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An Effective Method for Exact Reliability Analysis

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Abstract: In this study, an effective method, we called PTM-FEM, is proposed to evaluate the probability of failure in analytical form, instead of approximation methods like FORM/SORM, no series expansion is involved in this expression. This method is based on the Finite Element Method (FEM) to get the expression of the response of stochastic systems then to linearize step-by-step this expression and finally to apply the probabilistic transformation method (PTM) of random variables to obtain the probability density function of the response. After that, the computation of the failure probability is straightforward. An application of 25-bar truss structure is provided to illustrate this method.

Key words: Probability of failure, probabilistic transformation method, reliability analysis, FORM, SORM

INTRODUCTION

The problem of reliability analysis of stochastic mechanical systems is of central importance in the safety assessment of structures. In a stochastic system, a large number of random variables influences the performance of the system, e.g., Young's modulus, external loads … The performance of the system is evaluated by a best-estimate code. Consider a performance criterion Y of the system depending on the input variables $X_1, X_2,\ldots,X_n$. the function $Y = g(X_1, X_2,\ldots,X_n)$ is a random variable to be determined.

In order to get the information about the uncertainty of Y, a number of FE runs have to be performed. For each of these runs, all identified uncertain parameters are varied simultaneously.

According to the exploitation of the result of these studies, the uncertainty on the response can be evaluated either in the form of an uncertainty range, or in the form of a probability density function (pdf).

UNCERTAINTY RANGE

A two-sided confidence interval [m, M] of a response Y, for a fractile α and a confidence level β is given by:

$$P\{P(m \le Y \le M) \ge \alpha\} \ge \beta$$

Such a relation means that one can affirm, with at the most $(1-\beta)$ percent of chances of error, that at least α percents values of the response Y lie between the values m and M (Glaeser, 2000). To calculate the limits m and M, the technique usually used is a method of random simulation combined with the formula of Wilks (1941).

The advantage of using this technique is that the number of code calculation needed is independent of the number of uncertain parameters. However for reliability evaluation, this method is not very useful because it is difficult, indeed impossible to interpret the two levels of probability (α and β) in term of reliability value for the system.

PROBABILITY DENSITY FUNCTION

The uncertainty evaluation in the form of a pdf gives richer information than a confidence interval. Once the pdf of the system response is determined, the reliability can be directly obtained for a given failure criterion. However, the determination of this distribution can be expensive in computing time. The following paragraphs describe the various methods available for this evaluation.

Method of monte-carlo: The method of Monte-Carlo (Rubinstein, 1981; Devictor, 1996) is used to build pdf, as well as to assess the reliability of components or structures or to evaluate the sensitivity of parameters. Monte Carlo simulation consists of drawing samples of the basic variables according to their probabilistic characteristics and then feeding them into the performance function. In this way, a sample of response $\{Y_j, j = 1,..,N\}$ is obtained.

The main advantage of the Monte-Carlo method is that this method is valid for static, but also for dynamic models and for probabilistic model with continuous or discrete variables. The main drawback of this method is that it requires often a large number of calculations and can be prohibitive when each calculation involves a long and onerous computer time.

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Response surface method: To avoid the problem of long computer time in the method of Monte-Carlo, it can be interesting to build an approximate mathematical model called response surface (Rajashekhar *et al*., 1993).

Experiments are conducted with design variables $X_1, X_2,\ldots,X_n$ a sufficient number of times to define the response surface to the level of accuracy desired. Each experiment can be represented by a point with coordinates $x_{1j}, x_{2j},\ldots,x_{nj}$ in an n-dimensional space. At each point, a value of y_i is calculated. The basic response procedure is to approximate by a simple mathematical model, such as an n^{th} order polynomial with undetermined coefficients.

When a response surface has been determined, the system reliability can be easily assessed with Monte Carlo simulation, in using the approximate mathematical model, but this response surface must be qualified. The practical problems encountered by the use of the response surface method are in the analysis of strongly non-linear phenomena where it is not obvious to find a family of adequate functions and in the analysis of discontinuous phenomena.

FORM/SORM: We present now specific methods usable for a direct evaluation of the reliability, without the need of defining the pdf of the system performance.

The performance function of a stochastic system according to a specified mission is given by:

M = performance criterion-given criterion limit = $g(X_1, X_2,\ldots,X_n)$ in which the X_i ($i = 1,\ldots,n$) are the n basic random variables (input parameters), with g(.) being now the functional relationship between the random variables and the failure of the system. The performance function can be defined such that the limit state, or failure surface, is given by M = 0. The failure event is defined as the space where M<0 and the success event is defined as the space where M>0. Thus a probability of failure can be evaluated by the following integral:

$$P_f = \iint\ldots\int f_X(x_1,\ldots,x_n)dx_1\ldots dx_n \quad (1)$$

where f_X is the joint density function of $x_1\ x_2,\ldots,\ x_n$ and the integration is performed over the region where M<0. Because each of the basic random variables has a unique distribution and they interact, the integral (1) cannot be easily evaluated. Two types of methods can be used to estimate the probability of failure: The Monte Carlo simulation and the approximate methods (FORM/SORM).

Direct monte carlo: Simulation techniques can be used to estimate the probability of failure defined in Eq. (1) (or its complement to 1, the reliability). Monte Carlo simulation (Fig. 1) consists of drawing samples of the basic variables

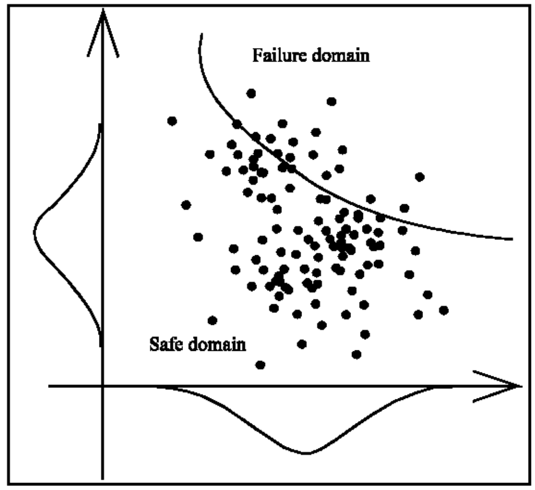


Fig. 1: Reliability assessment by Monte-carlo simulation

according to their probabilistic characteristics and then feeding then into the performance function. An estimate of the probability of failure P_f (Sundararajan 1995) can be found by:

$$\overline{P_f} = \frac{N_f}{N}$$

where N_f is the number of simulation cycles in which g(.) <0 and N the total number of simulation cycles. As N approaches infinity, P_f approaches the true probability of failure.

The first and second-order reliability methods (FORM/SORM) consist of 3 steps (Fig. 2):

- The transformation of the space of the basic random variables $X_1, X_2,\ldots,X_n$ into a space of standard normal variables.
- The research, in this transformed space, of the point of minimum distance from the origin on the limit state surface (this point is called the design point).
- An approximation of the failure surface near the design point:

FORM (First Order Reliability Method) consists in approaching the surface of failure by a hyper plane tangent to the failure surface at the design point (Madsen *et al*., 1986). Then an estimate of the failure probability is obtained by:

$$P_f = \Phi(-\beta)$$

where Φ is the cumulative Gaussian distribution of the standard normal law. β the reliability index according to Hasofer and Lind. The precision of this approximation depends on the non-linearity of the failure surface.

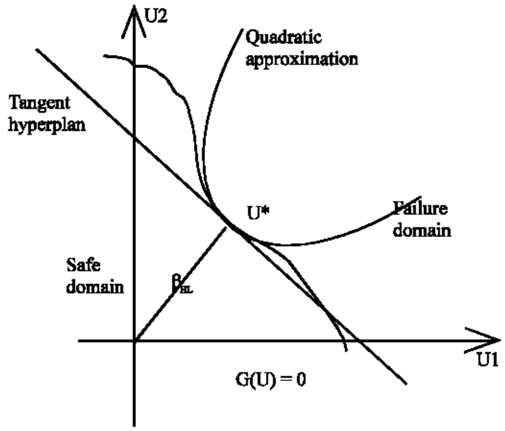


Fig. 2: Reliability assessment with FORM/SORM methods

If the linear approximation is not satisfactory, more precise evaluations can be obtained from approximations to higher orders of the failure surface at the design point. The approximation by a quadratic surface at the design point is called the SORM method (Second Order Reliability Method) (Melchers, 1999). The corresponding formula uses the knowledge of the (q-1) principal curves κ_i of the failure surface at the design point:

$$P_f \approx \Phi(-\beta_{HL}) \prod_{i=1}^{N-1} \frac{1}{\sqrt{(1+\beta_{HI}\kappa_i)}}$$

This result is known as asymptotically exact, in the sense that the approximation of the failure probability obtained is better for large reliability indexes. The computing time is influenced by the calculation of the matrix of the second-order derivatives.

The FORM and SORM methods are approximate methods, but their accuracy is generally good for small probabilities. The analytical properties enable the method to yield relatively inexpensive sensitivity factors. However, the basic random variables and the failure function must be continuous. For small order probabilities, FORM/SORM is extremely efficient, compared to simulation methods.

PTM-FEM TECHNIQUE

The Probabilistic Transformation Methods (PTM) evaluate the Probability Density Function (pdf) of a function by multiplying the input pdf by the Jacobean of the inverse function. The idea of PTM is based on the following formula (Hogg and Craig, 1978).

$$f_U(u) = f_P(p) \cdot |J_{p,U}| = f_P(p) \cdot \left| \frac{\partial \varphi^{-1}(u)}{\partial u} \right|$$

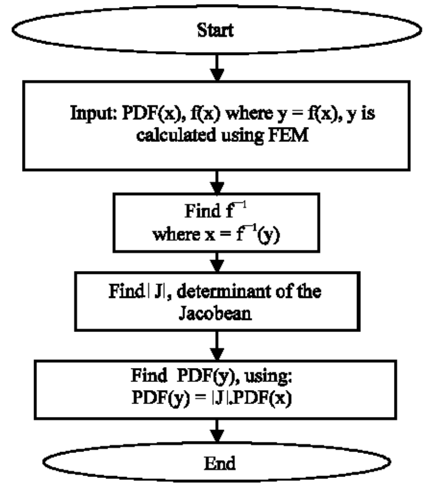


Fig. 3: Algorithm of PTM-FEM

Where p is the input parameter, u is the response (solution) and $\varphi^{-1}(u)$ is the inverse transformation, which can be determined either analytically or numerically.

The PTM-FEM technique, introduced recently by the authors (Kadry *et al*., 2006; Kadry and Chateauneuf, 2006), is a combination between the finite element method and the probabilistic transformation method. The steps of the PTM-FEM technique are as following (Fig. 3): Using FEM to get the stiffness matrix and the load vector. Once we have the relation function between the input and the output, the inverse function is required to calculate the determinant of the Jacobian transformation. Finally the pdf of the output is equal the determinant of the Jacobian multiplied by the pdf of the input.

The advantage of the PTM-FEM technique in the context of reliability analysis is the evaluation of the pdf, of the response in a closed-form as opposed of other numerical methods which give only the first and second moment of the response under some conditions.

Nonlinear stiffness terms: In the practice, the problem of nonlinearity in the structural stiffness matrix has been always faced; The difficulty arises when the nonlinear element is random. The proposed method in this situation is to substitute the element of stiffness matrix by using the Variable Changes Technique which in turn allows us to apply the PTM-FEM. The general term of stiffness matrix can be expressed by:

$$k_{ij} = AB^n \left(\sum_i C_i^{r_i} \right)^m$$

Let us consider the two principal cases (the other cases, for example, for the sign m are similar):

$r_i > 0$

In this case, the nonlinearity present in the numerator of the element of stiffness matrix, i.e., $C_i^{r_i}$ is random and $r_i>1$. It requires linearizing the numerator, to do that we suppose $\delta_i = C_i^{r_i}$ then by applying the PTM technique the pdf of δ_i,

$$f_{\delta_i}(\delta_i) = \frac{1}{r_i}\sqrt[r_i]{\delta_i^{1-r_i}}\, f_{C_i}(C_i)\ .$$

Again, we apply the PTM technique to get the pdf of $\gamma = \sum_i \delta_i$. Finally, we use the same procedure to find the pdf of γ^m.

$\underline{r_i \leq 0}$

In this case, the nonlinearity appears in the denominator of the Stiffness Matrix Term, i.e., $C_i^{r_i}$ is random and $r_i<1$. It requires linearizing the denominator, to do that we suppose $\delta_i = 1 / C_i^{r_i}$ then we apply the PTM technique to get the pdf of

$$\delta_i, f_{\delta_i}(\delta_i) = -\frac{1}{\delta^2 r_i}\sqrt[r_i]{\delta_i^{-1+r_i}}\, f_{C_i}(C_i)$$

again we apply the PTM technique to get the pdf of $\gamma = \sum_i \delta_i$. Finally, we use the same procedure to find the pdf of γ^m.

Illustrative example: If the Stiffness Matrix Term has the following form of

$$k_i = \frac{E_i I_i}{\Sigma_i \frac{1}{\ell_i}}$$

and the random variables are the lengths (l_i) of the analyzed structure, we then apply the same technique, suppose for example

$$M_i = \frac{1}{\ell_i}$$

and apply PTM to find the pdf of M_i, again we suppose $L = \Sigma_i M_i$ and find the pdf of L and finally we suppose

$$T = \frac{1}{L}$$

and then we evaluate its pdf. In this case the term of stiffness becomes linear in terms of the random variable $k_i = E_i I_i T$ then we can easily apply the PTM to find the pdf of k_i.

Application: In this application, we analyze the reliability of a twenty five-bar truss structure (Fig. 4) with random parameters (Young's modulus E, cross section S or and the horizontal load q).

The virtual work method allows us to calculate the displacement at a point using the following formula:

$$u = \sum_{i=1}^{n} \frac{N_i \overline{N_i}}{ES_i} L_i$$

where N_i is the normal effort due to the external load, $\overline{N_i}$ is the normal effort due to a unit load to the point and in the direction of searching displacement, E is the Young's modulus, S_i and L_i are respectively the section and the length of the bar i.

By symmetry, the cross sections of some bars are identical. We adopt the following distribution as shown in Table 1.

Table 2 represents the normal efforts on the bars.

The horizontal displacement u_{y2} at the node 2 is given by:

$$u_{y2} = \frac{q}{180000E}\left(\frac{0}{S_1} + \frac{7850940221}{S_2} + \frac{4285580903}{S_3} + \frac{73766125.44}{S_4} + \frac{1.464\,698\,375\times 10^{10}}{S_5} + \frac{6428099237}{S_6} \right)$$

with E the Young's modulus, q the Horizontal Load and S_i the cross-section of bar i.

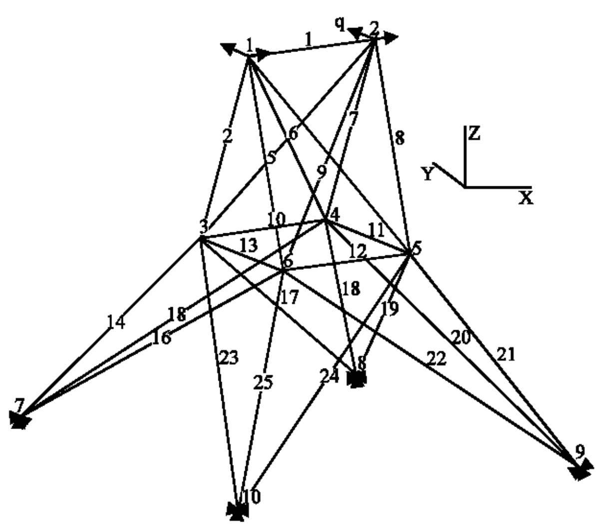


Fig. 4: 25-Bar truss structure

Table 1: Cross sections of bars

Bar	Section
1	S_1
2,5,7,8	S_2
3,4,6,9	S_3
10,11,12,13	S_4
14,18,21,25	S_5
15, 16, 17, 19, 20, 22, 23, 24	S_6

Table 2: Normal efforts on the bars

Bar	Vertical load	Horizontal load	Fx1 = 1	Fy1 = 1	Fz1 = 1	Fx2 = 1	Fy2 = 1	Fz2 = 1	Length Li
1	118496	1.57898E-10	-0.44778	3.7646E-17	-0.116842	0.44778	3.7646E-17	-0.116842	18000
2	-182632	-108058	0.39318	-0.88916	0.4508	0.31882	-0.04387	-0.08319	25632
3	-103094	-181168	-0.48044	-0.65356	0.101654	-0.38958	0.053606	0.101654	31321
4	-103094	24564	-0.48044	0.65356	0.101654	-0.38958	-0.053606	0.101654	31321
5	-182632	236220	0.39318	0.88916	0.4508	0.31882	0.04387	-0.08319	25632
6	-103094	-34814	0.38958	0.053606	0.101654	0.48044	-0.65356	0.101654	31321
7	-182632	-227820	-0.31882	-0.04387	-0.08319	-0.39318	-0.88916	0.4508	25632
8	-182632	99670	-0.31882	0.04387	-0.08319	-0.39318	0.88916	0.4508	25632
9	-103094	191418	0.38958	-0.053606	0.101654	0.48044	0.65356	0.101654	31321
10	-2112.2	27160	0.021874	0.075444	-0.013032	-0.021874	0.075444	-0.013032	18000
11	25438	25416	0.142222	-7.529E-17	-0.010987	0.140172	3.7646E-17	-0.071498	18000
12	-2112.2	-27160	0.021874	-0.075444	-0.013032	-0.021874	-0.075444	-0.013032	18000
13	25438	-25416	-0.140172	7.5292E-17	-0.071498	-0.142222	0	-0.010987	18000
14	-261700	-84148	0.57096	-0.52328	0.35794	0.57524	-0.6071	0.022956	32031
15	-142550	-129638	-0.147116	-0.034886	-0.041108	-0.154788	-0.54426	0.24192	43474
16	-136254	138918	0.2779	0.17921	0.17797	0.27976	0.122694	0.009951	43474
17	-142550	-78852	0.154788	-0.54426	0.24192	0.147116	-0.034886	-0.041108	43474
18	-261700	-322780	-0.57524	-0.6071	0.022956	-0.57096	-0.52328	0.35794	32031
19	-136254	-30232	-0.27976	0.122694	0.009951	-0.2779	0.17921	0.17797	43474
20	-136254	-70148	-0.27976	-0.122694	0.009951	-0.2779	-0.17921	0.17797	43474
21	-261700	116470	-0.57524	0.6071	0.022956	-0.57096	0.52328	0.35794	32031
22	-142550	133196	0.154788	0.54426	0.24192	0.147116	0.034886	-0.041108	43474
23	-136254	-38536	0.2779	-0.17921	0.17797	0.27976	-0.122694	0.009951	43474
24	-142550	75296	-0.147116	0.034886	-0.041108	-0.154788	0.54426	0.24192	43474
25	-261700	290460	0.57096	0.52328	0.35794	0.57524	0.6071	0.022956	32031

$$u_{y2} = \frac{q}{ES}(184918.7)$$

We suppose that the young's modulus E is uniformly distributed in the range [1, 2] and the cross-section S is uniformly distributed in the range [3, 4]. We notice the nonlinearity in the response expression (u_{y2}) in terms of E and S. using our proposed technique of linearization to linearize u_{y2}. Let M = ES, using our proposed technique (PTM-FEM) to evaluate the pdf of M leads to:

$$f_M(M) = \begin{cases} \log M - \log 3, & 3 < M < 4 \\ \log 4 - \log 3, & 4 < M < 6 \\ 3\log 2 - \log M, & 6 < M < 8 \end{cases}$$

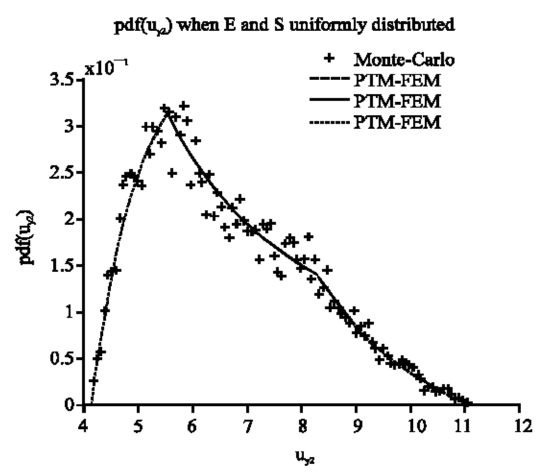
Again, let us suppose now N = 1/M, the PTM-FEM applied for N gives:

$$f_N(N) = \begin{cases} \frac{1}{N^2}(-\log N - \log 3), & \frac{1}{4} < N < \frac{1}{3} \\ \frac{1}{N^2}(\log 4 - \log 3), & \frac{1}{6} < N < \frac{1}{4} \\ \frac{1}{N^2}(3\log 2 + \log N), & \frac{1}{8} < N < \frac{1}{6} \end{cases}$$

Finally, the pdf of using PTM-FEM technique:

$$f_{u_{y2}}(u_{y2}) = \begin{cases} \frac{A}{u_{y2}^2}\left(-\log\frac{u_{y2}}{A} - \log 3\right), & \frac{A}{4} < u_{y2} < \frac{A}{3} \\ \frac{A}{u_{y2}^2}(\log 4 - \log 3), & \frac{A}{6} < u_{y2} < \frac{A}{4} \\ \frac{A}{u_{y2}^2}\left(3\log 2 + \log\frac{u_{y2}}{A}\right), & \frac{A}{8} < u_{y2} < \frac{A}{6} \\ A = 184918.7q \end{cases}$$

Numerical values: q = 18N.

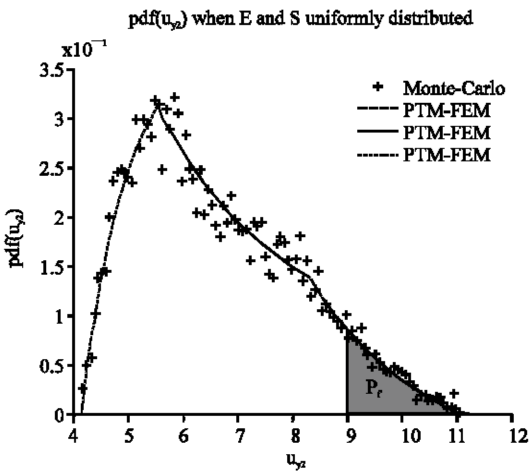


Reliability analysis: Let us suppose now the allowed displacement is u_{y2}, it is requested to find the failure probability $P_f = P(u \geq u_{y2})$, which is as follows:

$$P_f = \int_9^\infty f_{u_{y2}}(u_{y2})du_{y2} = \frac{1849187.2}{u_{y2}^2}$$

$$\left(-\log\frac{u_{y2}}{1849187.2} - \log 3\right) = 0.14$$

This result is confirmed by 10000 Monte Carlo simulations giving 0.1396.



CONCLUSION

In this article, a technique for the evaluation in exact form of the probability of failure is developed. Compared to other numerical methods like FORM/SORM, Monte-Carlo and Response surface method, no-series expansion is involved in this expression. Our method is validated using 10000 Monte-Carlo simulations through the analysis of 25-bar truss structure with random parameters.

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