deflection of plate for problem No. 2

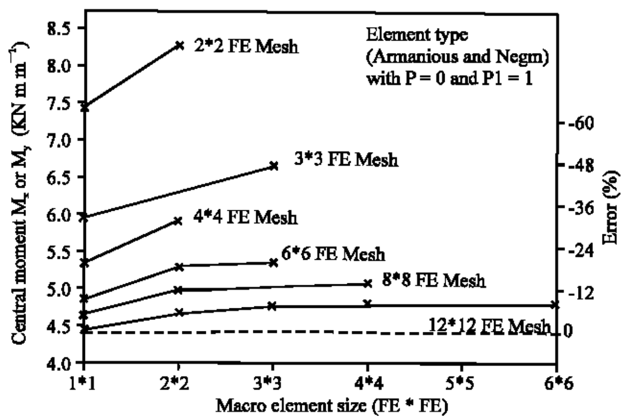

Fig. 10: Central moment M_x or M_y for problem No. 2

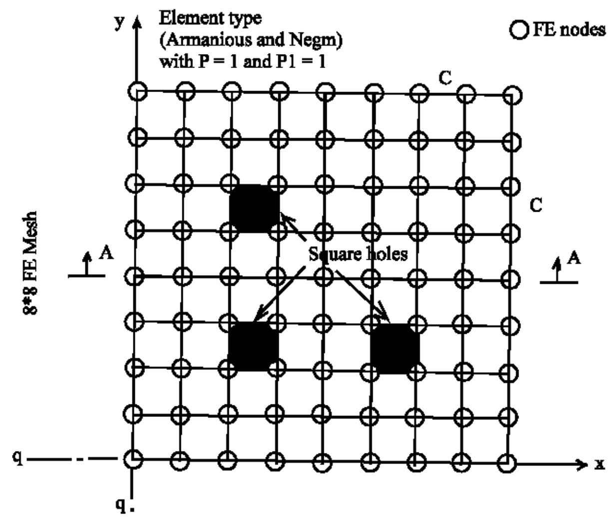

Fig. 11: Quarter of plate for problem No. 3

Problem No. 3: The analysis of thin square clamped isotropic plate with square holes under a central concentrated load as shown in Fig. 11.

The following data are given to this problem:

L, E G_{xy} and μ are as before in problem-2.
t = 0.18 m.
P_z = 242.223 KN.

Due to symmetry only one quarter of plate is analyzed.

The quarter of the plate has three square holes as shown in Fig. 11. Each hole is represented by a conventional FE which will be a sub-element inside a ME when analysis is done with the equivalent energy theory. Such FE is defined by assuming its thickness equal to zero. Stress concentration beside these holes are not considered in the analysis, which is done using the Armanios and Negm element in its optimum case for isotropic plates, i.e., P = 1 and P1 = 1

Table 4: Details for problem No. 3

Mesh	ME size (FE*FE)	No. of nodes	Total degree of freedom	Reduction in degree of freedom (%)
8*8 conventional FE	-	81	243	-
4*4 equivalent ME	2*2	25	75	69.136
2*2 equivalent ME	4*4	9	27	88.889
1*1 equivalent ME	8*8	4	12	95.062

Table 4 shows the details of the original FE model and the equivalent ME models.

Figure 11-14 shows the deflection along the x-axis and section A-A, respectively. The maximum errors are at the center of the plate, as shown in Table 5.

These errors are measured for the (8*8) conventional FE analysis.

Table 5: Central deflections and their corresponding errors for problem No. 3

Mesh	Central deflection (mm)	Error (%)
8*8 conventional FE	5.884508	-
4*4 ME	5.855182	0.50
2*2 ME	5.841002	0.74
1*1 ME	5.504848	6.45

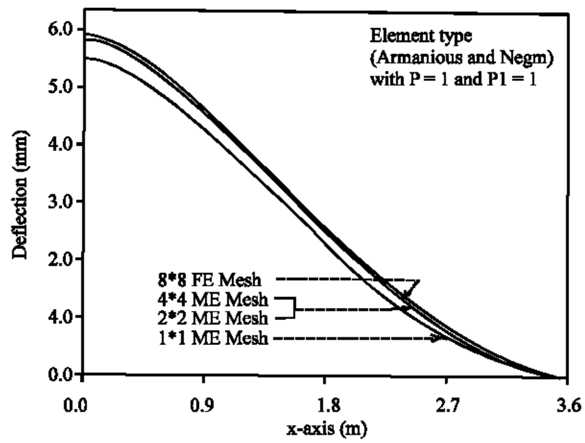

Fig. 12: x-axis deflection for problem No. 3

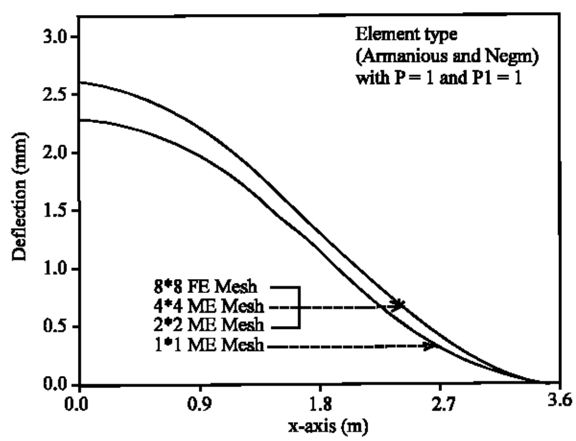

Fig. 13: Deflection along section A-A for problem No. 3

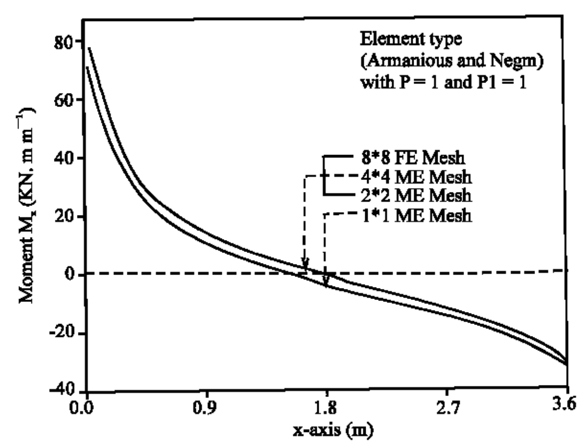

Fig. 14: Moment M_x along x-axis for problem No. 3

RESULTS AND DISCUSSION

The three solved problems showed that using the macro-elements in the analysis reduced the number of equations to be solved. When the size of the macro-element used is of moderate, excellent results are achieved with good amount of reduction in degree of freedom and computer time.

But when the size of the macro-element is large still acceptable results are achieved with substantial reductions in degree of freedom as shown in Table 3-5.

CONCLUSIONS

New rectangular plate bending macro-element based on Armaios and Negm rectangular finite element were developed.

The solved examples demonstrated that using these macro-elements in the analysis largely reduced the total number of degree of freedom required to model a certain structure. This in turn reduced the total number of equations to be solved.

Reduction in total number of equations reduced computer time and memory space for storage. This will allow personal computers to analyze relatively large structures.

At the same time these ME provided accurate results. In addition, finite elements of different sizes, thicknesses and material properties can easily be used inside the macro-elements if required in the analysis.

NOTATIONS

The following symbols are used in this study:

D	: Flexural rigidity.
E_x, E_y	: Moduli of elasticity along x and y direction of the plate, respectively.
FE	: Finite element.
{F}	: Element nodal load vector.
Fr	: Free edge of plate.
G_{xy}, G_{xz} and G_{yz}	: Shear moduli in the z, y and x planes, respectively.
[K]	: The stiffness matrix.
L	: Side length of a square plate.
m	: A subscript refers to the macro-element structure.
ME	: Macro-element
{M}	: The vector of generalized stresses at a point.
Nu, Nu_{xy}, Nu_{yx}	: Poisson's rations.
{N}	: Vector of shape functions.
o	: A subscript refers to the original (finite element) structure.

P_z : Concentrated force applied on the plate in the z direction.
P_{zi} : Concentrated force at node i of an element, in the z direction.
{q} : Element nodal displacement vector.
Q_z : Uniformly distributed load applied on the plate in the z direction (force per unite area).
S, T : Local coordinates of a point in the x and y directions, respectively.
S_i T_i : Local coordinates of node i of an element, in the x and y directions, respectively.
SS : Simply supported edge of plate.
t : Thickness of plate.
[T] : The transformation matrix needed in macro-element construction.
w_i : Vertical displacement at node i of an element, in the z direction.
x, y, z : Global coordinates.
x and y : First derivatives of certain function with respect to x and y, respectively.

REFERENCES

Armanios, E.A. and H.M. Negm, 1983. An improved rectangular element for plate bending analysis. Computers and Structures, 16: 677-686.