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Effect of the Logarithmic Transformation on the Trend-cycle Component

Iheanyi S. Iwueze and Anthony C. Akpanta
Department of Statistics, Abia State University, P.M.B. 2000, Uturu, Abia State, Nigeria

Abstract: In this study we present some solutions to the problems encountered when a time series multiplicative model is linearized by taking the natural logarithm of the observed series. Problems associated with the trend-cycle component, with a view to achieving no alteration (if possible) to the fundamental nature of the original data, are discussed for a class of trending curves which includes the linear, quadratic and exponential growth curves. Necessary and sufficient conditions are derived for the parameters of the linear and quadratic curves. Numerical examples are given to illustrate the results obtained.

Key words: Multiplicative model, logarithmic transformation, trend-cycle component, growth curves

INTRODUCTION

A general multiplicative model for descriptive time series analysis would be of the form

$$X_t = T_t \; S_t \; C_t \; I_t \,, \; t = 1, 2, \ldots, n \qquad (1)$$

where, for time t, X_t denotes the observed value of the series, T_t is the trend, S_t the seasonal term, C_t the cyclic term and T_t is the irregular or random component of the series. If short period of time are involved, the cyclical component is superimposed into the trend (Chatfield, 1980) and we obtain a trend-cycle component denoted by M_t. The model (1) will be regarded to be adequate when the irregular component is purely random. For our purposes, I_t to be denoted by e_t are independent, identically distributed normal errors with mean 1 and variance $\sigma_1^2 > 0 \; [I_t = e_t \sim N(1, \sigma_1^2)]$. Model (1) can then be written as

$$X_t = M_t \; S_t \; e_t \,, \; t = 1, 2, \ldots, n \qquad (2)$$

Descriptive modeling of the multiplicative time series model (2) generally involve the logarithmic transformation. Upon linearization of (2), we obtain

$$\log_e X_t = \log_e M_t + \log_e S_t + \log_e e_t, t = 1, 2, \ldots, n \qquad (3)$$

or

$$Y_t = M_t^* + S_t^* + e_t^* \,, \; t = 1, 2, \ldots, n \qquad (4)$$

where $Y_t = \log_e X_t$, $M_t^* = \log_e M_t$, $S_t^* = \log_e S_t$ and $e_t^* = \log_e e_t$. It is important to note that (4) is the additive time series model.

Cogent reasons for utilizing data transformation, such as the logarithmic transformation of (4), are given in Hoerl (1954), Dolby (1963), Bond and Fox (2001) and Osborne (2002). Osborne (2002) warned that the use of data transformation must be done with care and never, unless there is a clear reason, because data transformations can alter the fundamental nature of the data, such as changing the measurement scale from interval or ratio to ordinal and creating curvilinear relationships, thereby making interpretation of results somewhat more complex.

The logarithmic transformation (4) obviously alters the fundamental nature of the multiplicative model (2). For the multiplicative model (2), it is convenient to assume that the sum of the seasonal components over a complete period is s. That is,

$$\sum_{j=1}^{s} S_{t+j} = s \qquad (5)$$

where s is the length of the periodic interval. For the additive model (4) the convenient variant assumption is that the sum of the seasonal components over a complete period is zero. That is,

$$\sum_{j=1}^{s} S_{t+j} = 0 \qquad (6)$$

For the additive model (4), the variant assumption on the random component e_t^* is that they are independent, identically distributed normal errors with mean zero and variance $\sigma_2^2 > 0 \; [e_t^* \sim N(0, \sigma_2^2)]$. Iwueze (2006), has established that the variant assumptions for the error term e_t^* are valid if and only if $\sigma_1 < 0.1$ and $\sigma_1 = \sigma_2$.

Corresponding Author: Iheanyi S. Iwueze, Department of Statistics, Abia State University, P.M.B. 2000, Uturu, Abia State, Nigeria

For most trending data it is common practice to fit a simple curve, such as exponential, straight line or quadratic curve through the data (Levenbach and Reuter, 1976). For data tending to some saturation level, the Gompertz and logistic curves have found widespread applications (Chaddha and Chitgopekar, 1971). The important curves for the trend-cycle component are:

Linear: $$M_t = a + bt \quad (7)$$

Quadratic: $$M_t = a + bt + ct^2 \quad (8)$$

Exponential: $$M_t = ae^{bt} \quad (9)$$

Modified exponential: $$M_t = a + be^{ct} \quad (10)$$

Gompertz: $$\log_e M_t = a - be^{ct},\ c<0 \quad (11)$$

Logistic: $$M_t = a/1 + be^{-ct} \quad (12)$$

Naturally, the logarithmic transformation of growth curves must alter the fundamental nature of the original data.

This study investigates the effect of the logarithmic transformation on the simple trending curves (exponential, straight line and quadratic curves) with a view to achieving no alteration (if possible) to the fundamental nature of the original data.

LOGARITHMIC TRANSFORMATION OF THE TREND-CYCLE COMPONENT

The motivation for this study on the effect of the logarithmic transformation on the trend-cycle component stems from the fact that if a trend-cycle component is exponential, the square root transformation does not alter the fundamental nature of the data as it will still be exponential. That is, if M_t is given by (9), then

$$M_t^* = \sqrt{M_t} = a^{\frac{1}{2}} e^{\frac{bt}{2}} = \alpha e^{\beta t} \quad (13)$$

where $\alpha = \sqrt{a}$, $\beta = b/2$.

Naturally, the logarithmic transformation of the trend-cycle component will: (1) be a linear function for the exponential trend curve (9) and (2) a logarithmic function for the other growth curves under study. Therefore, can we achieve the same feat-retaining the original form-after taking the logarithmic transformation of the linear and quadratic trend-cycle growth curves? The answer is yes, provided necessary conditions are placed on the parameters of the original curve.

Straight line curve: For the straight line curve, (7), we assume that

$$M_t^* = \log_e(a + bt) = \alpha + \beta t \quad (14)$$

$$\Rightarrow a + bt = e^{\alpha+\beta t} = e^{\alpha}\left[1+\beta t + \frac{\beta^2 t^2}{2!} + \frac{\beta^3 t^3}{3!} + \ldots\right] \quad (15)$$

Equating corresponding coefficients, we obtain

$$e^{\alpha} = a \Rightarrow \alpha = \log_e a,\ a > 0 \quad (16)$$

$$\beta e^{\alpha} = b \Rightarrow \beta = b/a,\ a>0 \quad (17)$$

Let

$$\Delta(a, b) = b/a,\ a>0 \quad (18)$$

The coefficients of $t^2, t^3, t^4, \ldots$ in (14) are quantities that depend on Δ(a, b) but vanishes more quickly than Δ(a, b) as Δ(a, b) → 0. These coefficients are said to be ‘ little o of Δ(a, b) ‘, written 0 (Δ(a, b)) (Tuckwell, 1988). That is,

$$e^{\alpha+\beta t} = a + bt + 0(\Delta(a, b)) \quad (19)$$

For example, $(\Delta(a, b))^d$, d = 2, 3, ... is o (Δ(a, b)). That is, as (Δ(a, b)) → 0, Eq. 14 will be satisfied. This is vividly illustrated in Table 1 for n = 100 data points. The coefficient of the multiple determination, R^2, is the square of the correlation between $M_t^* = \log_e(a + bt)$ and $\hat{M}_t^* = \alpha + \beta t$ with $0 \le R^2 \le 1$ (Draper and Smith, 1981). A perfect fit to the data for which $M_t^* = \hat{M}_t^*$ would give $R^2 = 1$ which from Table 1 means $-0.004 \le \Delta(a, b) \le 0.007$. However, allowing $R^2 \ge 0.99$, we obtain that $-0.006 \le \Delta(a, b) \le 0.01$. For $R^2 \ge 0.95$, $-0.008 \le \Delta(a, b) \le 0.06$ holds. We will concentrate on $R^2 \ge 0.99$ in this study.

Quadratic curve: For the quadratic curve, (8), we assume that

$$M_t^* = \log_e(a + bt + ct^2) = \alpha + \beta t + \gamma t^2 \quad (20)$$

$$\Rightarrow a + bt + ct^2 = e^{\alpha + \beta t + \gamma t^2}$$

$$= e^{\alpha}\left[1 + (\beta t + \gamma t^2) + \frac{(\beta t + \gamma t^2)^2}{2!} + \frac{(\beta t + \gamma t^2)^3}{3!} + \frac{(\beta t + \gamma t^2)^4}{4!} + \ldots\right] \quad (21)$$

That is,

Table 1: Computations illustrating changes/non-changes in form of the linear curve

$M_t^* = \log_e(a+bt) = \alpha+\beta t$

Assumed values		Theoretical		Estimated			
a	b	$\alpha = \log_e a$	$\beta = b/a$	$\hat{\alpha}$	$\hat{\beta}$	s	R^2
1.0000	1.0000	0.0000	1.0000	2.3183	0.0270	0.348	83.7
1.0000	0.9000	0.0000	0.9000	2.2253	0.0269	0.342	84.0
1.0000	0.8000	0.0000	0.8000	2.1226	0.0267	0.336	84.3
1.0000	0.7000	0.0000	0.7000	2.0078	0.0265	0.328	84.7
1.0000	0.6000	0.0000	0.6000	1.8778	0.0262	0.319	85.2
1.0000	0.5000	0.0000	0.5000	1.7278	0.0258	0.306	85.8
1.0000	0.4000	0.0000	0.4000	1.5502	0.0252	0.290	86.6
1.0000	0.3000	0.0000	0.3000	1.3324	0.0244	0.266	87.7
1.0000	0.2000	0.0000	0.2000	1.0497	0.0230	0.231	89.4
1.0000	0.1000	0.0000	0.1000	0.6440	0.0199	0.165	92.5
1.0000	0.0900	0.0000	0.0900	0.5918	0.0194	0.155	93.0
1.0000	0.0800	0.0000	0.0800	0.5365	0.0187	0.144	93.5
1.0000	0.0700	0.0000	0.0700	0.4778	0.0180	0.132	94.0
1.0000	0.0600	0.0000	0.0600	0.4153	0.0171	0.118	94.7
1.0000	0.0500	0.0000	0.0500	0.3487	0.0160	0.103	95.4
1.0000	0.0400	0.0000	0.0400	0.2775	0.0147	0.086	96.2
1.0000	0.0300	0.0000	0.0300	0.2019	0.0129	0.065	97.1
1.0000	0.0200	0.0000	0.0200	0.1230	0.0105	0.043	98.1
1.0000	0.0100	0.0000	0.0100	0.0464	0.0068	0.018	99.2
1.0000	0.0090	0.0000	0.0090	0.0395	0.0063	0.015	99.3
1.0000	0.0080	0.0000	0.0080	0.0329	0.0056	0.013	99.4
1.0000	0.0070	0.0000	0.0070	0.0265	0.0052	0.010	99.5
1.0000	0.0060	0.0000	0.0060	0.0206	0.0047	0.008	99.6
1.0000	0.0050	0.0000	0.0050	0.0152	0.0040	0.006	99.7
1.0000	0.0040	0.0000	0.0040	0.0103	0.0033	0.004	99.8
1.0000	0.0030	0.0000	0.0030	0.0062	0.0026	0.003	99.9
1.0000	0.0020	0.0000	0.0020	0.0030	0.0018	0.001	99.9
1.0000	0.0010	0.0000	0.0010	0.0008	0.0010	0.000	100.0
1.0000	0.0009	0.0000	0.0009	0.0006	0.0009	0.000	100.0
1.0000	-0.0009	0.0000	-0.0009	0.0007	-0.0009	0.000	100.0
1.0000	-0.0010	0.0000	-0.0010	0.0009	-0.0011	0.000	100.0
1.0000	-0.0020	0.0000	-0.0020	0.0041	-0.0022	0.002	99.9
1.0000	-0.0030	0.0000	-0.0030	0.0101	-0.0036	0.005	99.8
1.0000	-0.0040	0.0000	-0.0040	0.0201	-0.0051	0.010	99.6
1.0000	-0.0050	0.0000	-0.0050	0.0354	-0.0068	0.018	99.2
1.0000	-0.0060	0.0000	-0.0060	0.0585	-0.0090	0.031	98.6
1.0000	-0.0070	0.0000	-0.0070	0.0936	-0.0116	0.052	97.7
1.0000	-0.0080	0.0000	-0.0080	0.1492	-0.0149	0.089	96.0
1.0000	-0.0090	0.0000	-0.0090	0.2470	-0.0199	0.164	92.5
1.0000	-0.0100	0.0000	-0.0100	0.4693	-0.0289	0.414	80.1

s is the estimated standard deviation about the regression line; R^2 is the coefficient of determination

$$
\begin{aligned}
e^{\alpha+\beta t+\gamma t^2}
&= e^{\alpha}\Bigg[1+\beta t+\left(\frac{\beta^2}{2}+\gamma\right)t^2+\left(\frac{\beta^3}{6}+\beta\gamma\right)t^3+\left(\frac{\beta^4}{24}+\frac{\beta^2\gamma}{2}+\frac{\gamma^2}{2}\right)t^4 \\
&+\left(\frac{\beta^5}{120}+\frac{\beta^3\gamma}{6}+\frac{\beta\gamma^2}{2}\right)t^5+\left(\frac{\beta^6}{720}+\frac{\beta^4\gamma}{24}+\frac{\beta^2\gamma^2}{4}+\frac{\gamma^3}{6}\right)t^6 \\
&+\left(\frac{\beta^7}{5040}+\frac{\beta^5\gamma}{120}+\frac{\beta^3\gamma^2}{12}+\frac{\beta\gamma^3}{6}\right)t^7+\left(\frac{\beta^8}{40320}+\frac{\beta^6\gamma}{720}+\frac{\beta^4\gamma^2}{48}+\frac{\beta^2\gamma^3}{12}+\frac{\gamma^4}{24}\right)t^8 \\
&+\left(\frac{\beta^9}{362880}+\frac{\beta^7\gamma}{5040}+\frac{\beta^5\gamma^2}{240}+\frac{\beta^3\gamma^3}{36}+\frac{\beta\gamma^4}{24}\right)t^9 \\
&+\left(\frac{\beta^{10}}{3628800}+\frac{\beta^8\gamma}{40320}+\frac{\beta^6\gamma^2}{1440}+\frac{\beta^4\gamma^3}{144}+\frac{\beta^2\gamma^4}{48}+\frac{\gamma^5}{120}\right)t^{10}
\end{aligned}
\tag{22}
$$

Equating corresponding coefficients, we obtain

$$e^{\alpha} = a \Rightarrow \alpha = \log_e a,\ a>0 \tag{23}$$

$$\beta e^{\alpha} = b \Rightarrow \beta = b/a,\ a>0 \tag{24}$$

$$e^{\alpha}\left[\frac{\beta^2}{2}+\gamma\right] = c \Rightarrow \gamma = \frac{c}{a}-\frac{1}{2}\left(\frac{b}{a}\right)^2,\ a>0 \tag{25}$$

Now we need some sufficient conditions for Eq. 20 to hold. We obtain these by equating the coefficients of t^3, t^4, t^5, ... in (22) to zero. Derived conditions are obtained by substituting values of α, β and γ given by (23), (24) and (25), respectively. After the relevant substitutions, we will obtain a polynomial of order p = 1, 2, 3, ... whose real zeros must be determined. Real zeros of order greater than two are obtained by graphical methods only.

- $$e^{\alpha}\left(\frac{\beta^3}{6}+\beta\gamma\right)=0 \Rightarrow \frac{\beta^2}{6}+\gamma=0 \Rightarrow \frac{c}{a}-\frac{1}{3}\left(\frac{b}{a}\right)^2=0$$

$$\Rightarrow c = \frac{b^2}{3a} = 0.33\left(\frac{b^2}{a}\right) \tag{26}$$

- $$e^{\alpha}\left(\frac{\beta^4}{24}+\frac{\beta^2\gamma}{2}+\frac{\gamma^2}{2}\right)=0 \Rightarrow \left(\frac{\beta^4}{24}+\frac{\beta^2\gamma}{2}+\frac{\gamma^2}{2}\right)=0 \Rightarrow c^2=\frac{b^4}{6a^2}$$

$$\Rightarrow c = \frac{\pm}{\sqrt{6}}\left(\frac{b^2}{a}\right) = \pm 0.41\left(\frac{b^2}{a}\right) \tag{27}$$

- $$e^{\alpha}\left(\frac{\beta^5}{120}+\frac{\beta^3\gamma}{6}+\frac{\beta\gamma^2}{2}\right)=0 \Rightarrow \left(\frac{\beta^4}{120}+\frac{\beta^2\gamma}{6}+\frac{\gamma^2}{2}\right)=0$$

$$\Rightarrow \left(\frac{1}{2a^2}\right)c^2-\left(\frac{b^2}{3a^3}\right)c+\frac{b^4}{20a^4} = 0$$

$$\Rightarrow c = 0.23\left(\frac{b^2}{a}\right) \text{ or } c = 0.44\left(\frac{b^2}{a}\right) \tag{28}$$

- $$e^{\alpha}\left(\frac{\beta^6}{720}+\frac{\beta^4\gamma}{24}+\frac{\beta^2\gamma^2}{4}+\frac{\gamma^3}{6}\right) = 0$$
  $$\Rightarrow \left(\frac{\beta^6}{720}+\frac{\beta^4\gamma}{24}+\frac{\beta^2\gamma^2}{4}+\frac{\gamma^3}{6}\right)=0$$

$$\Rightarrow \frac{c^3}{2}-\left(\frac{b^4}{4a^2}\right)c+\frac{b^6}{15a^3} = 0 \tag{29}$$

Let $c = \frac{b^2}{xa}$ be a root of (29). Substituting this value of c into (29), we obtain

$$4x^3 - 15x^2 + 30 = 0 \quad (30)$$

$\Rightarrow$ x = −1.2275 or x = 2.1990 or x = 2.7785

$$\Rightarrow c = -0.81\left(\frac{b^2}{a}\right) \text{ or } c = 0.36\left(\frac{b^2}{a}\right) \text{ or } c = 0.45\left(\frac{b^2}{a}\right) \quad (31)$$

- $$e^{\alpha}\left(\frac{\beta^7}{5040}+\frac{\beta^5\gamma}{120}+\frac{\beta^3\gamma^2}{12}+\frac{\beta\gamma^3}{6}\right)=0$$
$$\Rightarrow\left(\frac{\beta^6}{5040}+\frac{\beta^4\gamma}{120}+\frac{\beta^2\gamma^2}{12}+\frac{\gamma^3}{6}\right)=0$$

$$\Rightarrow \frac{c^3}{3}-\left(\frac{b^2}{3a}\right)c^2+\left(\frac{b^4}{10a^2}\right)c-\frac{b^6}{126a^3} = 0 \quad (32)$$

Let $c = \frac{b^2}{xa}$ be a root of (32). Substituting this value of c into (32), we obtain

$$5x^3 - 63x^2 + 210x - 210 = 0 \quad (33)$$

$\Rightarrow$ x = 2.1531 or x = 2.4346 or x = 8.0123

$$c = 0.12\left(\frac{b^2}{a}\right) \text{ or } c = 0.41\left(\frac{b^2}{a}\right) \text{ or } c = 0.46\left(\frac{b^2}{a}\right) \quad (34)$$

- $$e^{\alpha}\left(\frac{\beta^8}{40320}+\frac{\beta^6\gamma}{720}+\frac{\beta^4\gamma^2}{48}+\frac{\beta^2\gamma^3}{12}+\frac{\gamma^4}{24}\right)=0$$

$$\Rightarrow\left(\frac{\beta^8}{40320}+\frac{\beta^6\gamma}{720}+\frac{\beta^4\gamma^2}{48}+\frac{\beta^2\gamma^3}{12}+\frac{\gamma^4}{24}\right)=0 \quad (35)$$

Repeating the procedure of substituting the values of β and γ into (35) and letting $c = \frac{b^2}{xa}$, we obtain

$$33x^4 - 196x^3 + 420x^2 - 420 = 0 \quad (36)$$

Equation 36 has only two real roots given by $x = -0.827$ and $x = 1.426$. It follows that

$$c = -1.21\left(\frac{b^2}{a}\right) \text{ or } c = 0.70\left(\frac{b^2}{a}\right) \quad (37)$$

- $$e^{\alpha}\left(\frac{\beta^9}{362880}+\frac{\beta^7\gamma}{5040}+\frac{\beta^5\gamma^2}{240}+\frac{\beta^3\gamma^3}{36}+\frac{\beta\gamma^4}{24}\right)=0$$

$$\Rightarrow\left(\frac{\beta^9}{362880}+\frac{\beta^7\gamma}{5040}+\frac{\beta^5\gamma^2}{240}+\frac{\beta^3\gamma^3}{36}+\frac{\beta\gamma^4}{24}\right)=0 \quad (38)$$

Repeating the procedure of substituting the values of β and γ into (38) and letting $c = \frac{b^2}{xa}$, we obtain

$$7x^4 - 360x^3 + 2268x^2 - 5040x + 3780 = 0 \quad (39)$$

$\Rightarrow$ x = 2.1033 or x = 2.6043 or x = 2.2164 or x = 44.5060

$$\Rightarrow c = 0.02\left(\frac{b^2}{a}\right) \text{ or } c = 0.38\left(\frac{b^2}{a}\right) \text{ or } c = 0.45\left(\frac{b^2}{a}\right)$$

$$\text{or } c = 0.48\left(\frac{b^2}{a}\right) \quad (40)$$

- $$e^{\alpha}\left(\frac{\beta^{10}}{3628800}+\frac{\beta^8\gamma}{40320}+\frac{\beta^6\gamma^2}{1440}+\frac{\beta^4\gamma^3}{144}+\frac{\beta^2\gamma^4}{48}+\frac{\gamma^4}{24}\right)=0$$

$$\Rightarrow\left(\frac{\beta^{10}}{3628800}+\frac{\beta^8\gamma}{40320}+\frac{\beta^6\gamma^2}{1440}+\frac{\beta^4\gamma^3}{144}+\frac{\beta^2\gamma^4}{48}+\frac{\gamma^4}{24}\right)=0 \quad (41)$$

Repeating the procedure of substituting the values of β and γ into (41) and letting $c = \frac{b^2}{xa}$, we obtain

$$1553x^5 - 5940x^4 + 20160x^3 - 25200x^2 + 15120 = 0 \quad (42)$$

Equation 42 has only one real root given by x = -0.6136. It follows that

$$c = -1.63\left(\frac{b^2}{a}\right) \quad (43)$$

and so on.

From our analysis so far, it does appear that Eq. 20 holds if $\Delta(a,b) \to 0$ and

$$c = k\left(\frac{b^2}{a}\right), \text{ for some real k} \quad (44)$$

Table 2 gives an investigation of the behaviour of k for the case where a = 1.0, b = 0.01, $c = k\left(\frac{b^2}{a}\right) = 0.0001k$, $\alpha = \log_e a = 0.00$, $\beta = b/a = 0.01$ and $\gamma = \frac{c}{a} - \frac{1}{2}\left(\frac{b}{a}\right)^2 =$ c-0.00005. It is clear that for $R^2 \geq 0.99$, k has so many

Table 2: Determination of k and c for Δ(a, b) = 0.01

k	c	γ	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	s	R^2
400.0	0.04000	0.03995	0.52261	0.11503	-0.00063	0.2012	98.4
300.0	0.03000	0.02995	0.37832	0.11013	-0.00060	0.1744	98.7
200.0	0.02000	0.01995	0.20266	0.10246	-0.00054	0.1375	99.1
100.0	0.01000	0.00995	-0.01866	0.08746	-0.00043	0.0806	99.6
50.0	0.00500	0.00495	-0.14062	0.07075	-0.00031	0.0401	99.9
*	*	*	*	*	*	*	*
1.0	0.00010	0.00005	-0.01230	0.01170	-0.00001	0.0035	100.0
0.9	0.00009	0.00004	-0.01019	0.01139	-0.00001	0.0029	100.0
0.8	0.00008	0.00003	-0.00812	0.01111	-0.00001	0.0023	100.0
0.7	0.00007	0.00002	-0.00608	0.01084	-0.00001	0.0017	100.0
0.6	0.00006	0.00001	-0.00407	0.01057	-0.00001	0.0011	100.0
0.5	0.00005	0.00000	-0.00211	0.01031	-0.00001	0.0005	100.0
0.4	0.00004	-0.00001	-0.00020	0.01005	-0.00001	0.0001	100.0
0.3	0.00003	-0.00002	0.00165	0.00981	-0.00002	0.0006	100.0
0.2	0.00002	-0.00003	0.00341	0.00957	-0.00002	0.0011	100.0
0.1	0.00001	-0.00004	0.00507	0.00935	-0.00002	0.0016	100.0
0.0	0.00000	-0.00005	0.00662	0.00914	-0.00002	0.0020	100.0
-0.1	-0.00001	-0.00006	0.00801	0.00895	-0.00003	0.0024	100.0
-0.2	-0.00002	-0.00007	0.00923	0.00878	-0.00003	0.0028	100.0
-0.3	-0.00003	-0.00008	0.01024	0.00863	-0.00003	0.0031	100.0
-0.4	-0.00004	-0.00009	0.01100	0.00851	-0.00004	0.0032	99.9
-0.5	-0.00005	-0.00010	0.01144	0.00842	-0.00005	0.0033	99.9
-0.6	-0.00006	-0.00011	0.01151	0.00838	-0.00005	0.0032	99.9
-0.7	-0.00007	-0.00012	0.01111	0.00838	-0.00006	0.0029	99.9
-0.8	-0.00008	-0.00013	0.01016	0.00844	-0.00007	0.0024	99.9
-0.9	-0.00009	-0.00014	0.00851	0.00856	-0.00008	0.0019	99.9
-1.0	-0.00010	-0.00015	0.00600	0.00877	-0.00009	0.0018	99.9
-1.1	*	*	*	*	*	*	*

* Means not applicable

Table 3: Summary of values of k and c for different values of -0.006 ≤ Δ(a,b) ≤ 0.010

a	b	Δ(a,b)	k	c
1.00	0.0100	0.0100	0.40	0.0000400
1.00	0.0090	0.0090	0.40	0.0000324
1.00	0.0080	0.0080	0.39	0.0000250
1.00	0.0070	0.0070	0.38	0.0000186
1.00	0.0060	0.0060	0.37	0.0000133
1.00	0.0050	0.0050	0.36	0.0000090
1.00	0.0040	0.0040	0.36	0.0000058
1.00	0.0030	0.0030	0.35	0.0000032
1.00	0.0020	0.0020	0.35	0.0000014
1.00	0.0010	0.0010	0.34	0.0000003
1.00	0.0009	0.0009	0.34	0.0000003
1.00	-0.0009	-0.0009	0.33	0.0000003
1.00	-0.0010	-0.0010	0.33	0.0000003
1.00	-0.0020	-0.0020	0.32	0.0000013
1.00	-0.0030	-0.0030	0.32	0.0000029
1.00	-0.0040	-0.0040	0.31	0.0000050
1.00	-0.0050	-0.0050	0.31	0.0000078
1.00	-0.0060	-0.0060	0.30	0.0000108

values for which Eq. 23-25 hold. Using values of s, the estimated standard deviation about the regression line, it is clear from Table 2 that the optimal condition of $c = 0.40\left(\frac{b^2}{a}\right)$ will be used when $\Delta(a, b) = 0.01$.

Optimum values of k for $-0.006 \le \Delta(a, b) \le 0.01$ were similarly determined and summarized in Table 3. It is evident that for $-0.006 \le \Delta(a, b) \le 0.001$, optimal value of k is 0.3 while for $0.002 \le \Delta(a, b) \le 0.01$, the optimal value of k is 0.4. Equation 23-25 hold perfectly for these optimal values of k.

NUMERICAL EXAMPLES

US beer production: Table 4 shows 32 consecutive quarters of US beer production, in millions of barrels, from the first quarter of 1975 to the fourth quarter of 1982. As shown in Table 4 and Fig. 1, the series is clearly seasonal with a slight upward trend. There is an upsurge of the series almost of equal magnitude in the second and third quarters and a sharp drop (again of almost equal magnitude) in the first and fourth quarters. The yearly standard deviations are stable while the seasonal standard deviations show a steady decline, indicating that the series needs some transformation to make the seasonal effect additive and stabilize the variance. Concentrating on the seasonal averages and standard deviations, we use Barttlet (1947) transformation method to obtain the slope of the linear relationship between the logarithms of the standard deviations and averages. In our own case, a slope of 0.81 was obtained (which is approximately 1), which suggest a logarithmic transformation.

Wei (1989), ignoring the stochastic trend in the series, used 30 observations of the original series for ARIMA model construction. Based on the forecasting performance of his models, he settled on the model

$$\left(1-B^4\right)X_t = \underset{(\pm 0.09)}{1.49} + \left(1-\underset{(\pm 0.16)}{0.87}B^4\right)e_t \quad (45)$$

Table 4: US Quarterly beer production in millions of barrels, between 1975 and 1982

	Quarter						
Year	I	II	III	IV	Total	Average	SD
1975	36.14	44.60	44.15	35.72	160.61	40.1525	4.8822
1976	36.19	44.63	46.95	36.90	164.67	41.1675	5.4287
1977	39.66	49.72	44.49	36.54	170.41	42.6026	5.7629
1978	41.44	49.07	48.98	39.59	179.08	44.7700	4.9711
1979	44.29	50.09	48.42	41.39	184.19	46.0475	3.9476
1980	46.11	53.44	53.00	42.52	195.07	48.7675	5.3491
1981	44.61	55.18	52.24	41.66	193.69	48.4226	6.3378
1982	47.84	54.27	52.31	41.83	196.25	49.0625	5.5217
Total	336.28	401.00	390.54	316.15	1443.97		
Average	42.0350	50.1250	48.8175	39.5188		45.1241	
SD	4.4228	4.0659	3.4967	2.7413			

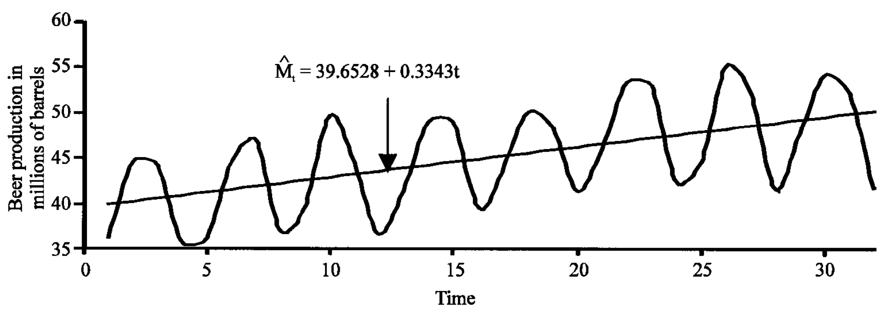

Fig. 1: US beer production, in millions of barrels, between 1975 and 1982

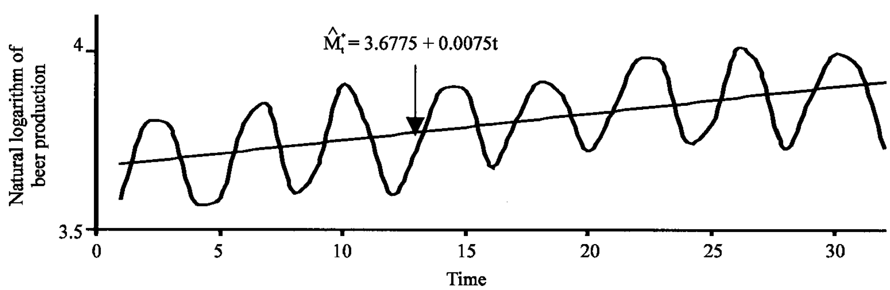

Fig. 2: Natural logarithm of US beer production

with $\hat{\sigma}_e^2 = 2.39$. Iwueze and Nwogu (2004), using their Buys-Ballot modeling procedure, fitted the descriptive model (2) to the 30 observations to obtain: $\hat{M}_t = 38.9484 + 0.3894t$, $\hat{S}_1 = 0.9467$, $\hat{S}_2 = 1.1200$, $\hat{S}_3 = 1.0712$, $\hat{S}_4 = 0.8620$ and error standard deviation of 0.03. In terms of forecasts, Iwueze and Nwogu (2004) claimed that their multiplicative model outperforms the ARIMA model of Wei (1989).

Fitting the multiplicative model (2) to the entire series X_t, we obtain: $\hat{M}_t = 39.6084 + 0.3343t$, $\hat{S}_1 = 0.9404$, $\hat{S}_2 = 1.1146$, $\hat{S}_3 = 1.0784$, $\hat{S}_4 = 0.8666$ and $\hat{\sigma}_1 = 0.0303$. Alternatively, fitting the additive model (3) to the transformed series $Y_t = \log_e X_t$, we obtain: $\hat{M}_t^* = 3.6775 + 0.0075t$, $\hat{S}_1^* = -0.0442$, $\hat{S}_2^* = 0.1027$, $\hat{S}_3^* = 0.0826$, $\hat{S}_4^* = -0.1411$ and $\hat{\sigma}_2 = 0.0324$. It is evident that $\Delta(a, b) = 0.3243/39.6084 = 0.0084$ and as expected from earlier results, $Y_t = \log_e X_t$ will have a linear trend-cycle component given by $M_t^* = \alpha + \beta t$ with $\hat{\alpha} = \log_e \hat{a} = \log_e 39.6084 = 3.6790$ and $\hat{\beta} = \hat{b}/\hat{a} = 0.3243/39.6084 = 0.0084$.

International airline passengers data: Monthly passenger totals (measured in thousands) in International Air Travels quoted by Box *et al.* (1994) and listed as Table 5 and Fig. 3 is our next example. The plot of the series which shows a marked seasonal pattern and linear trend is shown in Fig. 2. Box *et al.* (1994) have fitted the multiplicative seasonal ARIMA $(0, 1, 1) \times (0, 1, 1)_{12}$ model to the natural logarithms of the Airline data. Our interest here is to measure the trend-cycle component of the original and natural logarithms of the data. The Barttlet (1947) transformation method strongly supports the

Table 5: International Airline Passengers: Monthly Totals (thousands of passengers) January 1949-December 1960

	Month												
Year	Jan.	Feb.	Mar.	Apr.	May.	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Average	SD
1949	112.0	118.0	132.0	129.0	121.0	135.0	148.0	148.0	136.0	119.0	104.0	118.0	126.7	13.7
1950	115.0	126.0	141.0	135.0	125.0	149.0	170.0	170.0	158.0	133.0	114.0	140.0	139.7	19.1
1951	145.0	150.0	178.0	163.0	172.0	178.0	199.0	199.0	184.0	162.0	146.0	166.0	170.2	18.4
1952	171.0	180.0	193.0	181.0	183.0	218.0	230.0	242.0	209.0	191.0	172.0	194.0	197.0	23.0
1953	196.0	196.0	236.0	235.0	229.0	243.0	264.0	272.0	237.0	211.0	180.0	201.0	225.0	28.5
1954	204.0	188.0	235.0	227.0	234.0	264.0	302.0	293.0	259.0	229.0	203.0	229.0	238.9	34.9
1955	242.0	233.0	267.0	269.0	270.0	315.0	364.0	347.0	312.0	274.0	237.0	278.0	284.0	42.1
1956	284.0	277.0	317.0	313.0	3180.0	374.0	413.0	405.0	355.0	306.0	271.0	306.0	328.2	47.9
1957	315.0	301.0	356.0	348.0	355.0	422.0	465.0	467.0	404.0	347.0	305.0	336.0	368.4	57.9
1958	340.0	318.0	362.0	348.0	363.0	435.0	491.0	505.0	404.0	359.0	310.0	337.0	381.0	64.5
1959	360.0	342.0	406.0	396.0	420.0	472.0	548.0	559.0	463.0	407.0	362.0	405.0	428.3	69.8
1960	417.0	391.0	419.0	461.0	472.0	535.0	622.0	606.0	508.0	461.0	390.0	432.0	476.2	77.7
Average	241.8	235.0	270.2	267.1	271.8	311.7	351.3	351.1	302.4	266.6	232.8	261.8	280.3	-
SD	101.0	89.6	100.6	107.4	114.7	134.2	156.8	155.8	124.0	110.7	95.2	103.1	-	120.0

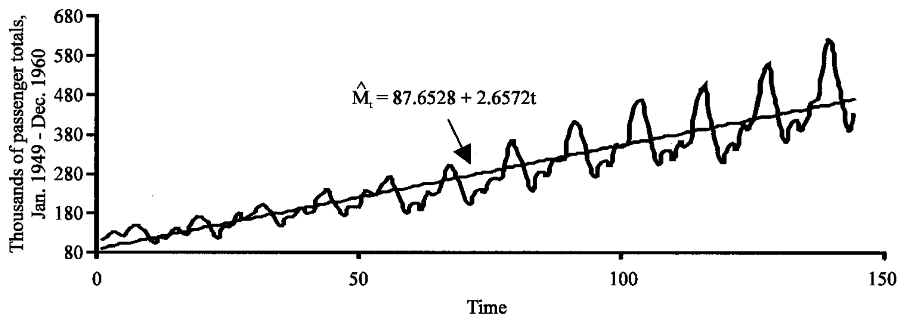

Fig. 3: Totals of international airline passengers in thousands, Jan. 1949-Dec. 1960

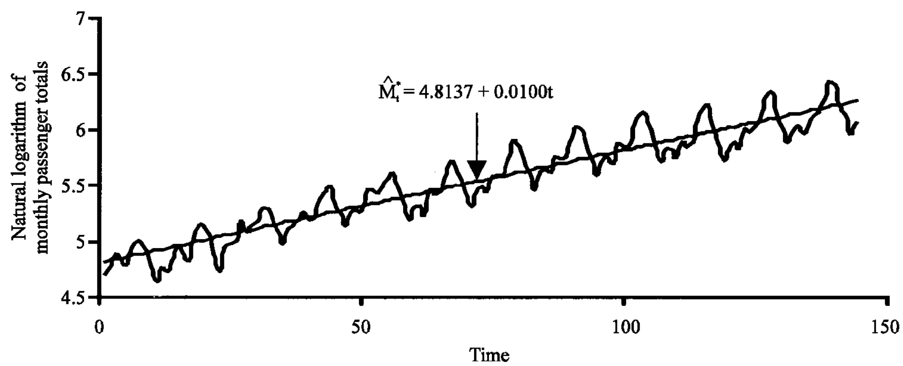

Fig. 4: Natural logarithm of totals of international airline passengers

logarithmic transformation since the linear relationship between the logarithms of the standard deviations and averages is 1.31 for the years and 1.27 for the months.

Fitting the trend-cycle line to the original series X_t, we obtain: $\hat{M}_t = 87.6528 + 2.6572t$, $R^2 = 0.85$, which implies that $\Delta(a, b) = 0.0303$. However, fitting the trend-cycle line to the transformed series $Y_t = \log_e X_t$, we obtain: $\hat{M}^*_t = 4.8137 + 0.0100t$, $R^2 = 0.90$ (Fig. 4). Using $\Delta(a, b) = 0.0303$ and the methods for the determination of Table 1, we fit a line to $\log_e \hat{M}_t = \log_e 87.6528\,(1.00 + 0.0303t)$ and obtain the following: $\hat{\alpha} = 4.7751$, $\hat{\beta} = 0.0106$ and $R^2 = 0.95$. If the fit to the trend-cycle component of $Y_t = \log_e X_t$ is perfect, we would have $\alpha = \log_e 87.6528 = 4.4734$ and $\beta = 2.65/87.6528 = 0.0303$.

Nigeria consumer price index: Our final example that has to do with a quadratic trend-cycle component is the consumer price index (January 1970-December 1979) in Nigeria quoted by Iwueze and Akpanta (2006) and listed as Table 6 and Fig. 5. Iwueze and Akpanta (2006) after performing time series decomposition on $Y_t = \log_e X_t$, also fitted the seasonal multiplicative ARIMA

Table 6: Nigeria consumer price index (January 1970-December 1979)

	Month													
Year	Jan.	Feb.	Mar.	Apr.	May.	June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Average	SD
1970	10.30	10.30	10.40	10.70	10.90	11.00	11.00	11.00	11.20	11.00	11.10	11.20	10.842	0.334
1971	11.60	11.80	12.00	12.10	12.60	13.10	13.50	12.80	12.80	12.90	12.80	12.90	12.575	0.572
1972	12.90	13.20	13.30	13.00	13.50	13.40	13.10	12.60	12.50	13.80	12.40	12.40	13.008	0.460
1973	12.60	13.10	13.10	13.40	13.70	14.00	14.00	13.90	13.60	13.60	13.60	13.60	13.517	0.413
1974	14.70	14.70	14.80	15.50	15.30	15.30	15.70	15.60	15.80	15.60	16.00	16.10	15.425	0.481
1975	17.10	18.10	18.90	19.30	20.50	21.30	21.50	21.90	22.00	22.00	22.60	23.10	20.692	1.910
1976	23.90	24.60	24.20	24.40	24.60	25.00	25.10	25.60	25.60	26.30	25.50	25.10	24.992	0.688
1977	27.10	26.60	27.50	28.30	29.60	30.70	31.70	32.80	31.80	32.20	33.00	34.00	30.442	2.545
1978	31.00	32.10	32.90	33.70	35.00	35.40	35.20	35.30	35.50	35.90	35.80	36.10	34.492	1.668
1979	35.70	36.80	37.50	38.30	39.10	39.40	39.40	39.10	39.10	39.10	39.20	39.10	38.483	1.194
Average	19.69	20.13	20.46	20.87	21.48	21.86	22.02	22.06	21.99	22.24	22.20	22.36	21.447	-
SD	9.06	9.31	9.55	9.81	10.13	10.28	10.31	10.54	10.46	10.49	10.65	10.76	-	9.687

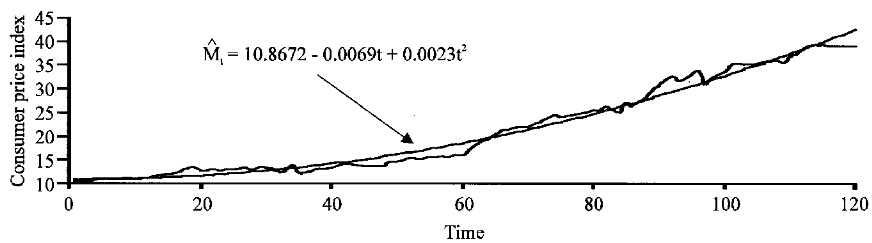

Fig. 5: Nigeria consumer price index (January 1970-December 1979)

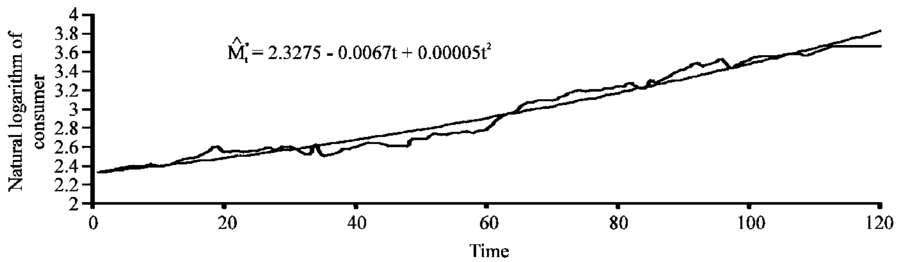

Fig. 6: Natural logarithm of Nigeria's consumer price index (January 1970-December 1979)

0 (2, 2)x(0, 1, 1)$_{12}$ time series model on $Y_t = \log_e X_t$. The Barttlet (1947) transformation method strongly supports the logarithmic transformation since the linear relationship between the logarithms of the standard deviations and averages is 1.27 for the years and 1.27 for the months.

Fitting the trend-cycle quadratic curve to the original series X_t, we obtain: $\hat{M}_t = 10.8672 - 0.0069t + 0.0023t^2 = 10.8672(1.0 - 0.00064t + 0.00021t^2)$, $R^2 = 0.98$ which implies that a = 1.0, b = − 0.00064, c = 0.00021 and $\Delta(a, b) = -0.00064$, $k = c/(b^2/a) = 512.6952 \approx 513$. However, fitting the trend-cycle quadratic curve to the transformed series $Y_t = \log_e X_t$, we obtain: $\hat{M}^*_t = 2.3275 + 0.0067t + 0.00005t^2$, $R^2 = 0.97$ (Fig. 6).

If the quadratic curve fits the trend-cycle component of $Y_t = \log_e X_t$ perfectly, then $\alpha = \log_e 10.8672 = 2.3857$, $\beta = -0.0069/10.8672 = -0.0006$ and $\gamma = 0.0023/10.8672 - (1/2)(-0.0069/10.8672)^2 = 0.0002$. Obtained results differ significantly with respect to β and γ. Using $\Delta(a, b) = -0.0006$ and the methods for the determination of Table 2 when a = 1.0, b = -0.00064 and $c = k\left(\frac{b^2}{a}\right)$, we observe that the optimal value of k is 0.33, which implies that the constant c ought to be 0.000000135. This accounts for the observed differences. Alternatively, if we fit a quadratic curve to $\log_e \hat{M}_t = \log_e 10.8672(1.00 - 0.00064t + 0.00021t^2)$, we obtain the following: $\hat{\alpha} = 2.3274$, $\hat{\beta} = 0.0062$ $\hat{\gamma} = 0.00006$ and $R^2 = 0.977$.

CONCLUSION

We have examined the effect of the logarithmic transformation on two simple trending curves (straight line and quadratic curves), with a view to achieving no alteration to the fundamental nature of the original data. The basic approach in our analysis was to assume that the transformed data has the same tend-cycle curve form as the magnitudes of the original time series data. To achieve this, we have placed necessary and sufficient conditions on the parameters of the original curve. For the straight line, the necessary condition is that $a > 0$ and $\Delta(a,b) = b/a \rightarrow 0$. For the quadratic curve, the sufficient condition is that $a > 0, \Delta(a, b) = b/a \rightarrow 0$ and $c = k\left(\frac{b^2}{a}\right)$. The real constant k is determined with the help of the coefficient of the multiple determination, R^2 and the estimated standard deviation about the regression line, s. Optimal values of k were also obtained for $-0.006 \leq \Delta(a,b) \leq 0.01$.

In fitting trend curves to the logarithmically transformed time series data, we need the sufficient conditions derived to tell us not only the appropriate form of the trend curve but also help us provide good estimates of the numerical values of its parameters. These values can then be checked later by trend analysis of the transformed data.

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