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Algebraic Structure of Lattices of SK Partitions

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Abstract: In the context of Complexity Reduction in Lattice-Based Information Retrieval, the reduction process must preserve the algebraic structure of a lattice. The SK (Sharma-Kaushik)-Lattices are known to be of high applicational value. Hence the present study is aimed at detailing the 'Algebraic Structure of SK-Lattice. In the context of Information-Theoretic approach to coding, the n-tuples of integers are relevant. Therefore, the Algebraic properties are obtained for lattices of SK-partitions, which are characterized as n-tuples of integers. Such lattices are shown to satisfy the Jordan-Dedekind chain condition and the modular-identity. Right residuals and deficits are also considered.

Key words: Lattice-based information retrieval, algebraic structure of a lattice, Sharma-Kaushik partition lattices, Jordan-Dedekind chain condition, Modular identity

INTRODUCTION

In the context of Complexity Reduction in Lattice-Based Information Retrieval (Ming and Wang, 2005), the reduction process must preserve the algebraic structure of a lattice (Kalinin and Spatzier, 2005; Karen and Vogel, 2005). The SK (Sharma-Kaushik)-Lattices are known to be of high applicational value. Hence the present study is aimed at detailing the algebraic structure of SK-lattice. In the context of Information-Theoretic approach to coding, the n-tuples of integers are relevant. Therefore, the Algebraic properties are obtained for lattices of SK-partitions, which are characterized as n-tuples of integers.

Lattices of sets have been studied to a significant extent. Lachlan (1968) investigated the lattice of recursively enumerable sets and characterized the hh-simple sets as co-infinite r.e. sets whose r.e. sets form a Boolean algebra. The lattice of partitions of a set is also a topic of interest. Haiman (1994) presented a construction realising a continuous partition lattice as a lattice of measurable partitions.

SK-partitions were introduced by Sharma and Kaushik (1977), in connection with metrics in Coding Theory. SK-partitions can be characterized as n-tuples of integers and so a lattice of SK-partitions can hence be considered to be a lattice of sets of n-tuples. We show that a lattice of SK-partitions satisfy the well-known Modular Identity and Jordan-Dedekind Chain Condition (Stern, 1999) and is a cl-semigroup. We also obtain conditions satisfied by the right-residual and deficit of SK-partitions and present a condition for the existence of a certain right residual. Most of these results are obtained directly by considering the structure of SK-partitions.

DEFINITIONS AND NOTATION

The following definitions and notations are found in Lattice Theory by Birkhoff (1967).

Definition: If a and b are elements of a partially ordered set P, then a is said to cover b if $a < b$ and $a > x > b$ is not satisfied by any x in P.

Notation: 0 denotes the least element and 1 the greatest element of P.

Definition: An element which covers 0 is called an atom.

Definition: If $a \geq b$ in a partially ordered set P, the set of x satisfying $a \geq x \geq b$ is called the closed interval [b, a]. The elements x satisfying $a \geq x \geq b$ are said to be between a and b.

Definition: Intervals which can be written as $[x \wedge y, x]$ and $[y, x \vee y]$ are called transposes.

Definition: A lattice having the property that every non-empty bounded set has a greatest lower bound and a least upper bound is called conditionally complete.

Definition: The modular identity is: If $x \leq z$, $x \vee (y \wedge x) = (x \vee y) \wedge x$ for elements x, y, z of a lattice L.

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Definition: Jordan-Dedekind chain condition: All finite connected chains between fixed end-points have the same length.

Definition: An element of a lattice L is called join irreducible if $x \vee y = a \Rightarrow x = a$ or $y = a \ \forall x, y \in L$.

Definition: A multiplicative lattice or m-lattice is a lattice L with a binary operation satisfying $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$.

- A zero of an m-lattice L is an element 0 satisfying $0 \wedge x = 0x = x0 = 0 \forall x \in L$.
- A unity of L is an element e satisfying $ex = xe = x \forall x \in L$.
- An infinity of L is an element I satisfying $I \vee x = Ix = xI = I \forall x \in L$.
- L is called commutative if $xy = yx \ \forall x, y \in L$. L is called associative if $x(yz) = (xy)z \ \forall x, y, z \in L$.

If L is conditionally complete and satisfies the unrestricted distributive laws $a \vee b_a = \vee (ab_a)$ and $(\vee a_a)b = \vee(a_a b)$, it is called a complete m-lattice, or cm-lattice. An associative lattice with unity is called a lattice-ordered semigroup or l-semigroup and if complete it is called a cl-semigroup.

Definition: Let G be any m-lattice. The right-residual h:k of h by k is the largest x (if it exists) satisfying $xk \leq h$; the left-residual h:k of h by k is the largest y satisfying $ky \leq h$.

We will also require the following notations.

Notation: We denote the SK-partition $P = \{B_0, B_1,..., B_{m-1}\}$ by $((1,b_1,b_2,...,b_{m-1}))$, where $b_i = |B_i|$ = number of elements of B_i; $i = 1,2,..., m-1$.

Notation: The set of all SK-partitions will be denoted by F_P and the set of SK-partitions with m classes will be denoted by $F_{P,m}$.

Definition: The dimension function d is defined on F_p by

$$d\left(\left(1, b_1, b_2,..., b_{m-1}\right)\right) = \sum_{j=1}^{m-1} \frac{b_j}{2}$$

THE MODULAR IDENTITY AND JORDAN-DEDEKIND CHAIN CONDITION

Lemma 1: Let $x, y, a \in F_{P,m} \ni x$ and y cover a and $x \neq y$. Then $x \vee y$ covers x and y.

Proof: Let
$a = ((1,a_1,a_2,...,a_{m-1}))$.
$x = ((1,x_1,x_2,...,x_{m-1}))$.
$y = ((1,y_1,y_2,...,y_{m-1}))$.

$$\text{Then } \exists u, v \in \{1,2,..., m-1\} \ni a_i = \begin{cases} x_i & , i \neq u \\ x_i - 2 & , i = u \end{cases}$$

and

$$a_i = \begin{cases} y_i & , i \neq v \\ y_i - 2, & i = v \end{cases}$$

$$\text{Let } x \vee y = \left(\left(1, z_1, z_2,..., z_{m-1}\right)\right)$$

$$\text{Then } z_i = \begin{cases} a_i, & i \neq u, v \\ a_u + 2 = x_u, & i = u \\ a_v + 2 = y_v, & i = v \end{cases}$$

$\therefore x_i = a_i = z_i$, $i \neq u, v$; $x_i = a_i+ 2\ z_i$, $i = u$; $x_i + 2 = a_i + 2 = z_i$, $i = v$. Hence $x \vee y$ covers x. Similarly, $x \vee y$ covers y.

Lemma 2: Let $x, y, a \in F_{P,m} \ni a$ cover x and y and $x \neq y$. Then x and y cover $x \wedge y$.

Proof: Let
$a = \left(\left(1, a_1, a_2,...a_{m-1}\right)\right)$
$x = \left(\left(1, x_1, x_2,...x_{m-1}\right)\right)$
$y = \left(\left(1, y_1, y_2,..., y_{m-1}\right)\right)$

$$\text{Then } \exists \ u, v \in \{1,2,..., m-1\} \ni a_i = \begin{cases} x_i & , i \neq u \\ x_i + 2, & i = u \end{cases}$$

$$\text{and } a_i = \begin{cases} y_i & , i \neq v \\ y_i + 2, & i = v \end{cases}$$

$$\text{Let } x \wedge y = \left(\left(1, z_1, z_2,..., z_{m-1}\right)\right) \text{ Then } z_i = \begin{cases} a_i, & i \neq u, v \\ a_i - 2, & i = u \\ a_i - 2, & i = v \end{cases}$$

$\therefore x_i = a_i = z_i$, $i \neq u, v$; $x_i = a_i - 2 = z_i$, $i = u$; $x_i = a_i = z_i + 2$, $i = v$. Hence x covers $x \wedge y$. Similarly, y covers $x \wedge y$.

Theorem 1: $(F_{P,m}, \leq_s)$ satisfies the modular identity and the Jordan-Dedekind chain condition.

Proof: This follows immediately from Theorem 1 of Lattice Theory by Birkhoff.

TRANSPOSES

Theorem 2: Let [l, x] and [y, n] be intervales ∋

$$x = ((1, x_1, x_2, ..., x_{m-1}))$$
$$l = ((1, l_1, l_2, ..., l_{m-1}))$$
$$u = ((1, u_1, u_2, ..., u_{m-1}))$$
$$y = ((1, y_1, y_2, ..., y_{m-1}))$$
$$x_i = u_i ; i = a_1, a_2, ... a_r.$$
$$l_j = y_j ; j = a_1, a_2, ..., a_r.$$
$$x_i \neq l_i ; i = a_1, a_2, ..., a_r$$

And

$$x_i = l_i;\ i \neq a_1, a_2, ... a_r,\ i \in \{1, 2, ... m-1\}$$
$$u_i = y_i ; i \neq a_1, a_2, ..., a_r, i \in \{1, 2, ..., m-1\}$$
$$x_i \leq y_i ; i \neq a_1, a_2, ..., a_r, i \in \{1, 2, .., m-1\}$$

Where, $a_1, a_2, ..., a_r$ are fixed elements of {1,2,...,m-1}. Then, [1,x] and [y,u] are transposes.

Proof: For $i \in \{1,2,...,m-1\} \ni i \neq a_1,a_2,...,a_r$, $l_j = x_i \leq y_i$.
Hence, $l_i = \min [x_i, y_i\}$ (1)
For $i = a_1,a_2,...a_{m-1}$; $1_i = y_i \leq u_i = x_i$
Hence, $l_i = \min [x_i, y_i\}$. (2)
From Eq. 1 and 2, $l = x \wedge y$. For $i \in \{1,2,...m-1\} \ni$
$i \neq a_1,a_2,...,a_r$ with $u_i = y_i \geq x_i$
$\therefore u_i = \max \{x_i,y_i\}$ (3)
For $i \in \{1,2,...,m-1\} \ni i = a_1,a_2,...,a_{m-1}$; $u_i = x_i \geq l_i = y_i$
$\therefore u_i = \max \{x_i,y_i\}$ (4)
From Eq. 3 and 4, $u = x \vee y$.

JOIN-IRREDUCIBLE ELEMENTS

Theorem 3: Each join-irreducible element of ($F_{P,m} \leq_s$) is of the form ((1,2,2,...,2,a,...,a)), where a ∈ {2,4,6,...}.

Proof: Let $((1,2,2,...,2,a_1,a_2,...,a_s))$ be any join-irreducible element of $F_{p,m}$. Suppose that $a_1 < a_2$.

Then, $((1,2,2,...,2,a_1,a_2,...,a_s)) = ((1,2,2,...,2,a_1,a_1,...,a_1)) \vee ((1,2,2,...,2,a_2,a_3,...,a_s))$
r twos r twos r + 1 twos

So, $((1,2,2,...,2,a_1,a_2,...,a_s))$ is not join-irreducible, contradiction. Hence, $a_1 = a_2$.

Similarly, it can be established that $a_2 = a_3 = ... a_s$. Any element of $F_{p,m}$ can be expressed is terms of join-irreducible elements of $F_{p,m}$, as shown in the following theorem.

Theorem 4: Let $\alpha = ((1, a_{11}, a_{12}, ..., a_{1r_1}, a_{21}, a_{22}, .., a_{2r_2} ..., a_{s1}, a_{s2}, ..., a_{sr_s}))$ be an arbitrary element of $F_{p,m}$.

Then,

$$\alpha = \bigvee_{i=1}^{s} ((1, 2, 2, ..., 2, a_{i1}, a_{i1}, ..., a_{i1}));\ t_i \text{ twos},$$

Where:

$$t_i = \begin{cases} o & i = 1 \\ r_1 + r_2 + ... + r_{i-1} & i = 2, 3, ... \end{cases}$$

Proof: Obvious.

MULTIPLICATION AND THE RIGHT-RESIDUAL

Definition: Multiplication in $F_{P,m}$ defined by;

$$((1, a_1, a_2, ..., a_{m-1}))((1, b_1, b_2, ..., b_{m-1})) = \left(\left(1, \frac{a_1 b_1}{2}, \frac{a_2 b_2}{2}, ..., \frac{a_{m-1} b_{m-1}}{2}\right)\right)$$

Theorem 5: ($F_{P,m}$, $\leq_s$) is a cl-semigroup.

Proof:

- Clearly the unity e of $(F_{P,m}, \le_s)$ is ((1,2,2,...,2)).
- $(F_{P,m}, \le_s)$ is associative.
- $(F_{P,m}, \le_s)$ satisfies the unrestrictive distributive laws.
- $(F_{P,m}, \le_s)$ is conditionally complete, since if A is a bounded subset of $F_{P,m}$, then $\vee_{a_i \in A} a_i$ is the least upper bound of A and $\wedge_{a_i \in A} a_i$ is the greatest lower bound of A.

Theorem 6: $\forall$ P, Q $\in (F_{p,m}, \le_s)$, $(P \wedge Q)(P \vee Q) = PQ$

Proof: Let
$P = ((1,a_1,a_2,...,a_{m-1}))$
$Q = ((1,b_1,b_2,...,b_{m-1}))$.

We assume that $a_i < b_i$, $i \in \{1,2,...,m-1\}$.
Then the (i+1) th entry of $(P \wedge Q)(P \vee Q) = 1/2(a_i)(b_i)$ = (i + 1)th entry of PQ (I)
Similarly, if $a_i > b_i$ or $a_i = b_i$ (I) can still be shown to be true.

Theorem 7: Let
$P = ((1,a_1,a_2,...,a_{m-1}))$
$Q = ((1,a_1,a_2,...,a_{m-1}))$ be elements of $F_{p,m}$. Then $\exists R \in F_{p,m} \ni P = QR$ if and only if

- $Q \le_s P$
- b_j is a divisor of $2a_j$; j = 1,2,...,m-1
- $\frac{q_1}{b_i} \le \frac{a_2}{b_2} \le .... \le \frac{a_{m-1}}{b_{m-1}}$

Proof: Let Suppose P = QR, where $R = ((1,c_1,c_2,...,c_{m-1}))$

Then, $((1,a_1,a_2,...,a_{m-1})) = ((1,b_1,b_2,...,b_{m-1}))((1,c_1,c_2,...,c_{m-1}))$

$$\therefore a_j = \frac{b_j c_j}{2}; j = 1,2,...,m-1.$$

$$\therefore b_j = \frac{2}{c_j} a_j \le a_j, \text{ since } c_j \ge 2$$

- Clearly Q ≤ P.
- Also, $\frac{2a_j}{b_j} = c_j$ so b_j is a divisor of $2a_j$
- Also, $\frac{a_j}{b_j} = \frac{c_j}{2}$

$$\therefore \frac{a_1}{b_1} \le \frac{a_2}{b_2} \le ... \le \frac{a_{m-1}}{b_{m-1}}, \text{ since } \frac{c_1}{2} \le \frac{c_2}{2} \le ... \le \frac{c_{m-1}}{2}$$

Conversely, suppose that the three conditions are true.

$$\text{Let } R = ((1, c_1, c_2, ..., c_{n-1})),$$

Where:

$$c_j = \frac{2a_j}{b_j}$$

Then, clearly $R \in F_{p,m}$ and P = QR.

Theorem 8: Let P, Q be as in the above theorem. If $a_i = kb_i$, $k \in \{2,3,...\}$, then QR = P, where $R = ((1,c_1,c_2,...,c_{m-1}))$ and $c_i = 2k$.

Proof: If QR = P, then

$$\frac{b_i c_i}{2} = a_i = kb_i$$

Hence, $c_i = 2k$.

Theorem 9: Let P, Q, R be as in the above theorem. If $a_i = b_i^n$ (n = 2,3,...) and QR = P, then $c_i = 2b_i^{n-1}$.

Proof: If QR = P, then

$$\frac{b_i l_i}{2} = a_i = b_i^n.$$

Hence, $c_i = 2b_i^{n-1}$.

THE RIGHT RESIDUAL

Theorem 10: Let
$P_k = ((1, a_{k1}, a_{k2}, ..., a_{k(m-1)})); k = 1,2,...,\lambda$
$Q = ((1, b_1, b_2, ..., b_{m-1}))$

be elements of $(F_{p,m}, \le_s)$. Then $(\wedge P_k)$: Q exists if and only if

$$b_i \le \min\{a_{1i}, a_{2i}, ..., a_{(m-1)i}\} \quad (5)$$

Proof: Suppose $(\wedge P_k$:Q) exists. Then $\exists$ positive, even integers $c_1,c_2,...,c_{m-1}$ such that $R = ((1,c_1,c_2,...,c_{m-1})$ and

$$RQ \le \wedge P_k. \text{ Hence } \frac{c_i b_i}{2} \le \min\{a_{1i}, a_{2i}, ..., a_{(m-1)i}\}$$
$$\text{and } b_i \le \min\{a_{1i}, a_{2i}, ..., a_{(m-1)i}\}$$

Conversely, suppose Eq. 5 is true. Then,

$$\frac{2b_i}{2} \le 2\min\{a_{1i}, a_{2i}, .., a_{(m-1)i}\}. \text{ Hence } TQ \le \wedge P_k,$$

Where:

$$T = ((1, 2, 2, ..., 2))$$

Hence, ($\wedge P_k$: Q) exists.

Theorem 11:
Let

$P_i = ((1, a_{i1}, a_{i2}, ..., a_{i(m-1)})), i = 1, 2, ..., u$

$Q = ((1, b_1, b_2, ..., b_{m-1}))$

$R = (\wedge P_i : Q), = ((1, c_1, c_2, ..., c_{m-1}))$ and

$R_i = P_i : Q = ((1, c_{i1}, c_{i2}, ..., c_{i(m-1)}))$ Then, $(\wedge P_i) : Q = \wedge (P_i; Q)$

Proof: Since R = ($\wedge$ P_i): Q, QR $\le$ $P_1 \wedge P_2 \wedge ... \wedge P_u$ and $c_1, c_2, ..., c_{m-1}$ are the largest positive, even integers satisfying:

$c_1 \le \frac{2}{b_1}\min\{a_{11}, a_{21}, ..., a_{(m-1)1}\}$

$c_2 \le \frac{2}{b_2}\min\{a_{12}, a_{22}, ..., a_{(m-1)2}\}$

$...; c_{m-1} \le \frac{2}{b_{m-1}}\min\{a_{1(m-1)}, a_{2(m-1)}, ..., a_{(m-1)(m-1)}\}$ and

$c_1 \le c_2 \le .... \le c_{m-1}$.

Also, since

$$R_i = P_i : Q, \; QR_i \le P_i; \; c_{i1}, c_{i2}, ..., c_{i(m-1)}$$

are the largest positive even integers satisfying

$$c_{i1} \le \frac{2a_{i1}}{b_1}; c_{i2} \le \frac{2a_{i2}}{b_2}; c_{i(m-1)} \le \frac{2a_{i(m-1)}}{b_{m-1}} \text{ and}$$
$$c_{i1} \le c_{i2} \le ... \le c_{i(m-1)}.$$

Let

$d_1 = \min\{c_{11}, c_{21}, ..., c_{(m-1)1}\}$

$d_2 = \min\{c_{12}, c_{22}, ..., c_{(m-1)2}\}$

$d_{m-1} = \min\{c_{1(m-1)}, c_{2(m-1)}, .., c_{(m-1)(m-1)}\}$

We will show that:

$$d_1 \le \frac{2}{b_1}\min\{a_{11}, a_{21}, ..., a_{(m-1)1}\} \quad (6)$$

$$d_2 \le \frac{2}{b_2}\min\{a_{12}, a_{22}, ..., a_{(m-1)2}\} \quad (7)$$

$$\vdots$$

$$d_{m-1} \le \frac{2}{b_{m-1}}\min\{a_{1(m-1)}, a_{2(m-1)}, ..., a_{(m-1)(m-1)}\} \quad (m+4)$$

$$d_1 \le d_2 \le ... \le d_{m-1} \quad (m+5)$$

and that $d_1, d_2, ..., d_{m-1}$ are the largest positive, even integers that satisfy Eq. 6, 7,…, m+5.
Equation (m+5) is obviously true.
Let $D_1, D_2, ..., D_{m-1}$ be positive even integers satisfying $d_1 \le D_1$, $d_2 \le D_2, ..., d_{m-1} \le D_{m-1}$.

$D_1 \le \frac{2}{b_1}\min\{a_{11}, a_{21}, ..., a_{(m-1)1}\}$

$D_2 \le \frac{2}{b_2}\min\{a_{12}, a_{22}, .., a_{(m-1)2}\}$

$\vdots$

$D_{m-1} \le \frac{2}{b_{m-1}}\min\{a_{1(m-1)}, a_{2(m-1)}, ..., a_{(m-1)(m-1)}\}$ and

$D_1 \le D_2 \le ... \le D_{m-1}$.

We show by contradiction, that $d_1 = D_1$, $d_2 = D_2, ..., d_{m-1} = D_{m-1}$
Suppose that $c_{1(m-1)} < D_{m-1}$.

Then $((1, c_{11}, c_{12}, ..., c_{1(m-1)})) < ((1, c_{11}, c_{12}, ..., D_{1(m-1)}))$

Hence $((1, c_{11}, c_{12}, ..., D_{(m-1)}))$ would be P :Q, not $((1, c_{11}, c_{12}, ..., c_{1(m-1)}))$, contradiction.

$$\therefore \; c_{1(m-1)} \not< D_{m-1}$$

Similarly, $\therefore$ $c_{1(m-1)} \not< D_{m-1}$ for any i.

$$\therefore D_{m-1} \le \min\{c_{1(m-1}, c_{2(m-1)}, ..., c_{(m-1)(m-1)}\} = d_{m-1}$$

Since $d_{m-1} \le D_{m-1}$, this means that $D_{m-1} = d_{m-1}$.
Suppose that $c_{1(m-2)} < D_{m-2}$.

Then $((1, c_{11}, c_{12}, .., c_{1(m-2)}, c_{1(m-1)})) < ((1, c_{11}, c_{12}, .., D_{m-2}, c_{1(m-1)}))$

Now $c_{1(m-3)} \le c_{1(m-2)} < D_{m-2} \le D_{m-1} = d_{m-1} \le c_{1(m-1)}$.

Hence, $((1, c_{11}, c_{12}, ..., c_{1(m-1)}))$ is not

$P_1 : Q$, a contradiction. $\therefore$ $c_{1(m-1)} \not< D_{m-2}$

Similarly, $c_{i(m-2)} \not< D_{m-2}$ for $i \in \{2,3,\ldots,m-1\} \therefore D_{m-2} \le \min\{c_{1(m-2)}, c_{2(m-2)}, \ldots, c_{(m-1)(m-2)}\} = d_{m-2}$

This means that: $D_{m-2} = d_{m-2}$. Continuing like this, we can establish that $d_i = D_i$, $\forall i \in \{1,2,\ldots,m-1\}$.

Thus, the values we have chosen for $d_1, d_2, \ldots, d_{m-1}$ are the largest positive, even integers satisfying Eq. 6, 7,...,(m+5).

Hence, $c_1 = d_1, c_2 = d_2, \ldots, c_{m-1} = d_{m-1} \therefore (\wedge P_i) : Q = R = ((1, d_1, d_2, \ldots, d_{m-1})) = \wedge R_i = R_i = (P_i : Q)$.

Theorem 12:

Let $P_j = ((a_{i1}, a_{i2}, \ldots, a_{i(m-1)}))$; $i = 1,2,\ldots,h$ and $Q = ((1, b_1, b_2, \ldots, b_{m-1}))$ be elements of $F_{p,m}$ satisfying the following conditions:

- $Q \le P_i; i = 1,2,\ldots,h$
- b_j is a divisor of $2a_{ij}$; $j = 1,2,\ldots,m-1$ and $i = 1,2,\ldots,h$.
- $\frac{a_{i1}}{b_1} \le \frac{a_{i2}}{b_2} \le \ldots. \le \frac{a_{i(m-1)}}{b_{m-1}}$

Then $\left\{\left(\prod_{i=1}^{h} P_i\right) : Q\right\} = \left\{\prod_{i=1}^{h} (P_i : Q)\right\} \times Q^{h-1}$

Proof:

$$\text{LHS} = \left(\left(1, \frac{a_{11}a_{21}\ldots a_{h1}}{2^{h-1}}, \frac{a_{12}a_{22}\ldots a_{h2}}{2^{h-1}}, \ldots, \frac{a_{1(m-1)}a_{2(m-1)}\ldots a_{(m-1)(m-1)}}{2^{h-1}}\right)\right) : ((1, b_1, b_2, \ldots, b_{m-1}))$$

$$= \left(\left(1, \frac{a_{11}a_{21}\ldots a_{h1}}{2^{h-2}b_1}, \frac{a_{12}a_{22}\ldots a_{h2}}{2^{h-2}b_2}, \ldots, \frac{a_{1(m-1)}a_{2(m-1)}\ldots a_{h(m-1)}}{2^{h-2}b_{m-1}}\right)\right)$$

$$\text{RHS} = \left\{\prod_{i=1}^{h} ((1, a_{i1}, a_{i2}, \ldots, a_{i(m-1)})) : ((1, b_1, b_2, \ldots, b_{m-1}))\right\} \times Q^{h-1}$$

$$= \left\{\prod_{i=1}^{h} \left(\left(1, \frac{2a_{i1}}{b_1}, \frac{2a_{i2}}{b_2}, \ldots, \frac{2a_{i(m-1)}}{b_{m-1}}\right)\right)\right\} \times Q^{h-1}$$

$$= \left(\left(1, \frac{2a_{11}a_{21}\ldots a_{h1}}{b_1^h}, \frac{2a_{12}a_{22}\ldots a_{h2}}{b_2^h}, \ldots, \frac{a_{1(m-1)}a_{2(m-1)}\ldots a_{h(m-1)}}{b_{m-1}^h}\right)\right) \times \left(\left(1, \frac{b_1^{h-1}}{2^{h-2}}, \frac{b_2^{h-1}}{2^{h-2}}, \ldots, \frac{b_{m-1}^{h-1}}{2^{h-2}}\right)\right) = \text{LHS}$$

THE DEFICIT

Definition: We define the deficit $h \bullet\bullet k$ of h by k as the smallest x, if it exists, satisfying $h \le xk$.

Theorem 13: $(\vee P_i) \bullet\bullet Q = \vee(P_i \bullet\bullet Q)$

Proof :

Let $P_i = ((1, a_{i1}, a_{i2}, \ldots, a_{i(m-1)}))$; $Q = ((1, b_1, b_2, \ldots, b_{m-1}))$

$R_i = (P_i \bullet\bullet Q) = ((1, c_{i1}, c_{i2}, \ldots, c_{i(m-1)}))$; $R = (\vee P_i) \bullet\bullet Q = ((1, r_1, r_2, \ldots, r_{m-1}))$ and $(\vee P_i) \le RQ$

$\max\{a_{11}, a_{21}, \ldots, a_{(m-1)1}\} \le \frac{r_1 b_1}{2}$; $\max\{a_{12}, a_{22}, \ldots, a_{(m-1)2}\} \le \frac{r_2 b_2}{2}$;...

$\max\{a_{1(m-1)}, a_{2(m-1)}, \ldots, a_{(m-1)(m-1)}\} \le \frac{r_{m-1} b_{m-1}}{2}$ and $r_1 \le r_2 \le \ldots \le r_{m-1}$

Thus, $P_i \le R_i Q$. Also, $a_{i1} \le \frac{c_{i1}b_1}{2}; a_{i2} \le \frac{c_{i2}b_2}{2};...; a_{i(m-1)} \le \frac{c_{i(m-1)}}{2} b_{m-1}$.

Let $V = \left(\left(1, d_1, d_2, ..., d_{m-1}\right)\right) = \vee(P_i \bullet\bullet Q) = \vee R_i$

$\therefore d_1 = \max\left\{c_{11}, c_{21}, ..., c_{(m-1)1}\right\}; d_2 = \max\left\{c_{12}, c_{22}, ..., c_{(m-1)2}\right\}; ...; d_{m-1} = \max\left\{c_{1(m-1)}, c_{2(m-1)}, ..., c_{(m-1)(m-1)}\right\}$

$d_1 \le d_2 \le ... \le d_{m-1}$

$$\frac{2}{b_1}\max\left\{a_{11}, a_{21}, ..., a_{(m-1)1}\right\} \le r_1 \qquad (\alpha 1)$$

$$\frac{2}{b_2}\max\left\{a_{12}, a_{22}, ..., a_{(m-1)2}\right\} \le r_2 \qquad (\alpha 2)$$

$$\vdots$$

$$\frac{2}{b_{m-1}}\max\left\{a_{1(m-1)}, a_{2(m-1)}, ..., a_{(m-1)(m-1)}\right\}. \le r_{m-1} \qquad [\alpha(m-1)]; r_1 \le r_2 \le .... \le r_{m-1}$$

We claim that $d_1, d_2, ..., d_{m-1}$ are the smallest positive even integers satisfying the same inequalities as $r_1, ... r_{m-1}$, namely $\alpha 1, \alpha 2, ..., \alpha(m-1)$.

Suppose $\exists$ positive even integers $D_1, D_2, ..., D_{m-1} \ni D_1 \le d_1, D_2 \le d_2, ..., D_{m-1} \le d_{m-1}$

$$\frac{2}{b_1}\max\left\{a_{11}, a_{21}, ..., a_{(m-1)1}\right\} \le D_1 \qquad (\beta 1)$$

$$\frac{2}{b_2}\max\left\{a_{12}, a_{22}, ..., a_{(m-1)2}\right\} \le D_2 \qquad (\beta 2)$$

$$\vdots$$

$$\frac{2}{b_{m-1}}\max\left\{a_{1(m-1)}, a_{2(m-1)}, ..., a_{(m-1)(m-1)}\right\} \le D_{m-1} \qquad [\beta(m-1)];\ D_1 \le D_2 \le ..., \le D_{m-1}$$

Suppose that $D_1 \le c_{11}$; Consider $((1, D_1, c_{12}, c_{13}, ..., c_{1(m-1)})$. Now $D_1 \le \frac{2}{b_1} a_{11}$, from $\beta 1$.

Hence, $\left(\left(1, c_{11}, c_{12}, ..., c_{1(m-1)}\right)\right)$ is not $\left((P_1 : Q)\right)$, but $\left(\left(1, D_1, c_{12}, ..., c_{1(m-1)}\right)\right)$ is, contradiction. Thus, $D_1 \not< c_{11}$.

Similarly, $D_1 \not< c_{i1}$; $i = 2, 3, ..., m-1$. $\therefore D_1 \ge d_1$. Hence $D_1 = d_1$

Now suppose that $D_2 < c_{12}$.; Consider $\left(\left(1, c_{11}, D_2, c_{13}, c_{14}, ..., c_{1(m-1))}\right)\right)$.

Clearly < $D_2 \ge \frac{2}{b_2} a_{12}$ and $c_{11} \le D_1 \le D_2 < c_{12} \le c_{13} \le c_{14} \le ... \le c_{1(m-1)}$

Hence, $\left(\left(1, c_{11}, c_{12}, ..., c_{1(m-1)}\right)\right)$ is not $(P_1 : Q)$, but $\left(\left(1, c_{11}, D_2, c_{13}, c_{14}, ..., c_{1(m-1)}\right)\right)$ is, contradiction. Thus, $D_2 \not< c_{12}$.

Similarly $D_2 \not< c_{i2}$, $i = 2, 3, ..., m-1 \therefore D_2 \ge d_2$. and $\therefore D_2 = d_2$. Continuing like this, we can establish that:

$D_1 = d_1, D_2 = d_2, ..., D_{m-1} = d_{m-1}$. Hence $d_1, d_2, ..., d_{m-1}$ are the smallest positive even integers satisfying $\beta 1, \beta 2, ..., \beta(m-1)$.

$\therefore r_1 = d_1, r_2 = d_2, ..., r_{m-1} = d_{m-1}$. Hence $(\vee P_i) \bullet\bullet Q = \vee(P_i \bullet\bullet Q)$.

Theorem 14: $P \bullet\bullet (\wedge Q_i) = \vee(P \bullet\bullet Q_i)$

Proof:

Let $P = \left(\left(1, a_1, a_2, ..., a_{m-1}\right)\right); Q_i = \left(\left(1, b_{i1}, b_{i2}, ..., b_{i(m-1)}\right)\right); P \bullet\bullet Q_i = T_i = \left(\left(1, t_{i1}, t_{i2}, ..., t_{i(m-1)}\right)\right);$

$\vee(P \bullet\bullet Q_i) = T = \left(\left(1, t_1, t_2, ..., t_{m-1}\right)\right); P \bullet\bullet (\wedge Q_i) = S = \left(\left(1, s_1, s_2, ..., s_{m-1}\right)\right)$ Now $P \le T_i Q_i$ and so,

$a_1 \le \frac{t_{i1}b_{i1}}{2} \le t_1\frac{b_{i1}}{2} \le \frac{t_1}{2}\min\{b_{11}, b_{21}, ..., b_{(m-1)1}\}; a_2 \le \frac{t_{i2}b_{i2}}{2} \le \frac{t_2 b_{i2}}{2} \le \frac{t_2}{2}\min\{b_{12}, b_{22}, ..., b_{(m-1)2}\}; ...;$

$a_{m-1} \le \frac{t_{i(m-1)}b_{i(m-1)}}{2} \le \frac{t_{m-1}b_{i(m-1)}}{2} \le \frac{t_{m-1}}{2}\min\{b_{1(m-1)}, b_{2(m-1)}, ..., b_{(m-1)(m-1)}\}; t_{i1} \le t_{i2} \le ..., \le t_{i(m-1)}$

Also, $t_1 = \max\{t_{11}, t_{21}, ..., t_{(m-1)1}\}; t_2 = \max\{t_{12}, t_{22}, ..., t_{(m-1)2}\}; ...; t_{m-1} = \max\{t_{1(m-1)}, t_{2(m-1)}, ..., t_{(m-1)(m-1)}\}$.

Now $P \le S(\wedge Q_i)$. Hence, $s_1, s_2, ..., s_{m-1}$ are the smallest positive even integers satisfying:

$a_1 \le \frac{s_1}{2}\min\{b_{11}, b_{21}, ..., b_{(m-1)1}\}; a_2 \le \frac{s_2}{2}\min\{b_{12}, b_{22}, ..., b_{(m-1)2}\}; ...; a_{m-1} \le \frac{s_{m-1}}{2}\min\{b_{1(m-1)}, b_{2(m-1)}, ..., b_{(m-1)(m-1)}\}$

$s_1 \le s_2 \le ... \le s_{m-1}$, Let $D_1 \le t_1, D_2 \le t_2, ..., D_{m-1} \le t_{m-1}$, $\ni D_1, D_2, ..., D_{m-1}$ be positive even integers

$$a_1 \le \frac{D_1}{2}\min\{b_{11}, b_{21}, ..., b_{(m-1)1}\} \qquad (\lambda 1)$$

$$a_2 \le \frac{D_2}{2}\min\{b_{12}, b_{22}, ..., b_{(m-1)2}\} \qquad (\lambda 2)$$

$$\vdots$$

$a_{m-1} \le \frac{D_{m-1}}{2}\min\{b_{1(m-1)}, b_{2(m-1)}, ..., b_{(m-1)(m-1)}\}$ $[\lambda(m-1)]$ and $D_1 \le D_2 \le .... \le D_{m-1}$

Suppose that $D_1 < t_{i1}$. Consider $((1, D_1, t_{i2}, ..., t_{i(m-1)}))$. This would be $P \bullet\bullet Q_i$, not T_i, contradiction.

Hence $D_1 \not< t_{i1}$, $\forall i = 1, 2, ..., m-1$. $\therefore D_1 = t_1$. Now suppose that $D_2 < t_{i2}$. Consider,

$((1, t_{i1}, D_2, t_{i3}, t_{i4}, ..., t_{i(m-1)}))$. Clearly, $a_2 \le \frac{D_2 b_{i2}}{2}, D_2 < t_{i2} \le t_{i3}$ and $t_{i1} \le t_1 = D_1 \le D_2$.

$\therefore ((1, t_{i1}, D_2, t_{i3}, t_{i4}, ..., t_{i(m-1)}))$ is $P \bullet\bullet Q_i$, not T_i, a contradiction. $\therefore D_2 \ge t_{i2}$

Hence $D_2 \ge t_2$ and since $D_2 \le t_2, D_2 = t_2$ Continuing like this, we obtain $D_1 = t_1, D_2 = t_2 ..., D_{m-1} = t_{m-1}$.

Hence, $t_1, t_2, ..., t_{m-1}$ are the smallest values of $D_1, D_2, ..., D_{m-1}$ satisfying $\lambda_1, \lambda_2, ..., \lambda_{m-1}$.

Hence, $s_1 = t_1, s_2 = t_2, ..., s_{m-1} = t_{m-1}$. $\therefore P \bullet\bullet(\wedge Q_i) = \vee(P \bullet\bullet Q_i)$.

Theorem 15: $P \le QR$ and $P \bullet\bullet Q \le R$ are equivalent.

Proof:
Let $P = ((1, a_1, a_2, ..., a_{m-1}))$, $Q = ((1, b_1, b_2, ..., b_{m-1}))$, $R = ((1, c_1, c_2, ..., c_{m-1}))$ and let $P \bullet\bullet Q = D$.
now D is the smallest element of $F_p \ni P \le QD$. Hence, $P \le QR \Leftrightarrow D \le R \Leftrightarrow P\bullet\bullet Q \le R$.

Theorem 16: Let $P\bullet\bullet Q = X$ and let $R \in F_{p,m} \ni P \le QR$. Then, $X \le R$

Proof: X is the smallest element of $F_{p,m}$ satisfying $P \le QX$. We show that X and R must be related.
If $X \not\le$, R and $X \not\ge R$, then $X \wedge R < X$. Since, $P \le QR$ and $P \le QX$, $P \le Q(X \wedge R)$. Hence, X is not $P\bullet\bullet Q$ since $(X \wedge R) < X$, contradiction. Hence X and R must be related. Hence, clearly $X \le R$.

Theorem 17: $(P \bullet\bullet Q) \bullet\bullet R = P \bullet\bullet (QR)$

Proof:
Let $P\bullet\bullet Q = X$ and $P\bullet\bullet Q)\bullet\bullet R = V$. Then, X is the smallest element of $F_{p,m} \rightarrow P \le QX$.
Also, V is the smallest element of $F_{p,m} \ni X \le RV$. $\therefore P \le QX \le QRV$. Now, suppose that $W \le V$ and $P \le QRW$. Then, $X \le RW$. $\therefore R \le W$. Since, $W \le V$ this means that $V = W$. Hence, V is the smallest element of $F_{p,m} \ni P \le QRV$. $\therefore V = P\bullet\bullet(QR)$. $\therefore (P\bullet\bullet Q)\bullet\bullet R = V = P\bullet\bullet(QR)$.

Theorem 18: $(P \bullet\bullet Q) \bullet\bullet R = (P \bullet\bullet R) \bullet\bullet Q$.

Proof: From Theorem 17 $(P \bullet\bullet Q) \bullet\bullet R$ is the smallest element T of $F_{p,m}$ satisfying $P \leq QRT$.

From Theorem 16 $(P \bullet\bullet R) \bullet\bullet Q$ is the smallest element V of $F_{p,m}$ satisfying $P \leq RQV$.
Hence the result.

CONCLUSIONS

The details characterizing the Algebraic Structures of the SK-Lattices are of immense usefulness in the context of the recently-growing interest in the area of the contemporarily modern Approach to Image Retrieval Based on Concept Lattices, as also in useful reference to Complexity reduction in Lattice-based information retrieval, which requires that the reduction process must preserve the algebraic structure of a lattice (Kalinin and Spatzier, 2005; Karen and Vogel, 2005).

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