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Solving Singular Initial Value Problems in the Second-order Ordinary Differential Equations

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Abstract: Singular initial value problems, linear and nonlinear, homogeneous and nonhomogeneous, are investigated by using Taylor series method. The solutions are constructed in the form of a convergent series. A general formula is established. The approach is illustrated with few examples.

Key words: Singular initial value problems, taylor series method, second order linear and nonlinear ODE, lane-emaden-type equations

INTRODUCTION

In recent years, the studies of singular initial value problems in the second-order Ordinary Differential Equations (ODEs) have attracted the attention of many mathematicians and physicists. One of the equations describing this type is the Lane-Emden-type equations formulated as.

$$y'' + \frac{2}{x}y' + f(y) = 0,\ 0 < x \le 1 \qquad (1)$$
$$y(0) = A,\ y'(0) = B$$

On the other hand, studies have been carried out on another class of singular initial value problems of the form

$$y'' + \frac{2}{x}y' + f(x,y) = g(x), 0 < x \le 1 \qquad (2)$$
$$y(0) = A,\ y'(0) = B$$

where A and B are constants, f (x, y) is a continuous real valued function and g (x) $\in$ c [0,1]. Equation 2 differs from the classical Lane-Emden-type Eq. 1 for the function f (x, y) and the inhomogeneous term g (x).

Equation 1 with specializing f (y) was used to model several phenomena in mathematical Physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres and theory of thermionic currents Chandrasekhar (1976) and Davis (1962). Due to the significant applications of Lane-Emden-type equations in the scientific community, various forms of f (y) have been investigated in many research works. A discussion of the formulation of these models and the physical structure of the solutions can be found in Chandrasekhar (1976), Davis (1962), Shawagfeh (1993), Adomian (1986) and Wazwaz (2001). Most algorithms currently in use for handling the Lane-Emden-type problems are based on either series solution or perturbation techniques. Wazwaz (2001) has given a general study to construct exact and series solution to Lane-Emden-type equations by employing the Adomian decomposition method. Moreover, a generalization was developed in Wazwaz (2001) by replacing the coefficient 2/x of y' (x) by n/x.

It is important to note that (2), with boundary conditions, has attracted many mathematicians and has been studied from various points of view. Russell and Shampine (1975) have investigated (2) for the liner function f (x, y) = ky+h (x) and have proved that a unique solution exists if h (x) $\in$ C [0,1] and $-\infty<k\le \pi^2$. Three-point difference methods of second order have been used in Russell and Shampine (1975). Moreover, three-point difference method of second order have been also used by Chawal and Katt I(1984), Chawla *et al.* (1986) and Iyengar and Jain (1987). However, Jain and Jain (1989) derived three-point difference method of four and six order to solve this problem. The numerical results obtained in Jain and Jain (1989) demonstrate o (h^4) and o (h^6) convergence of the method. Recently, El- sayed (1999) used a multi-integral method to investigate the nonlinear problem (2) with two boundary conditions.

The Eq. 1 can not have a taylor series expansion directly over the interval in which a solution is desired. For example, if

$$y'' + \frac{2}{x}y' + y = 0\ ,$$

then y'' and higher derivatives do not exist at x =0. It is the aim of this study to study the singular problem (2), with initial condition and to make further progress beyond the achievements made so far in this regard. Next aim consists in testing the proposed algorithm in handling a generalization of this type of problems. A difficult

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element in the analysis of this type of equations is the singularity behavior that occurs at x = 0.

THE METHOD

In theory, the infinite Taylor series can be used to evaluate a function, given its derivative function and its value at some point, consider the nonlinear first-order ODE:

$$y' = f(x, y),\ y(x_0) = y_0$$

The Taylor series for y (x) at $x = x_0$ is

$$y(x) = y(x - x_0) + y'(x - x_0)x + \frac{y''(x - x_0)}{2!}x^2 + \dots$$

We solving the Eq. 2 by the following steps :

(a)
$$y'' + \frac{2}{x}y' + f(x,y) = g(x), 0 < x \le 1 \qquad (3)$$
$$y(0) = A, y'(0) = B,$$

(b)
$$\text{from (a)}\ y' = \frac{1}{2}(xg(x)) - \frac{1}{2}(xf(x,y)) - \frac{1}{2}xy''$$

(c)
$$y'' = \frac{1}{3}(xg(x))' - \frac{1}{3}(xf(x,y))' - \frac{1}{3}xy'''$$
$$y''' = \frac{1}{4}(xg(x))'' - \frac{1}{4}(xf(x,y))'' - \frac{1}{4}xy^{(4)}$$
$$y^{(4)} = \frac{1}{5}(xg(x))''' - \frac{1}{5}(xf(x,y))''' - \frac{1}{5}xy^{(5)}$$
$$\vdots$$
$$\vdots$$
$$y^{(k)} = \frac{1}{k+1}(xg(x))^{(k-1)} - \frac{1}{k+1}(xf(x,y))^{(k-1)} - \frac{1}{k+1}xy^{(k+1)},\ k = 1, 2, 3, 4....$$

(d) Hence x = 0. Y (0) = A, y (0) = B, we can fined.
$y''(0)$, $y'''(0)$, $y^{(4)}(0)$,..., $y^{(k)}(0)$.

(e) By taylor series method with x = 0. We have

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \dots$$

substituting (e) the initial condition and values found in (d) we obtain the solution.

NUMERICAL ILLUSTRATIONS

Example 1: We consider the linear singular initial value problem:

$$y'' + \frac{2}{x}y' + y = 6 + 12x + x^2 + x^3, \qquad (4)$$
$$y(0) = 0, y'(0) = 0.$$

By (3), Eq. 4 becomes

$$y' = \frac{1}{2}[6x + 12x^2 + x^3 + x^4] - \frac{1}{2}xy - \frac{1}{2}xy'',$$

than

$$y'' = \frac{1}{3}[6 + 24x + 3x^2 + 4x^3] - \frac{1}{3}[xy' + y] - \frac{1}{3}xy''',$$
$$y''' = \frac{1}{4}[24 + 6x + 12x^2] - \frac{1}{4}[xy'' + 2y'] - \frac{1}{4}xy^{(4)},$$
$$y^{(4)} = \frac{1}{5}[6 + 24x] - \frac{1}{5}[xy''' + 3y''] - \frac{1}{5}xy^{(5)},$$
$$y^{(5)} = \frac{24}{6} - \frac{1}{6}[xy^{(4)} + 4y'''] - \frac{1}{6}xy^{(6)},$$
$$\vdots$$
$$\vdots$$

Hence x = 0, y (0) = 0, $y'(0) = 0$ we can find

$$y''(0) = 2,\ y'''(0) = 6,\ y^{(4)} = 0,\ y^{(5)} = 0,\ y^{(6)} = 0, \dots$$

by Taylor series method with x = 0 we have

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + y'''(0)\frac{x^3}{3!} + \dots$$

substituting the initial condition and the values find, we obtain the solution $y = x^2 + x^3$.

Example 2: We consider the linear singular initial value problem

$$y'' + \frac{2}{x}y' = 2(2x^2 + 3)y,$$
$$y(0) = 1, y'(0) = 0.$$
$$y' = \frac{1}{2}[(4x^3 + 6x)y] - \frac{1}{2}xy'',$$
$$y'' = \frac{1}{3}[(12x^2 + 6)y + (4x^3 + 6x)y'] - \frac{1}{3}xy''',$$
$$y''' = \frac{1}{4}[24xy + 2(12x^2 + 6)y' + (4x^3 + 6x)y''] - \frac{1}{4}xy^{(4)},$$
$$y^{(4)} = \frac{1}{5}[24xy' + 24y + 48xy' + 3(12x^2 + 6)y'' + (4x^3 + 6x)y'''] - \frac{1}{5}xy^{(5)},$$
$$y^{(5)} = \frac{1}{6}[96y' + 144xy'' + 4(12x^2 + 6)y''' + (4x^3 + 6x)y^{(4)}] - \frac{1}{6}xy^{(6)},$$
$$y^{(6)} = \frac{1}{7}[240y'' + 240xy''' + 5(12x^2 + 6)y^{(4)} + (4x^3 + 6x)y^{(5)}] - \frac{1}{7}xy^{(7)},$$
$$\vdots$$
$$\vdots$$

Hence x = 0, y (0) = 1, $y'(0) = 0$ we can find
$y''(0) = 2$, $y'''(0) = 0$, $y^{(4)}(0) = 12$, $y^{(5)}(0) = 0$, $y^{(6)}(0) = 120, \ldots$

by Taylor series method with x = 0 we have

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + y'''(0)\frac{x^3}{3!} + \ldots$$

substituting the initial condition and the values find, we obtain the solution

$$y(x) = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \ldots$$

and in the closed form $y(x) = e^{x2}$.

GENERALIZATION

A generalization of the Lane-Emaden- like Eq. 1 has been studied by Wazwaz (2001). We replace the standard coefficient of y' by $\frac{n}{x}$, for real n. In other words, a general equation,

$$y'' + \frac{n}{x}y' + f(x, y) = g(x), n \in R, \tag{5}$$

With initial conditions

$$Y(0) = A, \; y'(0) = B,$$

can be formulated. It is convenient to consider a modification to the approach presented before in order to enable us to handle (5). It is convenient to introduce the derivative of order k. From Eq. 5,

$$y' = \frac{1}{n}(xg(x)) - \frac{1}{n}(xf(x,y)) - \frac{1}{n}xy''$$

we can get the higher derivatives for y in the following

$$\begin{aligned} y'' &= \frac{1}{n+1}(xg(x))' - \frac{1}{n+1}(xf(x,y))' - \frac{1}{n+1}xy''', \\ y''' &= \frac{1}{n+2}(xg(x))'' - \frac{1}{n+2}(xf(x,y))'' - \frac{1}{n+2}xy^{(4)}, \\ y^{(4)} &= \frac{1}{n+3}(xg(x))''' - \frac{1}{n+3}(xf(x,y))''' - \frac{1}{n+3}xy^{(5)}, \\ &\vdots \\ &\vdots \\ y^{(k)} &= \frac{1}{n+(k-1)}(xg(x))^{(k-1)} - \frac{1}{n+(k-1)}(xf(x,y))^{(k-1)} \\ &\quad - \frac{1}{n+(k-1)}xy^{(k+1)} \; k = 1,2,3,\ldots \end{aligned} \tag{6}$$

Hence x = 0, y (0) = A, $y'(0) = B$, we can find

$$y''(0), y'''(0), y^{(4)}(0), \ldots, y^{(k)}(0).$$

By taylor series method with x = 0. We have

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \ldots$$

substituting the initial condition and the values find, we obtain the solution.

To illustrate the generalization discussed above, we discuss this example:

Example 3. We consider the linear initial value problem:

$$y'' + \frac{8}{x}y' + xy = x^5 - x^4 + 44x^2 - 30x, \quad y(0) = 0, y'(0) = 0. \tag{7}$$

By (6), Eq. 7 becomes

$$y' = \frac{1}{8}[x^6 - x^5 + 44x^3 - 30x^2] - \frac{1}{8}(x^2y) - \frac{1}{8}xy''$$

than

$$\begin{aligned} y'' &= \frac{1}{9}[6x^5 - 5x^4 + 132x^2 - 60x] - \frac{1}{9}(x^2y' + 2xy) - \frac{1}{9}xy''' \\ y''' &= \frac{1}{10}[30x^4 - 20x^3 + 264x - 60] - \frac{1}{10}[x^2y'' + 4xy' + 2y] - \frac{1}{10}xy^{(4)} \\ y^{(4)} &= \frac{1}{11}[120x^3 - 60x^2 + 264] - \frac{1}{11}[x^2y''' + 6xy'' + 6y'] - \frac{1}{11}xy^{(5)} \\ y^{(5)} &= \frac{1}{12}[360x^2 - 120x] - \frac{1}{12}[x^2y^{(4)} + 8xy''' + 12y''] - \frac{1}{12}xy^{(6)} \\ y^{(6)} &= \frac{1}{13}[720x - 120] - \frac{1}{13}[x^2y^{(5)} + 10xy^{(4)} + 20y'''] - \frac{1}{13}xy^{(7)} \\ &\vdots \\ &\vdots \end{aligned}$$

Hence x = 0, y (0) = 0, $y'(0) = 0$, we can find
$y''(0) = 0$, $y'''(0) = -6$, $y^{(4)}(0) = 24$, $y^{(5)}(0) = 0$, $y^{(6)}(0) = 0, \ldots 0$
by Taylor series method with x = 0 we have

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + y'''(0)\frac{x^3}{3!} + y^{(4)}(0)\frac{x^4}{4!} + \ldots$$

substituting the initial condition and the values find we obtain the solution $y(x) = x^4 - x^3$.

CONCLUSION

In the discussion it was shown that, with the proper use of the taylor series method, it is possible to obtain an analytic solution to a class of singular initial value problems, homogeneous or inhomogeneous. The difficulty in using a taylor series method directly to this type of equations, due to the existence of singular point at x = 0, is overcome here. The class of singular equations was generalized, by changing the coefficient of y' and the proposed technique was presented in a general way. This gives the proposed scheme a wider applicability.

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