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A Statistical Perspective for Improving Approximation by Modified Szasz Operator

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Abstract: We have proposed and studied another modification of the Szasz Operator that arises from a statistical perspective of the problem. The study is supported and illustrated by an empirical simulation study aimed at illustrating the potential numerical improvement for some well known simple functions.

Key words: Polynomial approximation modified Szasz-Mirakjan operator, simulated empirical study

INTRODUCTION

Szasz (1950) proposed the following generalization of the well known Bernstein's polynomials extending it to the infinite interval:

$$S_n(f;n) = [e^{-nx}\sum_{k=0}^{\infty} \{(nx)^k/k!\} f(k/n)]$$

for all $f \in C_{[0,1)}$

Heinz-Gerd Lehnhoff (1981), in particular, proposed the Modified SzaszMirakjan Operator:

$$S_n(f,x) = [\sum_{k=0}^{n} T_k f(k/n)]/[\exp(nx)]$$

for $x \in C[0,1]$, $f \in C[0,1)$.

Where, $T_k = (nx)^k/k!$, Ak = 0,....,n.
Motivated by the above modification, we have proposed analogously, though slightly differently, a Modified Szasz Operator, as follows:

$$MS[n] = \frac{(\sum_{k=o}^{n} T_k f(\frac{k}{n}))}{\sum_{k=0}^{n} T_k} \qquad (1.0.1)$$

The aforesaid Modified Szasz Operator MS[v] will approximate the function $\phi(\xi)$ using its values at equidistant 'knots' in the interval [0; 1]. This would be without loss of generality, as the approximation would also hold for $X[\alpha;\beta]$, and it holds conversely. Essentially, X [0;1] and X $[\alpha;\beta]$ are identical, for all practical purposes; they are linearly isometric as normed spaces, order isomorphic as lattices, and isomorphic as algebras (rings).

Further, we have proposed and studied, in what follows, a computerizable Iterative Algorithm with the motivation of having improved approximation by the aforesaid operator $M\Sigma[v]$, using the same information, namely, the values of the function at the stipulated knots.

MOTIVATING OBSERVATION AND THE ITERATIVE ALGORITHM

Before we go into the details of the Iterative Algorithm' for improved approximation by our Modified Szasz Operator, $M\Sigma[v]$, we observe the motivating fact seminal to its proposition. Whereas, all the approximating polynomials are concerned with the knots and the weight functions defined over these knots; none of them completely uses the information available about the unknown function (targeted for the Approximation), through the known values of the function at these knots. Such information could well be used in constructing or modifying the weight function, possibly gainfully.

In fact, as per the Statistical Perspective, such an information should be used gainfully in all the estimation problems and the approximation problem is an estimation problem per this perspective, as we are essentially estimating the unknown function through our weight function, defined at the chosen Knots for the approximation operator, at hand.

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In fact, if we con…ne ourselves to the polynomial approximation by Positive Linear Operators, we could well observe the fact that the weights may be interpreted as probabilities, in the context of using the Operator, say, Oν (ϕ) (ξ) the desirable/well known Statistical Property of Asymptotic Unbiasedness ensures that the Mathematical Expectation (the value on an average) of our approximating polynomial, namely the estimate Oν (ϕ) (ξ)) must approach the function, as the number of knots used, namely ν becomes very large:

$$E\, fO_{\nu}\, (\phi)\, (\nu) g\, !\, \phi\, (\xi);\ \text{as}\ \nu\, !\, 1:$$

In the above context and using the aforesaid Statistical Perspective of the approximation being an estimation problem, the estimated error could well be interpreted as the estimated Bias. Therefore, if we reduce this bias to make the estimate better, we should be accelerating the asymptotic convergence of the approximating polynomial. This will be feasible, inasmuch as we would be reducing the Error in approximation at each iteration, using the currently available estimate of the error to bring the approximating polynomial closer to the (unknown) function.

Let us denote the Error by E (ξ) then:

$$E(x) = MS[n](f)(x)-f(x)$$

However, E (ξ) is unknown since ϕ(ξ) is unknown. Therefore, we have to estimate it and we do so by using the same Modified Szasz polynomial MΣ[ν] (ϕ) (ξ): The only diference would consist in the fact that we have E in place of ϕ and analogously the values of this unknown Error Function are readily available through the difference between the Known and the Estimated values of the function at these knots: (k/n), respectively.

Hence, if we define the resultant estimating polynomial (of degree ν n, at most) by E [ν] (ϕ) (ξ), (keeping in mind implicitly that MΣ [ν] (ϕ) (ξ) is the approximating polynomial, without complicating the notations by explicit incorporation of this fact in our notation), we have

$$\begin{aligned} E[n]\,(f)\,(x) &\equiv MS[n]\ \{MS[n]\,(f)-f\} \\ &\equiv MS[n]^2\,(f)(x)-MS[n]\,(f)\,(x) \end{aligned}$$

Also, as we use this polynomial as an Estimated Bias and proceed with the correction, the resultant Improved Szasz approximating polynomial, at the first go/iteration, say I (1) MΣ [ν] (ϕ)) (ξ) will be:

$$\begin{aligned} I\,(1)\,MS\,[n]\,(f)\,(x) &= MS\,[n]\,(f)\,(x) - E\,[n]\,(f)\,(x) \\ &\quad 2\,MS\,[n]\,(f)\,(x) - MS\,[n]^2\,(f)\,(x) \qquad (1) \end{aligned}$$

$$[I-(I-MS\,[n]^2\,(f)\,(x) \qquad (2)$$

If we proceed exactly analogously for the Improved Szasz approximating polynomial at the second iteration, we will be led to:

$$\begin{aligned} &l\,(2)\,MS\,[n]\,(f)\,(x)l \\ &= l\,(1)\,MS\,[n]\,(f)\,(x) - MS\,[n]\,l\,(1)\,MS\,[n]\,(f)\,(x) - (f)\,(x) \\ &= 2\,MS\ [n]\ (f)\ (x) - MS\ [n]^2\ (f)\ (x) + \\ &\quad MS\ [n]^3\ (f)\ (x) + MS\ [n]\ (f)\ (x) \\ &= 3\,MS\ [n]\ (f)\ (x) - 3MS\ [n]^2\ (f)\ (x) \\ &= MS\ [n]^3\ (f) = [l- (l - MS\ [n]^3]\ (f)\ (x) \end{aligned}$$

Thus, in general, if we proceed exactly analogously for the Improved Szasz approximating polynomial at the iteration, we will be led to:

$$l\,(k)\,MS\,[n]\,(f)\,(x) = [lo - (l - MS\,[n])^{(k+1)}]\,(f)\,(x) \quad (2.0.3)$$

Now, we note that since it is intractable analytically to assess the achieved improvement in Approximation by the Modified Szasz Operator MΣ [ν] through the aforesaid Iterative Improvement Algorithm, we resort to an Empirical Simulation Study to obtain a numerical measure of the goodness of the Algorithm, as in the following section. It could well be noted that the Algorithm is evidently Computerizable for its execution.

THE EMPIRICAL SIMULATION STUDY

To illustrate the gain in efficiency by using our proposed Iterative Algorithm of Improvement of Modified Szasz Polynomial Approximation, we have carried out an Empirical Study. We have taken the example-cases of ν = 2; 4; 7 and 10, (i.e., ν+1 = 3; 5; 8 and 11 as knots) in the empirical study to numerically illustrate the relative gain in efficiency in using the Algorithm vis-à-vis the Modified Szasz Polynomial in each example-case of the ν value.

Essentially, the empirical study is a Simulation Empirical Study, because we would have to assume that the function being tried to be approximated, namely ϕ (ξ) (it will then be approximating the function f (x) in the interval [0;1], {the standard Conventional interval for the Approximation of the function f (x)'}) being known to us.

Once again, we have confined to the illustrations of the relative gain in efficiency by the iterative improvement to the approximation of the following four illustrative functions in the interval [0;1]:

$$\phi\,(\xi) = \exp\,(\xi);\ \ln\,(2 + \xi);\ \sin\,(2 + \xi);\ 10^{x}$$

These would be approximated in the interval [0; 1] by the Modified Szasz Operator MΣ [ν] and subsequently also approximated by using the Computerizable Iterative Improvement Algorithm.

To illustrate the POTENTIAL of improvement with our proposed Iterative Algorithm, we have considered THREE Iterations and the numerical values of four quantities, namely three Percentage Relative Errors (PRE) corresponding to our Improvement Iterations (# = 1 or 2 or 3), (PRE_I(#)ML[n]) and the Modified Szasz Polynomial (PRE_MS[n]). Also, we consider the corresponding three Percentage Relative Gains (PRG) by using our Proposed Iterative Algorithmic Modified Szasz Polynomial subsequent upon the approximation by Modified Szasz Polynomial, (PRG_I (#) MS[n]; # = 1(1)3). Now, these quantities are defined, as follows.

The Percentage Relative Error using Modified Szasz polynomial with n intervals in [0; 1],
i.e. [(k-1) = (n-1); k=ν]; k = 1 (1) ν:

$$PRE_{MS}(n) = \frac{\int_0^1 |(f(x) - MS[n](f)(x)dx}{\int_0^1 (f(x))} * 100$$

The Percentage Relative Errors respective to the Modified Szasz Polynomial and respective to the First, Second and the Third Algorithmic Improvement Iteration Polynomials have been tabulated respectively, for each of the examples and the number of approximation Knots/Intervals. Also the Percentage Relative Gains by using the proposed Algorithmic Improvement Iteration: I # (e.g., 1, or 2, or 3) Polynomials with the n intervals in [0; 1] over using solely the Modified Szasz Polynomial for the approximation of the (targeted) function, f, are tabulated in the appendix.

$$PRG_1 MS(n) = \frac{(PRE_{MS}[n] - PRE_1 MS[n]}{(PRE_{MS}[n])} * 100; \# = 1 \text{ or } 2 \text{ or } 3$$

These aforesaid SEVEN numerical quantities have been computed using Maple 10, for all the four illustrative functions mentioned in the preceding Section 3, for four values of ν, namely ν = 2; 4; 7 and 10. These values have been tabulated in the appendix. Table A.1-A.4 contain these quantities when the function f(x) has been taken as exp(x); ln (2+ξ); sin (2+ξ) and 10^{ξ}, respectively. The Percentage Relative Errors (PRE's) for our Algorithmic Iterative Polynomial Approximations are progressively lower with each subsequent iteration, as compared to that for the Modified Szasz Polynomial Approximation, for all the illustrative functions.

Consequently, the Percentage Relative Gains (PRG 0s) due to the use of our proposed Algorithmic Iterative Polynomial Approximations vis-à-vis the Modified Szasz Polynomial Approximation are also increasing progressively with each subsequent iteration, for all the illustrative functions.

Lastly, it is very heartening to note that when we use ten (n = 10) intervals, i.e., eleven knots for the polynomial approximation, the Percentage Relative Gain (PRG) becomes quite significant for the third iteration. Otherwise also, the speed of convergence is highly accelerated by the Iterative Algorithmic improvement in the Modified Szasz Polynomial, using the statistical perspective reducing the Bias in the Estimator/Approximating Polynomial. It is worth noting again that the Modified Szasz Operator is nothing but the weighted average of the data, i.e., the known values of the unknown function f at the ν+1 knots.

Table A.1: Algorithmic improvement efficiency [ϕ(ξ) = exp(ξ)]

Model # ν !	2	4	7	10
ПРЕ_MΣ [ν]	10:942	8:242	6:176	5:017
ПРЕ_I (1) MΣ [ν]	07:240	4:869	3:401	2:646
ПРЕ_I (2) MΣ [ν]	06:537	3:738	2:508	1:911
ПРЕ_I (3) MΣ [ν]	05:716	3:189	2:068	1:550
ПРГ_I (1)MΣ [ν]	33:832	40:930	44:932	47:251
ПРГ_I (2)MΣ [ν]	40:263	54:654	59:389	61:918
ПРГ_I (3)MΣ [ν]	47:757	61:313	66:518	69:101

Table A.2: Algorithmic improvement efficiency [ϕ(ξ) = Sin (2+(ξ)]

Model # ν !	2	4	7	10
ПРЕ_MΣ [ν]	16:096	11:879	8:693	6:933
ПРЕ_I (1) MΣ [ν]	10:015	6:541	4:367	3:299
ПРЕ_I (2) MΣ [ν]	8:753	4:702	3:002	2:231
ПРЕ_I (3) MΣ [ν]	7:600	3:803	2:390	1:757
ПРГ_I (1)MΣ [ν]	37:775	44:935	49:768	52:409
ПРГ_I (2)MΣ [ν]	45:617	60:419	65:467	67:824
ПРГ_I (3)MΣ [ν]	52:782	67:985	72:511	74:661

Table A. 3: Algorithmic improvement efficiency [ϕ(ξ) = ln (2+ξ)]

Model # ν !	2	4	7	10
ПРЕ_MΣ [ν]	6:719	4:170	2:723	2:042
ПРЕ_I (1) MΣ [ν]	2:135	1:310	0:873	0:665
ПРЕ_I (2) MΣ [ν]	1:545	0:902	0:598	0:452
ПРЕ_I (3) MΣ [ν]	1:352	0:720	0:471	0:352
ПРГ_I (1)MΣ [ν]	68:217	68:588	67:935	67:416
ПРГ_I (2)MΣ [ν]	77:001	78:380	78:021	77:866
ПРГ_I (3)MΣ [ν]	79:874	82:741	82:713	82:77

Table A. 4: Algorithmic improvement efficiency [ϕ(ξ) = 10^{ξ}]

Model # ν !	2	4	7	10
ПРЕ_MΣ [ν]	24:07	19:90	16:06	13:65
ПРЕ_I (1) MΣ [ν]	31:42	20:42	14:06	10:87
ПРЕ_I (2) MΣ [ν]	18:08	11:29	8:91	7:49
ПРЕ_I (3) MΣ [ν]	17:33	10:96	7:72	5:91
ПРГ_I (1)MΣ [ν]	27:02	2:06	12:45	20:37
ПРГ_I (2)MΣ [ν]	26:08	43:26	44:52	45:13
ПРГ_I (3)MΣ [ν]	29:08	44:92	51:93	56:70

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