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# Quadratic Frontier for the Production Possibility Set in Data Envelopment Analysis

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**Abstract:** Data Envelopment Analysis (DEA) applies simple-structured linear programs to evaluate the efficiency of similar Decision Making Units (DMUs). For large-scale problems with a huge number of DMUs, however, it is a computationally expensive method. Since in large-scale problems the piece-wise linear frontier of the Production Possibility Set (PPS) tends to be a smooth nonlinear surface, it is reasonable to consider such a nonlinear frontier in efficiency evaluations. In this study, a parametric DEA method is proposed to estimate a quadratic frontier for the PPS in constant returns-to-scale environment. Quadratic frontiers better fit the reality of large-scale problems. In addition, parametric approaches reduce the computations to a great extent. With a numerical example, the proposed approach is compared with the CCR model, which shows that the results of the proposed approach are good approximations, even for small-size problems.

**Key words:** Data envelopment analysis, quadratic frontier, large-scale problems, parametric method

## INTRODUCTION

Data Envelopment Analysis (DEA) is a method for comparing the performance of a group of similar Decision Making Units (DMUs). Since its innovation, a wide range of DEA applications have been reported by Birman *et al.* (2003) and Thanassoulis (1990). There are two classes of methods in DEA: parametric and non-parametric. In the parametric methods, the case of multiple inputs and single output is assumed and the parameters of a function like Cobb-Douglas are estimated. This function is considered as the efficiency frontier (production function) and the efficiency of each DMU is evaluated by comparing it with the points on this frontier. The non-parametric methods use an empirical piece-wise linear enveloping frontier as the production function. This frontier is not determined explicitly, but rather the features of such a frontier are implicitly embedded in the resulting models. The non-parametric methods are able to handle multiple-inputs-and-multiple-outputs problems. The following is the most famous DEA model CCR (Charnes *et al*., 1978).

Minimize $\theta$
subject to

$$\theta\hat{\mathbf{x}}_o - \sum_{j=1}^{n}\lambda_j\hat{\mathbf{x}}_j \geq 0,$$

$$\sum_{j=1}^{n}\lambda_j\hat{\mathbf{y}}_j \geq \hat{\mathbf{y}}_o,$$

$$\lambda_j \geq 0 \qquad j = 1,...,n,$$
$\theta$ unrestricted in sign,

Where:

$\hat{\mathbf{x}}_j$ = The input vector,
$\hat{\mathbf{y}}_j$ = The output vector of $DMU_j$ (j = 1,...,n),
$o \in \{1,....,n\}$ = The index of the DMU under consideration.

The above model is solved n times by changing the DMU under efficiency evaluation. The resulting efficiency scores ($\theta$) are used to obtain a ranking for DMUs based on their performances.

Although DEA uses well-known LP formulations, it can be a cumbersome method computationally. Especially in large-scale problems, i.e., ones with more than 1000 DMUs, its popularity decreases quickly. It is because obtaining efficiency scores requires solving as many large LP problems as the number of DMUs. Large-scale problems have been reported frequently in DEA literature (Barr and Durchholz, 1997). Several work of research for reducing the computational efforts in DEA have been carried out. Some of them propose ideas for decreasing the number or the size of LPs needing to be solved (Ali, 1993; Chen and Ali, 2002). For example, the following two interesting properties can be helpful. First, inefficient DMUs do not play any roll in efficiency evaluations of the other DMUs and hence all DMUs which are found inefficient can be removed in the remaining computations. Second, DMUs in the reference sets of an inefficient DMU are all efficient. Although, these features are relatively easy to be considered, they are usually unable to reduce the computations to a significant extent.

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Dulá and Helgason (1996) and Dula *et al.* (1997) introduced the concept of frame which is the minimal subset of DMUs needed to generate the Production Possibility Set (PPS) and proposed an algorithm to find the frame of a convex hull. Korhonen and Halme (1996) used an efficiency curve to search for efficient DMUs. Another approach for dealing with large-scale DEA problems, which is basically different from the aforementioned methods, is decomposition algorithm and was proposed by Barr and Durchholz (1997). In the decomposition algorithm, the set of observed DMUs is partitioned to smaller sets and each set is handled separately to reach the final results. Unfortunately, none of the above methods lead to a significant improvement in the computational efforts.

In this study, a parametric method is used to estimate a nonlinear frontier for the PPS. This method can be considered as an extension of the parametric methods from single output to multiple outputs. The motivation for this method is that in some problems, especially large-scale problems where the number of DMUs is very much greater than the total number of inputs and outputs, the efficient frontier tends to a nonlinear surface and, therefore, a nonlinear formulation better fits the production function. Furthermore, if a formula for the efficient frontier is at hand, efficiency estimations become extremely easy. Although the proposed method is appropriate for large-size problems, it can also be employed to streamline the DEA computations in small-size problem if the exact values of the classical efficiency scores are of importance to the decision maker. To this end, it will be shown that the efficient DMUs in the proposed approach are efficient in the CCR model, too. Knowledge of this, as well as of the approximate position of inefficient DMUs, can be used to make the frame of the reference set. In this study, constant returns-to-scale (CRS) is assumed and a PPS cross section is obtained by scaling the DMUs. A quadratic frontier instead of a piece-wise linear frontier is estimated for this cross section. This frontier is sufficient to perform all the efficiency evaluations. Since it is done by solving only one LP problem, the computations are a small percentage of what is needed in the traditional and also streamlined DEA methods.

## QUADRATIC EFFICIENT FRONTIER

Consider n DMUs, each one using $m \geq 2$ inputs to produce $s \geq 1$ outputs. The input and output vectors of $DMU_j$ ($j = 1,...,n$) are denoted by $\hat{\mathbf{x}}_j=(\hat{x}_{1j},...,\hat{x}_{mj})$ and $\hat{\mathbf{y}}_j=(\hat{y}_{1j},...,\hat{y}_{sj})$, respectively. In this study, constant returns-to-scale is assumed and hence the PPS is a cone with a piece-wise linear frontier. Figure 1 shows the PPS and its cross section for a case with two inputs and one output. Obviously, when $n >> (m+s)$, the efficiency frontier (top part of the PPS cone) tends to a nonlinear surface. Therefore, it is quite rational to search for a nonlinear enveloping surface as the efficiency frontier. Such a frontier must satisfy the following conditions:

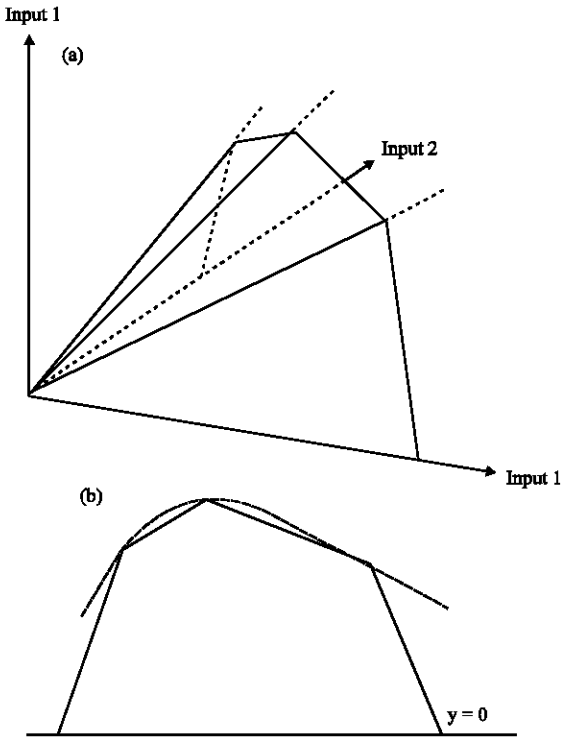


Fig. 1: (a) The production possibility set in CRS environment and (b) a cross section of the PPS and its nonlinear approximation graph

(i) It must envelop all DMUs from above with respect to outputs axes.
(ii) It must be cone-shaped.
(iii) Its output variables must be concave with respect to input variables.
(iv) It must have the minimum distance to all DMUs.

Clearly, conditions i, ii and iv are translations of the PPS axioms in DEA. Condition iii is needed because of two properties of the PPS: convexity and the possibility of existence of virtual DMUs with lower outputs and higher inputs. If $f : R^n \rightarrow R$ is a concave function, then $\text{hypograph}(f) = \{(\mathbf{x}, y) \mid \mathbf{x} \in R^n, y \in R, y \leq f(\mathbf{x})\}$ is a convex set. Therefore, condition ii and iii together impose the

convexity and the possibility conditions on the PPS for the one output case, but for the general case we face a surface instead of a function and hence the convexity of a group of variables ($y_r$'s) with respect to variables of the other group ($x_i$'s) should be considered. An interesting property of quadratic functions is that their concavity or convexity does not change in their domain. For quadric surfaces, concavity or convexity of a variable with respect to another variable is not necessarily unique at a point. However, it does not change for each function-mannered part of the surface. This feature is an important reason for using quadric surfaces here. Instead of finding an enveloping cone for the whole PPS, a surface for the frontier of its cross section with hyperplane $\sum_{i=1}^{m} x_i = k$ is found where k is an arbitrary scalar. Figure 1b shows such a cross section and a quadratic approximation for its frontier. To obtain this frontier, each DMU is scaled by an adequate constant such that it is projected on hyperplane $\sum_{i=1}^{m} x_i = k$ . This is allowed because of the CRS assumption. The multiplier for $DMU_j$ is $k/\sum_{i=1}^{m} x_i$. In this case, one of the inputs is dependent on the others and hence one of the inputs axes, say m' th input axis, can be omitted and the points are projected on $(m+s-1)$ -dimensional space.

Assume that $(\mathbf{x}_j, \mathbf{y}_j)$ which is an $(m + s-1)$ -vector, represents the data for $DMU_j$ after scaling. Consider the following quadric surface:

$$\phi(\mathbf{x},\mathbf{y}) = (\mathbf{x},\mathbf{y})A(\mathbf{x},\mathbf{y})^t + (\mathbf{x},\mathbf{y})\mathbf{b} = c,$$

Where:

- $\mathbf{x}$ = An $(m-1)$ -vector.
- $\mathbf{y}$ = An s-vector.
- $\mathbf{A}$ = The $(m+s-1)\times(m+s-1)$ matrix of coefficients of nonlinear terms.
- $\mathbf{b}$ = The $(m + s-1)$ column vector of coefficients of linear terms.
- $\mathbf{c}$ = A constant.

Although this representation of a quadratic polynomial leads to $(m + s-1)(m + s-2)/2$ excess coefficients, it simplifies the formulation format. In practice, the extended form is used instead.

**Definition 1. Enveloping Quadric Surface (EQS):** Quadric surface $\phi(\mathbf{x}, \mathbf{y}) = c$ envelops the set of points $D = \{(\mathbf{x}_j, \mathbf{y}_j) \mid \mathbf{x}_j \in R^{m-1}, \mathbf{y}_j \in R^s, j = 1,...,n)\}$ if $\phi(\mathbf{x}_j, \mathbf{y}_j) - c$ does not change sign for j = 1,...,n.

Since for most quadric surfaces, $\phi(\mathbf{x}, \mathbf{y}) \le c$ implies the points of the space surrounded by $\phi(\mathbf{x}, \mathbf{y}) = c$, $\phi(\mathbf{x}, \mathbf{y}) - c$ is assumed to be negative for all points of the PPS.

**Definition 2:** $\phi(\mathbf{x}, \mathbf{y}) = c$ is an EQS with the minimum output-oriented distance from points in D if for each EQS, say $\phi'(\mathbf{x}, \mathbf{y}) = c'$, we have;

$$\sum_{j=1}^{n} \theta_j \le \sum_{j=1}^{n} \theta'_j$$

Where:

$\theta_j = \text{minimum}\{\theta \mid \theta \ge 1, \phi(\mathbf{x}_j, \theta\mathbf{y}_j) = c\}$ and

$\theta'_j = \text{minimum}\{\theta' \mid \theta' \ge 1, \phi'(\mathbf{x}_j, \theta'\mathbf{y}_j) = c'\}$.

**Theorem 1:** Let $\phi(\mathbf{x}, \mathbf{y}) = c$ be an EQS of one sheet with minimum output-oriented distance to D such that $y_r$ variables are concave with respect to $x_i$ variables in the part of the surface that is used as the PPS frontier, that is

$$\frac{\partial^2 y_r}{\partial x_i^2} \le 0$$

for all $(\mathbf{x}, \mathbf{y}) \in D$. If $(\hat{\mathbf{x}}_j, \hat{\mathbf{y}}_j)$ is not a Pareto-efficient DMU, then $\phi(\mathbf{x}_j, \mathbf{y}_j) < c$.

**Proof:** First note that if $(\mathbf{x}_j, \mathbf{y}_j)$ does not lie on the surface then $\frac{\partial^2 y_r}{\partial x_i^2}$ at $(\mathbf{x}_j, \mathbf{y}_j)$ is of the same sign as $\frac{\partial^2 y_r}{\partial x_i^2}$ at the nearest point on the surface. Second, since $\phi$ is concave and is of one sheet, C (D), the convex hull of points in D, is enveloped by $\phi(\mathbf{x}, \mathbf{y}) = c$ because under these two conditions'

$$\text{hypograph}(\phi) = \{(\mathbf{x},\mathbf{y}) \in R^{m+s-1} \mid \exists \mathbf{y}' \ge \mathbf{y}, \phi(\mathbf{x},\mathbf{y}') = c\}$$

is convex. This shows that the sign of $\phi(\mathbf{x}, \mathbf{y}) - c$ does not change for C (D) and hence, since C (D) and the convex hull of the original-data DMUs correspond to each other, the benchmark for each DMU is also a member of C(D) after scaling. Assume $\phi(\mathbf{x}, \mathbf{y}) \le c$ for all points in D. We show that $\phi(\mathbf{x}_o, \mathbf{y}_o) < c$ if $(\hat{\mathbf{x}}_o, \hat{\mathbf{y}}_o)$ is a radial-inefficient DMU where $(x_o, y_o)$ denotes $(\hat{\mathbf{x}}_o, \hat{\mathbf{y}}_o)$ after scaling. Let $\theta > 1$ be a scalar such that $(\hat{\mathbf{x}}_o, \theta\hat{\mathbf{y}}_o)$ is radially efficient. The corresponding scaled point for $(\hat{\mathbf{x}}_o, \theta\hat{\mathbf{y}}_o)$ is $(\mathbf{x}_o, \theta\mathbf{y}_o)$, which is nearer to the surface with respect to $(\mathbf{x}_o, \mathbf{y}_o)$ and hence

$$\phi(\mathbf{x}_o,\mathbf{y}_o) < \phi(\mathbf{x}_o, \theta\mathbf{y}_o) \le c.$$

Now, assume that $(\hat{\mathbf{x}}_o, \hat{\mathbf{y}}_o)$ is a weak efficient DMU. Since $(\hat{\mathbf{x}}_o, \hat{\mathbf{y}}_o)$ must be an interior point of a weak efficient facet of the PPS, this point cannot lie on a quadric cone that envelops the PPS. This implies that scaled point $(\mathbf{x}_o, \mathbf{y}_o)$ does not lie on the surface $\phi(\mathbf{x}, \mathbf{y}) = c$ which is a cross section of such a cone □.

To find unknown coefficients of $\phi$ such that it envelops all scaled DMUs with minimum distance, the following LP problem is proposed:

$$\text{Minimize} \quad \sum_{j=1}^{n} \phi(\mathbf{x}_j,\mathbf{y}_j) \tag{1}$$

subject to

$$\phi(\mathbf{x}_j,\mathbf{y}_j) = (\mathbf{x}_j,\mathbf{y}_j)\mathbf{A}(\mathbf{x}_j,\mathbf{y}_j)^t + (\mathbf{x}_j,\mathbf{y}_j)\mathbf{b} \le 1, \quad j=1,\ldots,n,$$

**A**, **b** unrestricted in the sign of their entries,

where, c has been replaced with the arbitrary constant 1. Model (1) is an LP with $(m+s-1)^2+(m+s-1)=(m+s)(m+s-1)$ variables unrestricted in sign and n structural constraints. Since $n >> (m+s)$, its dual program is computationally preferable to solve:

$$\text{Minimize} \quad \sum_{j=1}^{n} w_j \tag{2}$$

subject to

$$\sum_{j=1}^{n} (\mathbf{x}_j,\mathbf{y}_j)^t(\mathbf{x}_j,\mathbf{y}_j)w_j = \sum_{j=1}^{n} (\mathbf{x}_j,\mathbf{y}_j)^t(\mathbf{x}_j,\mathbf{y}_j),$$

$$\sum_{j=1}^{n} (\mathbf{x}_j,\mathbf{y}_j)w_j = \sum_{j=1}^{n} (\mathbf{x}_j,\mathbf{y}_j),$$

$$w_j \ge 0 \qquad j=1,\ldots,n,$$

where $w_j$'s are dual variables. Note that $(\mathbf{x}_j, \mathbf{y}_j)$ is a row vector.

Conditions i ii and iv are embedded in model (1). Imposing condition (iii) on the model is not an easy task because it needs

$$\frac{\partial^2 y_{rj}}{\partial x_{ij}^2}$$

to be negative in sign for all i, r and j. These second order differentiation conditions lead to nonlinear constraints. One way to overcome this difficulty is considering special quadric surfaces with suitable shapes and concavity like elliptic paraboloid but this approach is too restrictive. Another way is considering some outer points with adequate variations such that they control the shape of the surface indirectly. These points should violate the constraints in (1). Denoting these points by $(\mathbf{x}'_1, \mathbf{y}_1')$,...,$(\mathbf{x}'_p, \mathbf{y}_p')$, the following constraints are added to model (1).

$$\phi(\mathbf{x}'_j, \mathbf{y}_j') \ge 1, \qquad j = 1,\ldots,p.$$

This approach is used in the illustrative example in the next section.

After determining quadric surface $\phi(\mathbf{x}, \mathbf{y}) = 1$ the efficiency score of $DMU_j$ is obtained by solving $\phi(\mathbf{x}_j, \theta\mathbf{y}_j) = 1$, which is a quadratic equation on $\theta$. If this equation has two answers, the one which is greater than 1 is the efficiency score. If both answers are greater than one, the smaller is chosen as the efficiency score.

**Comparing the computational complexity of EQS and CCR approaches:** In the efficiency evaluation with the CCR model, solving n LPs, each one with m+s constraints and n+m+s+1 variables is needed. Accepting $O(1d^2)$ as the average order of computational complexity of the simplex method for solving LPs with 1 variables and d constraints (Bazzara *et al.*, 1990), the computational complexity of the CCR approach is $O(n^2(m+s)^2)$ using the simplex algorithm. For the proposed approach we need $n(m-1)$ additions and $n(m+s)$ multiplications for scaling,

$$(n-1)\left(\frac{(m+s-1)}{2}+2(m+s-1)\right)$$

$$= (n-1)\left(\frac{(m+s-1)(m+s-2)}{2}+2(m+s-1)\right) = \frac{(n-1)(m+s-1)(m+s+2)}{2}$$

additions for computing the objective coefficients and

$$(p+n)\left(\frac{m+s-1}{2}\right)+(m+s-1) = \frac{(p+n)(m+s)(m+s-1)}{2}$$

multiplications for the entries of the constraints matrix where, p is the number of extra points used for controlling the shape of EQS and can be considered to be $5m(s-1)$. In summary, the complexity of initializing computations is of $O(n(m+s)^2)$. Therefore the total computations are of $O(n(m+s)^4)$ when the simplex method is employed to solve the LP problem (2) used in the approach. Since $n >> (m+s)$, the proposed approach leads to a significant reduction in the computations involved in the CCR approach.

**A comparison of the EQS approach with the other approaches for large-scale applications:** There are several approaches to streamline the DEA models in large-scale problems. Some of them like approaches proposed by Dulá and Helgason (1996), Dulá *et al.* (1997) and Korhonen and Halme (1996) strive to find the frame of the PPS (efficient DMUs). Unfortunately, these approaches are not easy-to-implement in practice while the method proposed in this paper performs efficiency evaluation by solving just one LP problem. Another method to deal with large-scale problems is the partitioning method. In general, the partitioning methods can reduce the computational time (not necessarily computational efforts) especially when they are performed on a parallel system. In other words, after partitioning the main problem, each part of the problem can be handled by a separate CPU. Barr and Durchholz (1997) developed a partitioning method for solving the DEA models. Although their method usually decreases the involved computations by

reducing the size of the LPs that must be solved, it increases the number of LPs to a great extent. Therefore the total saving in the amount of computations in this method is considerably less than that in the EQS approach. Furthermore, the growth of the computational errors cannot be avoided by using parallel systems in the partitioning approaches.

## ILLUSTRATIVE EXAMPLE

In order to illustrate the details of the EQS approach, an example of 15 DMUs which all use 2 inputs to produce one output is described here. Table 1 shows the data for this example. To scale DMUs, each one is multiplied by

$$\frac{10}{\hat{x}_{1j} + \hat{x}_{2j}}.$$

For instance, the first DMU, $(\hat{x}_{11},\hat{x}_{21},\hat{y}_{11}) = (13.68, 10.33, 12.96)$ is changed to $(x_{11}, y_{11}) = (5.698, 5.398)$ after scaling. Considering the quadric surface

$$\phi(x_1,y_1) = a_1x_1 + a_2x_1^2 + a_3y_1 + a_4y_1^2 + a_5x_1y_1 = 1,$$

the constraints of the model are computed as $\phi(x_{1j},y_{1j}) \le 1$ $(j=1...15)$. The objective function, which is the sum of these constraints, is

$$76.29a_1 + 463.81a_2 + 81.41a_3 + 464.08a_4 + 412.52a_5,$$

that must be maximized. For the concavity constraints, we consider the circle $(x_1-5)^2 + y_1^2 = r^2$ where, $r = 8.01$ is the maximum of $\|(x_{1j}, y_{1j}) - (5,0)\|$ over $j = 1,...,15$. The part of the circle with positive $x_r$'s and $y_r$'s, is a feasible solution for the problem. Now the following points on this circle are considered as the points that must not be enveloped by $\phi(x_1, y_1) = 1$.

$$\begin{aligned}
&(x'_{11},y'_{11}) = (1,\sqrt{r^2-16}+\varepsilon),\\
&(x'_{12},y'_{12}) = (3,\sqrt{r^2-4}+\varepsilon),\\
&(x'_{13},y'_{13}) = (5,r+\varepsilon),\\
&(x'_{14},y'_{14}) = (7,\sqrt{r^2-4}+\varepsilon) \quad \text{and}\\
&(x'_{15},y'_{15}) = (9,\sqrt{r^2-16}+\varepsilon),
\end{aligned}$$

where, $\varepsilon$ is a small positive scalar. By adding $\phi(x'_{1j}, y'_{1j}) \ge 1$ $(j = 1,...,5)$ to the model, the initializing process is completed.

The above procedure is coded and solved via Mathematica Ver. 5. The quadric surface evaluated by model (1) is

$$-0.2784x_1 + 0.0331x_1^2 + 0.395y_1 - 0.0206y_1^2 - 0.007x_1y_1 = 1.$$

Table 1: Comparing the CCR and the EQS approachs (output-oriented)

| | | | | CCR | | EQS | |
|---|---|---|---|---|---|---|---|
| DMU No. | Input 1 | Input 2 | Output 1 | Score | Rank | Score | Rank |
| 1 | 13.68 | 10.33 | 12.96 | 1.361 | 15 | 1.407 | 15 |
| 2 | 12.00 | 11.98 | 19.20 | 1.000 | 1 | 1.000 | 1 |
| 3 | 2.42 | 21.61 | 9.84 | 1.000 | 2 | 1.000 | 2 |
| 4 | 17.98 | 6.00 | 12.24 | 1.020 | 7 | 1.022 | 6 |
| 5 | 16.80 | 7.19 | 12.95 | 1.093 | 8 | 1.095 | 8 |
| 6 | 6.48 | 17.52 | 12.00 | 1.152 | 11 | 1.190 | 11 |
| 7 | 8.40 | 15.60 | 11.76 | 1.334 | 14 | 1.387 | 14 |
| 8 | 14.41 | 9.59 | 15.12 | 1.122 | 9 | 1.151 | 9 |
| 9 | 15.64 | 8.41 | 15.84 | 1.000 | 3 | 1.000 | 3 |
| 10 | 4.83 | 19.20 | 10.81 | 1.128 | 10 | 1.154 | 10 |
| 11 | 19.20 | 4.81 | 10.80 | 1.000 | 4 | 1.000 | 4 |
| 12 | 9.63 | 14.39 | 16.57 | 1.018 | 6 | 1.058 | 7 |
| 13 | 7.68 | 16.32 | 12.24 | 1.225 | 13 | 1.271 | 13 |
| 14 | 21.61 | 2.43 | 7.21 | 1.000 | 5 | 1.022 | 5 |
| 15 | 12.48 | 11.52 | 15.84 | 1.184 | 12 | 1.209 | 12 |

The CCR and EQS efficiency scores and the corresponding rank orders are ahown in Table 1. Although the condition $n >> (m + s)$ does not hold in this small-size problem, the efficiency scores evaluated by the EQS approach is a good approximation for those by the CCR approach. The maximum and average errors are 0.053 and 0.0219, respectively which are quite acceptable for the aim of ranking DMUs. The ranks resulted of the EQS and CCR approaches are exactly the same, except for $DMU_4$ and $DMU_{12}$ that have 1 position difference in the ranking. Since CCR-efficient DMUs $DMU_2$, $DMU_3$, $DMU_9$ and $DMU_{11}$ are evaluated as efficient in the EQS approach, they are Pareto-efficient by Theorem 1.

Now assume that the results of the EQS approach are available and the decision maker wishes to exploit these results to obtain the exact CCR scores. To this end, define $I_e = \{2, 3, 9, 11\}$ and $I_u = \{1,....,n\} - I_e$ as the index sets of DMUs with efficient and unknown status, respectively. First, solve the following modified CCR model for $o \in \{5, 12, 14\}$.

Minimize $\theta$

subject to

$$\begin{aligned}
&\theta\hat{\mathbf{x}}_o - \sum_{j\in I_e}^{n} \lambda_j\hat{\mathbf{x}}_j \ge \mathbf{0}, \qquad (3)\\
&\sum_{j\in I_e}^{n} \lambda_j\hat{\mathbf{y}}_j \ge \hat{\mathbf{y}}_o,\\
&\lambda_j \ge 0 \quad j = 1,....,n,\\
&\theta \quad \text{unrestricted in sign}
\end{aligned}$$

5, 12 and 14 are the indices of nearly-efficient DMUs in the EQS approach. Solving the above model shows that $DMU_{14}$ is CCR-efficient. 14 is removed from the $I_u$ and is added to $I_e$. Now, solve the model (3) for $o \in I_u$. This leads to the CCR results shown in Table 1. Note that the size of the constraint matrix in (3) is 3×5 which is very much

smaller than 3×16 the size of the constraint matrix in the classical CCR model. Furthermore, for the EQS-efficient DMUs solving model 3 is not required. Therefore, the results of the EQS approach can effectively streamline the CCR model.

## CONCLUSION

This study introduces an approach to find a quadratic frontier for the PPS in DEA. In contrast with non-parametric methods, this approach estimates a formula for the efficiency frontier that simplifies the process of evaluating efficiency scores. Therefore, this parametric DEA approach is ideal for large-scale problems which have smooth frontiers and the traditional DEA methods are computationally intensive to handle them. In the proposed approach, an enveloping quadric surface is characterized via solving an LP problem and a part of it with a suitable shape is used for efficiency estimations. The illustrative example shows that rank scores resulting from this approach are good approximations for the CCR efficiency scores even in small-size problems. In addition, the approach can be used as a means to expedite other DEA methods.

The study of computational complexity of the EQS approach shows that this approach decreases the number of LPs from n (the number of DMUs) to one. Each LP in the CCR approach has (m+s) constraints and (n+m+s+1) variables. The LP in the EQS approach has (m+s)(m+s-1) constraints and n variables. Therefore, the EQS approach leads to a great reduction in the computational efforts in large-scale problems in which $n >> (m + s)$.

Imposing some of the required conditions on the surface leads to nonlinear constraints. To make up for this drawback, some linear constraints are proposed to replace the nonlinear ones. Finding stronger linear constraints to be imposed on model (1) to assure the desired frontier properties can be considered for further investigations.

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