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On the Approximate Solution of Singular Integral Equations with Hilbert Kernel

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Abstract: The mechanical quadrature method has been applied to a certain class of nonlinear singular integral equations with Hilbert kernel in generalized Hölder spaces. The rate of convergence of approximate solution has been determined. Two examples have been introduced to show that the application of mechanical quadrature method to a class of, nonlinear and linear, singular integral equations gives accurate results. These results are very acceptable compared to the exact solution. The obtained results of the mechanical quadrature method are better than the obtained results of the Toeplitz matrix method and the product Nystrom method applied to the same classes to obtain an approximate solution. The error at the interior points has been calculated.

Key words: Nonlinear singular integral equations, mechanical quadrature method, rate of convergence, generalized Hölder space

INTRODUCTION

The theory of singular integral equations has been developed significant importance during the last years, it is arise in many problems of mathematical physics, such as the theory of elasticity, hydrodynamics, biological problems, population genetics and others. Also, nonlinear singular integral equations with Hilbert and Cauchy kernel and its related Rimann-Hilbert problems have been studied by Pogorzelski (1966), Guseinov and Mukhtarov (1980), Wolfersdorf (1985) and Wegert (1992).

Existence results and approximate solutions for certain classes of nonlinear singular integral equations are studied by Amer and Nagdy (1999 and 2000), Amer and Dardery (2005), Amer (1996, 2001 and 2005), Jinyuan (2000) and Junghanns and Weber (1993). The theory of approximation methods and its application for the solution of linear and nonlinear singular integral equations has been developed by Guseinov and Mukhtarov (1980), Kravchenko and Akilov (1982), Ladopoulos and Zisis (1996) and Mikhlin and Prossdorf (1980).

It is well known that the nonlinear singular integral equations are the much-complicated forms of the nonlinear integral equations. The mechanical quadrature method is one of the basic tools to investigate the approximate solutions of many classes of nonlinear and linear equations involving integral operator. In this research we applied the mechanical quadrature method to a certain class of Nonlinear Singular Integral Equation (NSIE) with Hilbert kernel in generalized Hölder spaces. The method has been applied to a nonlinear and a linear Singular Integral Equation (SIE) with known exact solution and the error has been calculated. The obtained results of the mechanical quadrature method of SIE are compared with the obtained results of the Toeplitz matrix method and the product Nystrom method that have been applied (Abdou *et al*., 2002), to obtain the approximate solution of the same problem.

FORMULATION OF THE PROBLEM

This study is devoted to investigate the approximate solution of the following nonlinear singular integral equation:

$$u(t) = F(t, u(t), v(t)) \tag{1}$$

where:

$$v(t) = \frac{\lambda}{2\pi}\int_0^{2\pi} g(t, \tau, u(\tau)) \cot\frac{\tau - t}{2} d\tau \tag{2}$$

in generalized Hölder spaces $H_{\varphi,m}$ and $H^{(N)}_{\varphi,m}$, λ is a numerical parameter, under the following assumptions:

Assumption 1: Suppose that the function $g(t, \tau, u, (\tau))$ is defined on the domain

$$D = \{(t, \tau, u);\ 0 \le t, \tau \le 2\pi,\ \ |u| \le M, M > 0\}$$

that has partial derivatives up to (m-1) - order and satisfy the following Hölder-Lipschitz condition for arbitrary

$$t_n, \tau_n \in [0, 2\pi], u_n \in [-M, M], (n = 1,2)$$

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$$\left|\frac{\partial^\beta g(t_1,\tau_1,u_1)}{\partial t^i \partial\tau^j \partial u^k} - \frac{\partial^\beta g(t_2,\tau_2,u_2)}{\partial t^i \partial\tau^j \partial u^k}\right| \le \eta(\beta) \left[\varphi(|t_1-t_2|)+\varphi^*(|\tau_1-\tau_2|)+|u_1-u_2|\right] \quad (3)$$

where, φ, φ* are non-decreasing functions belong to the class Φ, $i + j + k = \beta$ and $\beta = 0, 1, 2, ..., m-1$ and (η)β is a constant depends on β.

Assumption 2: Suppose that the function F (t, u(t), v(t)) is defined on the domain:

$$D^* = \{(t,u,v);\ 0 \le t \le 2\pi,\ |u| \le M, |v| \le M;\ M > 0\}$$

that has partial derivatives up to (m-1) order and satisfy the following condition for arbitrary

$$t_n \in [0, 2\pi], u_n, v_n \in [-M, M],\ M > 0,\ \ (n = 1,2)$$

$$\left|\frac{\partial^\upsilon F(t_1,u_1,v_1)}{\partial t^p \partial u^q \partial v^r} - \frac{\partial^\upsilon F(t_2,u_2,v_2)}{\partial t^p \partial u^q \partial v^r}\right| \le \xi(\upsilon)\left[\varphi_1(|t_1-t_2|)+|u_1-u_2|+|v_1-v_2|\right] \quad (4)$$

for $p + q + r = \nu$, $\nu = 0, 1, 2, ..., m-1$, where, $\varphi_1 \in \Phi$ and ξ (ν) is a constant depends on ν. Equation 1 with Cauchy kernel has been studied by the collocation method (Amer, 1966), the special cases of Eq. 1 have been found (Ladopoulos and Zisis, 1996; Saleh and Amer, 1987).

Some basic definitions and auxiliary results: In this section we introduce some definitions and results which will be used in the sequel.

Definition 1: (Guseinov and Mukhatarov, 1980; Mikhlin and Prossdorf, 1986).

- We denote by Φ to the class of all continuous almost increasing functions φ defined on (0, π] such that $\varphi(\delta) > 0,\ \lim_{\delta\to 0^+}\varphi(\delta) = 0$.
- We denote by Φ^m to the class of all functions φ ∈ Φ such that $0 < \delta_1 < \delta_2 < \pi$ implies $\delta_1^m \varphi(\delta_2) \le c(m)\delta_2^m \varphi(\delta_1)$, where, c(m) is a constant depends on m.
- We denote by $c_{2\pi}$ to the space of 2π-periodic continuous functions with the norm

$$\|u\|_c = \max_{t\in[-\pi,\pi]}|u(t)|$$

- The generalized Hölder space $H_{\varphi,m}$ is defined as

$$H_{\varphi,m} = \{u \in c_{2\pi} : \omega_u^m(\delta) = O(\varphi(\delta)),\ \varphi \in H\Phi^m\}$$

where, $\omega_u^m(\delta)$ is the modulus of continuity of order m of the function u and

$$H\Phi^m = \left\{\varphi \in \Phi^m : \int_0^\delta \frac{\varphi(\xi)}{\xi}d\xi + \delta^m \int_\delta^\pi \frac{\varphi(\xi)}{\xi^{m+1}}d\xi \le \tilde{c}(m)\,\varphi(\delta)\right\}$$

where, $\tilde{c}(m)$ is a constant depends on m.

- For $u \in H_{\varphi,m}$ we define

$$\|u\|_{\varphi,m} = \|u\|_c + \sup_{0<\delta\le\pi}\frac{\omega_u^m(\delta)}{\varphi(\delta)}$$

and

$$H_{\varphi,m}(M) = \{u \in H_{\varphi,m} : \|u\|_{\varphi,m} \le M,\ M > 0\}$$

as a subspace of $H_{\varphi,m}$

Definition 2: (Mosaev and Salaev, 1980; Saleh and Amer, 1987).

- Let the generalized Hölder space $H_{\varphi,m}^{(N)}$, $\varphi \in \Phi^m$, be the space of 2N-dimensional vectors $z = (z_0, z_1, ..., z_{2N-1})$ with the norm:

$$\|z\|_{\varphi,m}^{(N)} = \max\left\{\max_{i=0,\dots,2N-1}|z_i|,\ \max_{\substack{p\in X\\ p\ne 0}}\frac{\omega_X^m(z,p)}{\varphi(\pi p/N)},\ \max_{\substack{p\in X\\ p\ne 0}}\frac{\omega_Y^m(z,p)}{\varphi(\pi p/N)}\right\}$$

where:

$$\omega_X^m(z,p) = \max_{\substack{i\in[0,2N-1-mh]\cap X\\ h\in[0,p]\cap X}}|\Delta_h^m z_i|$$

and

$$\omega_Y^m(z,p) = \max_{\substack{i\in[0,2N-1-mh]\cap Y\\ h\in[0,p]\cap X}}|\Delta_h^m z_i|$$

are the modulus of continuity of order m of the vector z with respect to the two sets X = {0, 2,..., 2N-2}, Y = {1, 3, ..., 2N-1} and p ∈ X,

$$\Delta_h^m z_i = \sum_{k=0}^m (-1)^{m-k}\binom{m}{k} z_{i+kh}$$

For $z \in H_{\varphi,m}^{(N)}$ we define:

$$H_{\varphi,m}^{(N)}(M)=\left\{z\in H_{\varphi,m}^{(N)}:\ \|z\|_{\varphi,m}^{(N)}\le M,\ \ M>0\right\}$$

as a subspace of $H_{\varphi,m}^{(N)}$.

- We denote the norm in the space $L_p^{(N)}$ by

$$\|z\|_{L_p^{(N)}}=\left(\frac{\pi}{N}\sum_{k=0}^{2N-1}|z_k|^p\right)^{1/p},\quad p>1$$

Theorem 1 (Mosaev and Salaev, 1980; Saleh, 1984): Let $\varphi\in H\Phi^m$, then the operator

$$(Au)(t)=\frac{1}{2\pi}\int_0^{2\pi}u(\tau)\cot\frac{\tau-t}{2}d\tau$$

Transforms $H_{\varphi,m}(M)$ into $H_{\varphi,m}(\tilde{M})$, where:

$$\tilde{M}=M\left\{e_1(m)\int_0^{\pi}\frac{\varphi(\delta)}{\delta}d\delta+e_1(m)+e_2(m)\tilde{c}(m)\right\}$$

where, $e_1(m)$, $e_2(m)$ and $\tilde{c}(m)$ are constants depend on m.

Lemma 1 (Saleh and Amer, 1987): Let the condition (3) is satisfied and $u(t)\in H_{\varphi,m}$ then $g(t,\tau,u(\tau))\in H_{\varphi,m}$.

Lemma 2 (Saleh and Amer, 1990): Let the condition (4) is satisfied and $u(t),\ v(t)\in H_{\varphi,m}$ then $F(t,u,v)\in H_{\varphi,m}$.

The approximate solution in the space $H_{\varphi,m}$

Theorem 2: Let the function F(t, u(t), v(t)) satisfy the condition (4) and the function g(t, τ, u(τ)) satisfy the condition (3), then for $|\lambda|<\lambda_0$, (λ_0 sufficiently small), the Eq. 1 has a unique solution in $H_{\varphi,m}(M)$. The solution is uniformly convergent and can be obtained by the method of successive approximations.

Proof: Let u, v ∈ $H_{\varphi,m}(M)$. Then by Lemmas 1, 2 and Theorem 1, the operator

$$(Pu)(t)=F(t,u(t),v(t)) \tag{5}$$

transforms the space $H_{\varphi,m}(M)$ into the space $H_{\varphi,m}$ ($|\lambda|R$). Therefore if $|\lambda|R\le M$, the operator P transforms $H_{\varphi,m}(M)$ into itself. Using M. Riesz's Theorem (Kravchenko and Akilov, 1982; Saleh and Amer, 1987).

$$\|\tilde{u}\|_{L_p}\le\gamma(p)\|u\|_{L_p},\ p>1 \tag{6}$$

where:

$$\tilde{u}(t)=\frac{1}{2\pi}\int_0^{2\pi}u(\tau)\cot\frac{\tau-t}{2}d\tau$$

and from the conditions (3), (4), we obtain

$$\|Pu_1-Pu_2\|_{L_p}\le\xi(0)\left(1+|\lambda|\gamma(P)\eta(0)\right)\|u_1-u_2\|_{L_p} \tag{7}$$

Choosing

$$|\lambda|<\min\left\{\frac{M}{R},\frac{1-\xi(0)}{\gamma(P)\eta(0)\xi(0)}\right\}=\lambda_0$$

and

$$\xi(0)\left(1+|\lambda|\gamma(P)\eta(0)\right)<1,$$

then the operator P is a contraction mapping. From the completeness of $H_{\varphi,m}(M)$ in L_p, p>1, the Eq. 1 has a unique solution in the subspace $H_{\varphi,m}(M)$ and this solution can be found by the method of successive approximations.

The approximate solution in the space $H_{\varphi,m}^{(N)}$: By the mechanical quadrature formula (Saleh, 1984), the integral

$$(Iu)(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}u(\tau)\cot\frac{\tau-t}{2}d\tau \tag{8}$$

takes the following form:

$$(Iu)(t)=\frac{1}{N}\sum_{i=0}^{2N-1}u_i\sin^2\frac{t-t_i}{2}\cot\frac{t_i-t}{2} \tag{9}$$

where:

$$u_i=u(t_i),\ t_i=\frac{i\pi}{N}$$

the formula (9) at node points t_i takes the form:

$$(Iu)(t_j)=\frac{1}{2N}\sum_{\substack{i=0\\ i\ne j}}^{2N-1}u_i\,\alpha_{i-j}\cot\frac{t_i-t_j}{2} \tag{10}$$

where:

$$\alpha_{i-j}=\begin{cases}0 & ,\ i-j\ \text{even},\\ 2 & ,\ i-j\ \text{odd}.\end{cases}$$

Applying the quadrature formula (10) to the Eq. 1 at the node points, we obtain:

$$u(t_j) = F(t_j, u(t_j), \frac{\lambda}{2N}\sum_{\substack{i=0\\i\neq j}}^{2N-1} g(t_j, t_i, u(t_i))$$

$$\alpha_{i-j}\cot\frac{t_i - t_j}{2} + R_N(g, t_j))$$

where, $R_N(g, t_j)$ is the remainder term, $j = \overline{0, 2N-1}$. If we put $u(t_j) = z_j$ and the $R_N(g, t_j)$ is negligible, we obtain the following system of nonlinear algebraic equations:

$$z_j = F\left(t_j, z_j, \frac{\lambda}{2N}\sum_{\substack{i=0\\i\neq j}}^{2N-1} g(t_j, t_i, z_i)\alpha_{i-j}\cot\frac{t_i - t_j}{2}\right) \quad (11)$$

Lemma 3 (Guseinov and Mukhatarov, 1980; Amer, 2001): If the function g (t, τ, u) and its derivative g_t (t, τ, u) satisfy the condition (3), then the function:

$$\chi(t, \tau, u) = g(t, \tau, u) - g\tau, \tau, u)$$

satisfies the following condition

$$\left|\chi(t, \tau, u_1) - \chi(t, \tau, u_2)\right| \leq \eta(1)\left|t - \tau\right|\left|u_1 - u_2\right| \quad (12)$$

where:

$$u_1, u_2 \in [-M, M]$$

Theorem 4.1: Let the function F(t, τ, u) satisfy the condition (4) and the function g(t, τ, u) satisfy the condition (3), then the system (11) has a unique solution in the space $H_{\varphi,m}^{(N)}(M)$ for arbitrary N≥3 and this solution can be found by the method of successive approximations.

Proof: From definition 2, we have

$$H_{\varphi,m}^{(N)}(M) = \left\{ z \in H_{\varphi,m}^{(N)} : \| z \|_{\varphi,m}^{(N)} \leq M, \ M > 0 \right\}$$

where:

$$z = (z_0, z_1, \ldots, z_{2N-1})$$

Putting

$$Jz = \left(g(t_0, \tau_0, z_0), \ldots, g(t_{2N-1}, \tau_{2N-1}, z_{2N-1})\right)$$

since the space $H_{\varphi,m}^{(N)}(M)$ of vectors of bounded norms is a closed subspace of $L_p^{(N)}$ and the function g(t, τ, u) satisfies the condition of Lemmas 1, 2 and Theorem 2 hence $Jz \in H_{\varphi,m}^{(N)}(R')$.

Taking

$$P^{(N)}(z) = \left(F\left(t_0, z_0, K_0^{(N)}z\right), \ldots, F\left(t_{2N-1}, z_{2N-1}, K_{2N-1}^{(N)}z\right)\right)$$

$$K^{(N)}z = \left(K_o^{(N)}z, \ldots, K_{2N-1}^{(N)}z\right)$$

where:

$$K_j^{(N)}z = \frac{\lambda}{2N}\sum_{\substack{i=0\\i\neq j}}^{2N-1} g(t_j, t_i, z_i)\alpha_{i-j}\cot\frac{t_i - t_j}{2}$$

let

$$K^{(N)}z = \lambda E^{(N)}Jz$$

where:

$$E^{(N)}z = \left(E_0^{(N)}z, \ldots, E_{2N-1}^{(N)}z\right)$$

$$E_j^{(N)}z = \frac{1}{2N}\sum_{\substack{i=0\\i\neq j}}^{2N-1} z_{j,i}\,\alpha_{i-j}\cot\left(\frac{i-j}{2N}\right)\pi$$

and

$$\left\|E^{(N)}\right\|_{\varphi,m}^{(N)} \leq \theta(m), \text{ (Saleh and Amer, 1987)}$$

where, θ(m) is a constant depends on m. Thus we have:

$$\left\|K^{(N)}z\right\|_{\varphi,m}^{(N)} \leq |\lambda| R'\theta(m)$$

Now, let

$$z^{(k)} = \left(z_0^{(k)}, z_1^{(k)}, \ldots, z_{2N-1}^{(k)}\right) \in H_{\varphi,m}^{(N)}(M), \ k = 1, 2$$

Hence,

$$\left\|P^{(N)}z^{(1)} - P^{(N)}z^{(2)}\right\|_{L_p^{(N)}} = \left\|F\left(t, z^{(1)}, K^{(N)}z^{(1)}\right) - F\left(t, z^{(2)}, K^{(N)}z^{(2)}\right)\right\|_{L_p^{(N)}}$$

$$\leq \xi(0)\left(\left\|z^{(1)} - z^{(2)}\right\|_{L_p^{(N)}} + \left\|K^{(N)}z^{(1)} - K^{(N)}z^{(2)}\right\|_{L_p^{(N)}}\right) \quad (13)$$

since

$$\|K^{(N)}\|_{L_p^{(N)}} \le q(p),\ p>1, \text{(Saleh, 1984)} \quad (14)$$

where, q(p) is a constant depends on p.
Hence from condition (3), Lemma 3 and from (Saleh and Amer, 1987) we obtain:

$$\|K^{(N)}z^{(1)} - K^{(N)}z^{(2)}\|_{L_p^{(N)}} \le |\lambda| q(p)\eta(0)\|z^{(1)} - z^{(2)}\|_{L_p^{(N)}} + \left\{\frac{|\lambda|\eta(1)}{\pi^{1/q}}\left[\int_0^{11\pi/6}\left(\frac{x/2}{\sin x/2}\right)^q dx\right]^{1/q}\right\}\|z^{(1)} - z^{(2)}\|_{L_p^{(N)}} \quad (15)$$

substituting from inequality (15) into (13), we get:

$$\|P^{(N)}z^{(1)} - P^{(N)}z^{(2)}\|_{L_p^{(N)}} \le \xi(0)\|z^{(1)} - z^{(2)}\|_{L_p^{(N)}} + \xi(0)|\lambda|\left\{q(p)\eta(0) + \frac{\eta(1)}{\pi^{1/q}}\left[\int_0^{11\pi/6}\left(\frac{x/2}{\sin x/2}\right)^q dx\right]^{1/q}\right\} \|z^{(1)} - z^{(2)}\|_{L_p^{(N)}} \quad (16)$$

From boundedness of the operator $K^{(N)}$ in $L_p^{(N)}$ and by using the principle of contraction mapping at

$$|\lambda| < \min\left\{\frac{M}{R^{\cdot}\theta(m)}, \frac{1-\xi(0)}{\xi(0)}\left(q(p)\eta(0) + \frac{\eta(1)}{\pi^{1/q}}\left[\int_0^{11\pi/6}\left(\frac{x/2}{\sin x/2}\right)^q dx\right]^{1/q}\right)^{-1}\right\} \quad (17)$$

the system (11) has a unique solution in $H_{\varphi,m}^{(N)}(M)$ for arbitrary N≥3, hence the theorem is proved.

The rate of convergence of the approximate solution: From inequality (17), the Eq. 1 has a unique solution $u^*(t) \in H_{\varphi,m}(M)$ and the system (11) has a unique solution $z^* = \left(z_0^*, z_1^*, \ldots, z_{2N-1}^*\right) \in H_{\varphi,m}^{(N)}(M)$.

The relation

$$u_N^*(t) = F\left(t, u_N^*(t), \frac{\lambda}{N}\sum_{i=0}^{2N-1} g\left(t, t_i, z_i^*\right)\sin^2\frac{t-t_i}{2}\cot\frac{t_i - t}{2}\right) \quad (18)$$

at $t = t_j$ is called the approximate solution of the Eq. 1, $u_N^*(t) = z_j^*$, $(j = \overline{0, 2N-1})$. The norm of the difference of the vectors z* and u* where, $u^* = (u(t_0), u(t_1), \ldots, u(t_{2N-1}))$ in $L_p^{(N)}$ can be determined as follows:

Applying the quadrature formula (10) to Eq. 1 at node points t_j, we obtain

$$u^*(t_j) = F\left(t_j, u^*(t_j), \frac{\lambda}{2N}\sum_{i=0}^{2N-1} g(t_j, t_i, u^*(t_i))\,\alpha_{i-j}\cot\frac{t_i - t_j}{2} + R_N(g, t_j)\right) \quad (19)$$

putting $z^{(1)} = u^*, z^{(2)} = z^*$ in (16) and using the inequality (17), we get:

$$\|u^* - z^*\|_{L_p^{(N)}} \le \xi(0)\|u^* - z^*\|_{L_p^{(N)}} + |\lambda|\xi(0)\|R_N(g,t)\|_c + \xi(0)|\lambda|\left\{q(p)\eta(0) + \frac{\eta(1)}{\pi^{1/q}}\left[\int_0^{11\pi/6}\left(\frac{x/2}{\sin x/2}\right)^q dx\right]^{1/q}\right\}\|u^* - z^*\|_{L_p^{(N)}}$$

consequently, we have

$$\|u^* - z^*\|_{L_p^{(N)}} \le \frac{|\lambda|\xi(0)\|R_N(g,t)\|_c}{1 - \xi(0) - \xi(0)|\lambda|\left\{q(p)\eta(0) + \frac{\eta(1)}{\pi^{1/q}}\left[\int_0^{11\pi/6}\left(\frac{x/2}{\sin x/2}\right)^q dx\right]^{1/q}\right\}} \quad (20)$$

To evaluate $\|u^*(t) - u_N^*(t)\|_c$, we state the following two lemmas:

Lemma 4 (Saleh, 1984): Let $z = (z_0, z_1, \ldots, z_{2N-1}) \in H_{\varphi,m}^{(N)}(M)$, $\varphi \in \Phi^m$. Then for arbitrary natural number h, $0 < h < \frac{N}{2(m+1)}$, we get

$$\max_i |z_i| \le l(m,M)\left[\left(\frac{N}{h}\right)^{1/p}\|z\|_{L_p^{(N)}} + \varphi\left(\frac{\pi h}{N}\right)\right]$$

Lemma 5 (Saleh, 1984):

$$\frac{1}{N}\sum_{i=0}^{2N-1}\left|\sin^2\frac{t-t_i}{2}\cot\frac{t_i-t}{2}\right|\le 2(1+\pi)\big(1+\ln(2N)\big)$$

Applying formula (9) on the Eq. 1 and from Eq. 18, we obtain:

$$\left\|u^*(t)-u_N^*(t)\right\|_c\le\xi(0)\left\|u^*(t)-u_N^*(t)\right\|_c+\xi(0)\eta(0)|\lambda|\max_i\left|u^*(t_i)-z_i^*\right|$$
$$\left(\frac{1}{N}\sum_{i=0}^{2N-1}\left|\sin^2\frac{t-t_i}{2}\cot\frac{t_i-t}{2}\right|\right)+\xi(0)|\lambda|\left\|R_N\right\|_c.$$

Using Lemma 5, we have:

$$\left\|u^*(t)-u_N^*(t)\right\|_c\le\frac{2\xi(0)\eta(0)}{1-\xi(0)}|\lambda|(1+\pi)(1+\ln 2N)\max_i\left|u^*(t_i)-z_i^*\right|$$
$$+\frac{\xi(0)}{1-\xi(0)}|\lambda|\left\|R_N\right\|_c, \tag{21}$$

from Lemma 4 and inequality (20), we get:

$$\max_i\left|u^*(t_i)-z_i^*\right|\le l(m,M)\min_{2\le h\le N/2(m+1)}\left[\xi(0)|\lambda|\left(\frac{N}{h}\right)^{1/p}\left\|R_N(g,t)z\right\|_c+\varphi\left(\frac{\pi h}{N}\right)\right]$$

since

$$\left\|R_N(g,t)\right\|_c\le l(m)\varphi(\pi/N)\ln N,\text{(Mosaev and Salaev, 1980; Saleh, 1984)} \tag{22}$$

therefore,

$$\max_i\left|u^*(t_i)-z_i^*\right|\le 1(m,M)|\lambda|\min_{2\le h\le N/2(m+1)}\left[\xi(0)\left(\frac{N}{h}\right)^{1/p}\varphi\left(\frac{\pi}{N}\right)\ln N+\varphi\left(\frac{\pi h}{N}\right)\right]$$

taking $h = N^{\alpha}$, $p^{-}1 < \alpha < 1$, then

$$\max_i\left|u^*(t_i)-z_i^*\right|\le \text{const}\left[\xi(0)\big(\varphi(\pi/N)\ln N\big)N^{(p^{-1}-\alpha)}+\varphi\left(\frac{\pi}{N^{1-\alpha}}\right)\right] \tag{23}$$

consequently, from (21), (22) and (23), we obtain

$$\left\|u^*(t)-u_N^*(t)\right\|_c\le\frac{2\xi(0)\eta(0)}{1-\xi(0)}|\lambda|(1+\pi)(1+\ln 2N)\left[\xi(0)\left(\ln N/N^{\alpha-1/p}\right)\varphi(\pi/N)+\varphi\left(\pi/N^{1-\alpha}\right)\right]$$
$$+\frac{\xi(0)}{1-\xi(0)}|\lambda|\,l(m)\varphi(\pi/N)\ln N.$$

Hence,

$$\left\|u^*(t)-u_N^*(t)\right\|_c\le\text{const}\left(\ln^2(N)/N^{\alpha-p^{-1}}\right)$$

Now we present the following examples.

Example 1: Consider the integral equation:

$$u(t)=\frac{\lambda}{2\pi}\int_0^{2\pi} g(t,\tau,u(\tau))\cot\frac{\tau-t}{2}d\tau+f(t) \qquad (24)$$

where:

$$g\big(t,\tau,u(\tau)\big)=\sin t\,\sin u(\tau), \qquad f(t)=t-\sin t\cos t$$

It is easy to check that u* (t) = t is the exact solution of Eq. 24 at λ = 1. Applying the quadrature formula (10) to Eq. 24 at node points, we obtain the following system of nonlinear algebraic equations:

$$z_j^*=\frac{\lambda}{2N}\sum_{\substack{i=0\\ i\neq j}}^{2N-1}\sin t_j\sin z_i^*\,\alpha_{i-j}\cot\frac{t_i-t_j}{2}+t_j-\sin t_j\cos t_j$$

where:

$$t_j=j\pi/N,\ j=0,1,\ldots,2N-1$$

Table 1 displays the exact solution, the approximate solution and error between them for the Eq. 24 by using the mechanical quadrature method with N = 20, λ = 1 and at initial values $z_j^*=0$, $j=0,1,\ldots,2N-1$.

Now, we apply the mechanical quadrature method to a class of LSIE.

Example 2: Consider the integral equation:

$$u(t)-\lambda\int_{-\pi}^{\pi} u(\tau)\cot\frac{\tau-t}{2}d\tau=f(t) \qquad (25)$$

under the condition

$$u(\pm\pi)=0 \qquad (26)$$

where:

$$f(t)=\sin t-2\pi\cos t$$

It is clear that u*(t) = sin t is the exact solution of Eq. 25 at λ = 1. Applying the quadrature formula (10) to Eq. 25 under the condition (26) at the node points t_j, we obtain the following system of linear algebraic equations:

$$z_j^*=\frac{\lambda}{2N}\sum_{\substack{i=-N+1\\ i\neq j}}^{N-1} z_i^*\alpha_{i-j}\cot\frac{t_i-t_j}{2}+\sin t_j-2\pi\cos t_j$$

Table 1: The results for the Eq. 24 by using the mechanical quadrature method

t	u*	z*	E
0.0000000	0.0000000	0.0000000	0.000000000
0.1570796	0.1570796	0.1570795	1.490116E-07
0.3141593	0.3141593	0.3141592	2.980232E-08
0.4712389	0.4712389	0.4712395	5.960464E-07
0.6283185	0.6283185	0.6283190	4.172325E-07
0.7853982	0.7853982	0.7853984	2.384186E-07
0.9424778	0.9424778	0.9424775	2.980232E-07
1.099557	1.099557	1.099557	4.768372E-07
1.256637	1.256637	1.256638	4.768372E-07
1.413717	1.413717	1.413717	4.768372E-07
1.570796	1.570796	1.570797	4.768372E-07
1.727876	1.727876	1.727875	1.072884E-06
1.884956	1.884956	1.884955	5.960464E-07
2.042035	2.042035	2.042036	4.768372E-07
2.199115	2.199115	2.199115	2.384186E-07
2.356194	2.356194	2.356194	4.768372E-07
2.513274	2.513274	2.513274	2.384186E-07
2.670354	2.670354	2.670354	0.000000E+00
2.827434	2.827434	2.827434	4.768372E-07
2.984513	2.984513	2.984513	0.000000E+00
3.141593	3.141593	3.141593	2.384186E-07
3.298672	3.298672	3.298672	0.000000E+00
3.455752	3.455752	3.455752	0.000000E+00
3.612832	3.612832	3.612832	2.384186E-07
3.769911	3.769911	3.769912	2.384186E-07
3.926991	3.926991	3.926991	4.768372E-07
4.084071	4.084071	4.084071	4.768372E-07
4.241150	4.241150	4.241152	1.907349E-06
4.398230	4.398230	4.398227	2.384186E-06
4.555309	4.555309	4.555309	0.000000E+00
4.712389	4.712389	4.712389	0.000000E+00
4.869469	4.869469	4.869469	0.000000E+00
5.026548	5.026548	5.026549	4.768372E-07
5.183628	5.183628	5.183627	9.536743E-07
5.340708	5.340708	5.340708	4.768372E-07
5.497787	5.497787	5.497787	0.000000E+00
5.654867	5.654867	5.654868	9.536743E-07
5.811946	5.811946	5.811946	0.000000E+00
5.969026	5.969026	5.969025	9.536743E-07
6.126106	6.126106	6.126106	0.000000E+00

u* = The exact solution, z* = The approximate solution, E = The error

where:

$$t_j=j\pi/N,\ j=-N+1,\ldots,N-1$$

The condition u(±π) = 0 reduces the node points t_j to 2N-1 points.

Table 2 displays the exact solution, the approximate solution and the error between them for the Eq. 25 under the condition (26) by using the mechanical quadrature method with N = 20, λ = 1.

CONCLUSIONS

- Table 1 and 2 display that the mechanical quadrature method gives accurate results with respect to NSIE and LSIE, these results are very acceptable compared to the exact solution.

Table 2: The results of Eq. 25 by using the mechanical quadrature method

t	u*	z*	E
-0.298451E+01	-0.1564343E+00	-0.1564343E+00	1.192093E-07
-0.282743E+01	-0.3090165E+00	-0.3090165E+00	2.682209E-07
-0.267035E+01	-0.4539907E+00	-0.4539907E+00	2.980232E-07
-0.251327E+01	-0.5877851E+00	-0.5877851E+00	5.960464E-08
-0.235619E+01	-0.7071072E+00	-0.7071072E+00	4.768372E-07
-0.219911E+01	-0.8090172E+00	-0.8090172E+00	2.384186E-07
-0.204204E+01	-0.8910056E+00	-0.8910056E+00	8.940697E-07
-0.188496E+01	-0.9510571E+00	-0.9510571E+00	6.556511E-07
-0.172788E+01	-0.9876884E+00	-0.9876884E+00	0.000000E+00
-0.157080E+01	-0.1000000E+01	-0.9999997E+00	2.980232E-07
-0.141372E+01	-0.9876881E+00	-0.9876881E+00	2.384186E-07
-0.125664E+01	-0.9510569E+00	-0.9510569E\+00	3.576279E-07
-0.109956E+01	-0.8910068E+00	-0.8910068E+00	2.384186E-07
-0.942478E+00	-0.8090168E+00	-0.8090168E+00	1.788139E-07
-0.785398E+00	-0.7071068E+00	-0.7071068E+00	0.000000E+00
-0.628319E+00	-0.5877856E+00	-0.5877856E+00	3.576279E-07
-0.471239E+00	-0.4539903E+00	-0.4539903E+00	1.788139E-07
-0.314159E+00	-0.3090171E+00	-0.3090171E+00	1.192093E-07
-0.157080E+00	-0.1564343E+00	-0.1564343E+00	1.788139E-07
0.000000E+00	0.0000000E+00	-0.9783179E-07	9.783179E-08
0.157080E+00	0.1564345E+00	0.1564345E+00	5.960464E-08
0.314159E+00	0.3090171E+00	0.3090171E+00	1.192093E-07
0.471239E+00	0.4539906E+00	0.4539906E+00	5.960464E-08
0.628319E+00	0.5877854E+00	0.5877854E+00	1.192093E-07
0.785399E+00	0.7071070E+00	0.7071070E+00	2.384186E-07
0.942478E+00	0.8090170E+00	0.8090170E+00	0.000000E+00
0.109956E+01	0.8910061E+00	0.8910061E+00	4.768372E-07
0.125664E+01	0.9510570E+00	0.9510570E+00	4.172325E-07
0.141372E+01	0.9876889E+00	0.9876889E+00	5.364418E-07
0.157080E+01	0.1000000E+01	0.9999995E+00	5.364418E-07
0.172788E+01	0.9876882E+00	0.9876882E+00	1.192093E-07
0.188496E+01	0.9510567E+00	0.9510567E+00	2.384186E-07
0.204204E+01	0.8910065E+00	0.8910065E+00	5.960464E-08
0.219912E+01	0.8090163E+00	0.8090163E+00	7.152557E-07
0.235619E+01	0.7071071E+00	0.7071071E+00	3.576279E-07
0.251327E+01	0.5877852E+00	0.5877852E+00	0.000000E+00
0.267035E+01	0.4539907E+00	0.4539907E+00	2.980232E-07
0.282743E+01	0.3090197E+00	0.3090169E+00	1.192093E-07
0.298451E+01	0.1564343E+00	0.1564343E+00	1.341105E-07

u* = The exact solution, z* = The approximate solution, E = The error

Table 3: The results of Eq. 25 by using the Toeplitz matrix method and the product Nystrom method

t	u	$n_n^{(T)}$	$R^{(T)}$	$n_n^{(N)}$	$R^{(N)}$
-0.298451E+01	-0.156434E+00	-0.157566E+00	0.113183E-02	-0.157904E+00	0.146956E-02
-0.282743E+01	-0.309017E+00	-0.306774E+00	0.224258E-02	-0.307179E+00	0.183761E-02
-0.267035E+01	-0.453990E+00	-0.456815E+00	0.282412E-02	-0.454282E+00	0.291620E-03
-0.251327E+01	-0.587785E+00	-0.587984E+00	0.198949E-03	-0.587120E+00	0.665654E-03
-0.235619E+01	-0.707107E+00	-0.710086E+00	0.297899E-02	-0.706959E+00	0.147790E-03
-0.219911E+01	-0.809017E+00	-0.810573E+00	0.155627E-02	-0.808990E+00	0.265871E-04
-0.204204E+01	-0.891007E+00	-0.894161E+00	0.315436E-02	-0.890639E+00	0.367293E-03
-0.188496E+01	-0.951057E+00	-0.953450E+00	0.239302E-02	-0.951288E+00	0.231250E-03
-0.172788E+01	-0.987688E+00	-0.990983E+00	0.329504E-02	-0.987275E+00	0.413104E-03
-0.157080E+01	-0.100000E+01	-0.100286E+00	0.285655E-02	-0.100020E+01	0.197142E-03
-0.141372E+01	-0.987688E+00	-0.991010E+00	0.332117E-02	-0.987354E+00	0.334045E-03
-0.125664E+01	-0.951057E+00	-0.954070E+00	0.301384E-02	-0.951009E+00	0.476252E-04
-0.109956E+01	-0.891007E+00	-0.894194E+00	0.318768E-02	-0.890833E+00	0.173579E-03
-0.942478E+00	-0.809017E+00	-0.811927E+00	0.291043E-02	-0.808591E+00	0.425900E-03
-0.785398E+00	-0.707107E+00	-0.709991E+00	0.288390E-02	-0.707135E+00	0.285526E-04
-0.628319E+00	-0.587785E+00	-0.590376E+00	0.259122E-02	-0.586921E+00	0.864526E-03
-0.471239E+00	-0.453990E+00	-0.456418E+00	0.242748E-02	-0.454227E+00	0.236237E-03
-0.314159E+00	-0.309017E+00	-0.311125E+00	0.210820E-02	-0.307721E+00	0.129617E-02
-0.157080E+00	-0.156434E+00	-0.158292E+00	0.185783E-02	-0.156852E+00	0.417934E-03
0.000000E+00	0.000000E+00	-0.152180E-02	0.152180E-02	0.166220E-02	0.166220E-02
0.157080E+00	0.156434E+00	0.155205E+00	0.122941E-02	0.155886E+00	0.548042E-03
0.314159E+00	0.309017E+00	0.308119E+00	0.898460E-03	0.310933E+00	0.191594E-02
0.471239E+00	0.453990E+00	0.453386E+00	0.604871E-03	0.453382E+00	0.608263E-03

Table 3: Continued

t	u	$n_n^{(T)}$	$R^{(T)}$	$n_n^{(N)}$	$R^{(N)}$
0.628319E+00	0.587785E+00	0.587479E+00	0.306284E-03	0.589811E+00	0.202547E-02
0.785398E+00	0.707107E+00	0.707059E+00	0.481999E-04	0.706518E+00	0.588612E-03
0.942478E+00	0.809017E+00	0.809207E+00	0.190301E-03	0.810993E+00	0.197563E-02
0.109956E+01	0.891007E+00	0.891388E+00	0.381596E-03	0.890519E+00	0.487828E-03
0.125664E+01	0.951057E+00	0.951592E+00	0.535616E-03	0.952825E+00	0.176875E-02
0.141372E+01	0.987688E+00	0.988324E+00	0.635698E-03	0.987375E+00	0.313036E-03
0.157080E+01	0.100000E+01	0.100069E+01	0.686574E-03	0.100142E+01	0.142411E-02
0.172788E+01	0.987688E+00	0.988367E+00	0.678469E-03	0.987610E+00	0.784910E-04
0.188496E+01	0.951057E+00	0.951670E+00	0.613645E-03	0.952033E+00	0.976132E-03
0.204204E+01	0.891007E+00	0.891492E+00	0.485934E-03	0.891203E+00	0.196925E-03
0.219911E+01	0.809017E+00	0.809312E+00	0.294613E-03	0.809489E+00	0.472495E-03
0.235619E+01	0.707107E+00	0.707138E+00	0.308381E-04	0.707602E+00	0.495305E-03
0.251327E+01	0.587785E+00	0.587465E+00	0.320492E-03	0.587762E+00	0.234950E-04
0.267035E+01	0.453990E+00	0.453182E+00	0.808603E-03	0.454816E+00	0.825687E-03
0.282743E+01	0.309017E+00	0.307399E+00	0.161841E-02	0.308643E+00	0.373926E-03
0.298451E+01	0.156434E+00	0.152392E+00	0.404226E-02	0.157947E+00	0.151280E-02

u = The exact solution, $n_n^{(N)}$ = The approximate solution using product Nystrom method and $R^{(N)}$ = The error, $n_n^{(T)}$ = The approximate solution using Toeplitz matrix method and $R^{(T)}$ = The error

- The Toeplitz matrix method and the product Nystrom method have been applied to the same Eq. 25 under the condition (26) by Abdou *et al*. (2002), Table 3 displays the values of exact solution u(t) = sin t, approximate solution $u_n^{(T)}$ and the error $R^{(T)}$ at the interior points by using the Toeplitz matrix method with N = 20, λ = 1. Also it shows the approximate solution $u_n^{(N)}$ and the error $R^{(T)}$ at the interior points by using the product Nystrom method with N = 20, λ = 1.
- Its found that, the obtained results of the mechanical quadrature method are better than the obtained results of the Toeplitz matrix method and the product Nystrom method that have been applied in Abdou *et al*. (2002) to obtain the approximate solution of the same problem (25) under the same condition (26).

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