Journal of Applied Sciences

ISSN 1812-5654

Journal of Applied Sciences 7 (6): 830-834, 2007
ISSN 1812-5654

Partial Densities on the Rational Numbers

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Abstract: Conditions are obtained under which a partial density on a certain class of locally compact abelian groups can be extended to a density. These groups each consist of the integral multiples of a particular rational number, with the discrete topology. It is established that a collection of compatible measures on some of the quotient groups of one such group can be induced by a measure on a particular quotient group. This leads to the result that a partial density can be extended to a density when compatibility conditions are satisfied.

Key words: Partial density, extension to density

INTRODUCTION

Interest in probability measures on locally compact abelian groups has been sustained for a long period. Berg and Rubel (1969) studied the Banach algebra of densities on the compact quotients of a locally compact abelian group, where a density is a system of compatible measures on these compact quotients. They characterized groups G for which any density is induced by a measure on the semi-periodic compactification of G.

Semigroups of probability measures have also received attention. Siebert (1981) considered semigroups of probability measures and obtained results about the generating functional.

Carnal and Feldman (1995) studied probability measures on a locally compact abelian group. They found conditions under which the identification of a probability measure given the absolute value of its Fourier transform is possible up to a shift and a central symmetry. Carnel and Feldman (1997 and 2000) continued their study of probability distributions which can be identified when the absolute value of their Fourier transforms are known. Mauro Del Muto and Talamanca (2006) considered a Markov process on a locally compact, noncompact, totally disconnected, metrizable abelian group G and showed that the process may be obtained as a limit of discrete processes on discrete quotient groups of G.

Niederreiter and Sookoo (2000) obtained conditions under which a partial density on the group of integers can be extended to a density. Niederreiter and Sookoo (2002) also investigated conditions under which a partial density on a locally compact abelian group can be extended to a density. In this study, we obtain conditions for extending a partial density to a density when the LCA group is the group of integral multiples of a particular rational number, with the discrete topology.

DEFINITIONS AND NOTATIONS

Notation 1: Let Z denote the set of integers and R the set of real numbers.

Definition 1: Let $\frac{p}{q}$ and $\frac{r}{s}$ be rational numbers in lowest terms. $\frac{p}{q}$ is called an r-divisor of $\frac{r}{s}$ if $\frac{r}{s} \div \frac{p}{q}$ is a whole number.

Definition 2: The r-greatest common divisor (RGCD) of $\frac{p}{q}$ and $\frac{r}{s}$ is the largest

rational number $\frac{h}{k} \ni \frac{p}{q} \div \frac{h}{k}$ and $\frac{r}{s} \div \frac{h}{k}$ are whole numbers.

Definition 3: The r-least common multiple (RLCM) of two rational numbers $\frac{p}{q}$ and $\frac{r}{s}$ in lowest terms is the smallest rational number $\frac{u}{v} \ni \frac{u}{v} \div \frac{p}{q}$ and $\frac{u}{v} \div \frac{r}{s}$ are whole numbers.

Definition 4: $\frac{p}{q}$ and $\frac{r}{s}$ are called r-relatively prime if the greatest common divisor of p and r is 1 and the greatest common divisor of q and s is 1.

Notation 2: If p∈ {0,h,2h,...,I}, then p denotes the coset {...,-I+p,p,I+p,...} of IZ in hZ/IZ.

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A density (c.f. Berg and Rubel (1969)) on an LCA group G consists of a system of measures on subgroups of G of compact index satisfying compatibility conditions.

Notation 3: For a suiatable index set A, $\{H_\alpha | \alpha \in A\}$ denotes the set of all subgroups of G of compact index. $\{G_\alpha | \alpha \in A\}$ denotes the set of compact quotients of G, where

$$G_\alpha = G / H_\alpha, \alpha \in A.$$

Definition 5: Let D be a system of measures given by

$D = \{\mu_\alpha |\ \mu_\alpha$ is a probability measure on $G_\alpha,\ \alpha \in A\}$

D is called a density on G if the following condition is satisfied:

If ψ: $G_\beta \to G_\alpha$ is the natural homomorphism from G_β to a quotient G_α of G_β, then for any Borel set B in G_α, $\mu_\alpha(B) = \mu_\beta[\psi^{-1}(B)]$.

Next we define a partial density (Niederreiter, 1975).

Definition 6: Let G be an LCA group and $\{H_\alpha | \alpha \in A\}$ be the set of all subgroups of compact index of G. For a subset B of A, let

$P = \{\mu_\alpha | \mu_\alpha$ is a probability measure on $G_\alpha,\ \alpha \in B\}$ be a system of measures satisfying the following compatibility condition:

If $H_\alpha \supseteq H_{\beta 1}$ and $H_\alpha \supseteq H_{\beta 2}$ where $\alpha \in A$, $\beta_1 \in B$ and $\beta_2 \in B$, then $\mu_{\beta 1}$ and $\mu_{\beta 2}$ induce the same measure on G_α.

Then P is called a partial density on G.

RLCM AND RGCD OF RATIONALS

The following results will be needed to establish the results on systems of measures.

Theorem 1: If $\frac{h}{k}$ is the RGCD of $\frac{p}{q}$ and $\frac{r}{s}$, then h is the GCD of p and r and k is the LCM of q and s.

Proof: Since $\frac{h}{k}$ must be as large as possible, h must be as large as possible and k must be as small as possible. h has no common factor with k, so it must go into p. Similarly, h must go into r. Hence h must be the GCD of p and r. Also, q and s must be divisors of k. Hence k must be the LCM of q and s.

Theorem 2: The RLCM of $\frac{p}{q}$ and $\frac{r}{s}$ = {LCM of p and r}/ {GCD of q and s}

Proof: Let the RLCM of $\frac{p}{q}$ and $\frac{r}{s}$ be $\frac{u}{v}$ in lowest terms.

Then $\frac{u}{v} \div \frac{p}{q}$ is a whole number, that is, $\frac{u}{v} \times \frac{q}{p}$ is a whole number. Hence p must be a divisor of u and v must be a divisor of q. Similarly, r must be a divisor of u and v must be a divisor of s. Since $\frac{u}{v}$ must be as small as possible, u must be the LCM of p and r and v must be the GCD of q and s.

Corollary 1: RLCM of

$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n}$

= {LCM of $p_1, p_2, \ldots, p_n$}/ {GCD of $q_1, q_2, \ldots, q_n$}.

EXTENSION OF PARTIAL DENSITIES ON RATIONALS

Theorem 3: If $\frac{p}{q}$ and $\frac{r}{s}$ are r-relatively prime, μ_1 is a probability measure on $\frac{1}{qs}Z/\frac{p}{q}Z$ and μ_2 is a probability measure on $\frac{1}{qs}Z/\frac{r}{s}Z$, then there exists a measure μ on $\frac{1}{qs}Z/prZ$ which induces μ_1 and μ_2.

Proof: Let μ be the measure on

$$\{\frac{1}{qs}/\frac{p}{q}Z\} \times \{\frac{1}{qs}Z/\frac{r}{s}Z\}$$

given by $\mu = \mu_1 \times \mu_2$.

Then

$$\mu[A \times (\frac{1}{qs}Z/\frac{r}{s}Z)] = \mu_1(A)\mu_2(\frac{1}{qs}Z/\frac{r}{s}Z) = \mu_1(A),$$
$$\forall A \in \frac{1}{qs}Z/\frac{p}{q}Z$$

Similarly,

$$\mu[(\frac{1}{qs}Z/\frac{p}{q}Z)\times B]=\mu_1(\frac{1}{qs}Z/\frac{p}{q}/Z)\,\mu_2(B)=\mu_2(B),$$
$$\forall B\in\frac{1}{qs}Z/\frac{r}{s}Z.$$

Hence μ induces μ_1 and μ_2.

Let a and b be integers. Then

$$\frac{X}{qs}=\frac{a}{qs}\bmod\frac{p}{q}\text{ and }\frac{X}{qs}=\frac{b}{qs}\bmod\frac{r}{s}\Leftrightarrow$$

X = a + ups and X = b + vqr

where u and v are integers

$\Leftrightarrow$ X = r mod pqrs, where r is an integer uniquely determined mod pqrs by a and b, (from the Chinese Remainder Theorem)

$$\Leftrightarrow X=r+kpqrs, k\in Z\Leftrightarrow\frac{X}{qs}=\frac{r}{qs}+kpr$$
$$\Leftrightarrow\frac{X}{qs}=\frac{r}{qs}\bmod pr$$

Hence $\frac{1}{qs}Z/prZ\cong(\frac{1}{qs}Z/\frac{p}{q}Z)\times(\frac{1}{qs}Z/\frac{r}{s}Z)$

Hence μ can be considered as a measure on $\frac{1}{qs}Z/prZ$ and it is compatible with μ_1 and μ_2.

Remark 1 In the previous theorem, $\frac{1}{qs}$ is the RGCD of $\frac{p}{q}$ and $\frac{r}{s}$.

Theorem 4 Let RCGD $(\frac{p}{q},\frac{r}{s})=d.$ If μ_1 is a measure on $hZ/\frac{p}{q}Z$ and μ_2 is a measure on $hZ/\frac{r}{s}Z,$ where h is a divisor of $\frac{p}{q}$ and $\frac{r}{s}$ and $\{\mu_1,\mu_2\}$ is a partial density on hZ, then there exists a measure μ on hZ/IZ which induces μ_1 and μ_2, where I is the RLCM of $\frac{p}{q}$ and $\frac{r}{s}$.

Proof: We can reduce the problem to d/h simpler ones. In each case, we will have two r-relatively prime numbers, as in the previous theorem. The equations in which the measures of the cosets {ih,ih + d,ih +2d,...,ih + (I - d)}; I = 0,h,2h,...d-h, occur do not involve the measures of any other cosets. We show this as follows:

Let μ_1 take the values $x_0,x_1,\ldots,x_{\frac{p}{hq}-1}$ and μ_2 the values $y_0,y_1,\ldots,y_{\frac{r}{hs}-1}$. We must establish the existence of a measure μ on hZ/IZ such that

$$\mu(ih)+\mu(\frac{p}{q}+ih)+\mu(\frac{2p}{q}+ih)+\ldots+\mu(I-\frac{p}{q}+ih)$$
$$=x_i; i=0,1,2,\ldots,\frac{p}{hq}-1.$$

and

$$\mu(jh)+\mu(\frac{r}{s}+jh)+\mu(\frac{2r}{s}+jh)+\ldots+\mu(I-\frac{r}{s}+jh)$$
$$=\mu_j; j=0,1,2,\ldots,(\frac{r}{hs}-1)$$

Since μ_1 and μ_2 are compatible,

$$x_0+x_{\frac{d}{h}}+x_{\frac{2d}{h}}+\ldots+x_{\frac{p}{q}-\frac{d}{h}}=y_0+y_{\frac{d}{h}}+y_{\frac{2d}{h}}+\ldots+y_{\frac{r}{s}-\frac{d}{h}}=\alpha$$

say.

The set S_0 of equations in which $\mu(0)$, $\mu(d)$,$\mu(2d)$,...,$\mu(I-d)$ occur does not involve measures of any other cosets of hZ/IZ, so they can be solved independently.

If $\alpha = 0$, let $\mu(i) = 0$ for $I\in\{0,d,2d,\ldots,I-d\}$.

If $\alpha > 0$, consider the following problem:

Let λ_1 be a measure on $hZ/\frac{ph}{qd}Z,$ taking the values

$$\frac{x_0}{\alpha},\frac{x_{\frac{d}{h}}}{\alpha},\frac{x_{\frac{2d}{h}}}{\alpha},\ldots,\frac{x_{\frac{p}{q}-\frac{d}{h}}}{\alpha}$$

and let λ_2 be a measure on $hZ/\frac{rh}{sd}Z$, taking the values

$$\frac{y_0}{\alpha},\frac{y_{\frac{d}{h}}}{\alpha},\frac{y_{\frac{2d}{h}}}{\alpha},\ldots,\frac{y_{\frac{r}{s}-\frac{d}{h}}}{\alpha}.$$

λ_1 and λ_2 are probability measures, $\frac{ph}{qd}$ and $\frac{rh}{sd}$ are r-relatively prime and also h is the RGCD of $\frac{ph}{qd}$ and $\frac{rh}{sd}$. From the previous theorem, there exists a measure λ on $hZ/\frac{Ih}{d}Z$ such that λ Induces λ_1 and λ_2. Therefore, if we multiply each equation in S_0 by $\frac{1}{\alpha}$ and replace $\frac{\mu(id)}{\alpha}$ by $\lambda(ih); i = 0,1,2,...,\frac{I}{d}-1$, then the new set of equations obtained has at least one solution such that $\lambda(ih) \geq 0; i = 0,1,2,...,\frac{I}{d}-1.$ Hence S_0 has solutions for which $\mu(id) \geq 0; i = 0,1,2,...,\frac{I}{d}-1.$ We can show, in a similar way, that the set S_j of equations in which $\mu(jh), \mu(jh+d), \mu(jh+2d),...,\mu(jh+I\text{-}d)$ occur (where j is a fixed, arbitrary element of $\{1,2,...,\frac{d}{h}-1\}$ has nonnegative solutions. Hence the original set of equations has nonnegative solutions and so there exists a measure μ which induces μ_1 and μ_2.

Theorem 5 Let $\{\mu_{\frac{p_i}{q_i}} | i = 1,2,...,k\}$ be a partial density on hZ, where $\mu_{\frac{p_i}{q_i}}$ is a measure on $hZ/\frac{p_i}{q_i}Z$, and let R_a be the RLCM of $\frac{p_1}{q_1},\frac{p_2}{q_2},...,\frac{p_a}{q_a}$. If RGCD $(R_a,\frac{p_{a+1}}{q_{a+1}})$ is an r-divisor of at least one of $\frac{p_1}{q_1},\frac{p_2}{q_2},...,\frac{p_a}{q_a}; a \in \{1,2,...,k-1\}$, then there exists a measure on hZ/R_kZ which induces $\mu_{\frac{pi}{qi}}; i \in \{1,2,...,k\}$.

Proof There exists a measure υ_2 on hZ/TZ which induces $\mu_{\frac{p_1}{q_1}}$ and $\mu_{\frac{p_2}{q_2}}$, from the previous theorem.

Let T = RGCD $(R_2,\frac{p_3}{q_3})$. Then T is an r-divisor of at least one of $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$. Assume that T is an r-divisor of $\frac{p_1}{q_1}$ Since $\{\mu_{\frac{p_1}{q_1}} | i = 1,2,...,k\}$ is a partial density, $\mu_{\frac{p_1}{q_1}}$ and $\mu_{\frac{p_3}{q_3}}$ induce the same measure, ω, on hZ/TZ and since υ_2 induces $\mu_{\frac{p_1}{q_1}}$, υ_2 and $\mu_{\frac{p_3}{q_3}}$ also induce the measure ω on hZ/TZ. Hence υ_2 and $\mu_{\frac{p_3}{q_3}}$ comprise a partial density on hz/TZ. From the previous theorem, there exists a measure υ_3 on hZ/R_3Z which induces υ_2 and $\mu_{\frac{p_3}{q_3}}$, since the RLCM of R_2 and $\frac{p_3}{q_3}$ is RLCM $(\frac{p_1}{q_1},\frac{p_2}{q_2},\frac{p_3}{q_3})$. Clearly, υ_3 also induces $\mu_{\frac{p_1}{q_1}}$ and $\mu_{\frac{p_2}{q_2}}$. If we assume that T is an r-divisor of $\frac{p_2}{q_2}$, we reach this same conclusion. Now RGCD $(R_3,\frac{p_4}{q_4})$ is a divisor of at least one of $\frac{p_1}{q_1},\frac{p_2}{q_2},\frac{p_3}{q_3},\frac{p_4}{q_4}$. As above, we can show that there exists a measure υ_4 on hz/R_4Z which induces $\mu_{\frac{p_1}{q_1}},\mu_{\frac{p_2}{q_2}},\mu_{\frac{p_3}{q_3}}$ and $\mu_{\frac{p_4}{q_4}}$. Continuing like this, we can prove the existence of a measure υ_k on hZ/R_kZ which induces $\mu_{\frac{p_i}{q_i}}, i \in \{1,2,...,k\}$.

Remark 2 If μ_4 is a measure on Z/4Z and μ_6 is a measure on Z/6Z, then for compatibility it is necessary that

$$\mu_6(1+6Z)+\mu_6(3+6Z)+\mu_6(5+6Z) = \mu_4(1+4Z)+\mu_4(3+4Z)$$

since

$$(1+6Z)\cup(3+6Z)\cup(5+6Z) = (1+4Z)\cup(3+4Z)$$

as we illustrate on the following portion of the real line.

	*		**		***		*		**		***
0	1	2	3	4	5	6	7	8	9	10	11
	#		##		#		##		#		##

*, ** and *** represent (1+6Z), (3+6Z) and (5+6Z), respectively and # and ## represent (1+4Z) and (3+4Z), respectively.

Theorem 6: Let P = $\{\mu_{\frac{p_i}{q_i}} | i \in B\}$ be a partial density on hZ such that RGCD $(R_a,\frac{p_{a+1}}{q_{a+1}})$ is a divisor of at least one of $\frac{p_1}{q_1},\frac{p_2}{q_2},...,\frac{p_a}{q_a}$, for every $a \in B$. Then P can be extended to a density on hZ.

Proof: Let $N_{\frac{p_i}{q_i}}$ be the set of continuous, real-valued functions on hZ with period $\frac{p_i}{q_i}$ for each $i \in B$. Also, let

$$M = \{\sum_{i=1}^{n} k_i f_i \,|\, f_i \in N_{\frac{p_i}{q_i}}, k_i \in R; i = 1, 2, ..., n\}.$$

Define L on M as follows:

$$L(f) = \sum_{i=1}^{n} \int_{Z/\frac{p_i}{q_i}Z} k_i f_i d\mu_{\frac{p_i}{q_i}}$$

where

$$f = \sum_{i=1}^{n} k_i f_i; f_i \in N_{\frac{p_i}{q_i}}, k_i \in R$$

and $\mu_{\frac{p_i}{q_i}}$ is the measure on $hZ/\frac{p_i}{q_i}Z$ in P; $i \in \{1,2,...,n\}$.

Let $g \in M \ni g \geq 0$. Then $g = g_1 + g_2 + ... + g_a$, where $a \in B$ and g_i has period $\frac{p_i}{q_i}, i \in \{1, 2, ..., a\}$.

From the previous theorem, there exists a probability measure μ on hz/R_aZ which induces $\mu_{\frac{p_i}{q_i}}; i = 1, 2, ..., a$.

Hence

$$L(g) = \sum_{i=1}^{a} \int_{Z/\frac{p_i}{q_i}Z} g_i d\mu_{\frac{p_i}{q_i}} = \sum_{i=1}^{a} \int_{hZ/R_aZ} g_i d\mu = \int_{hZ/R_aZ} g d\mu \geq 0,$$

since $g \geq 0$.

Hence L is positive. Theorem 3.3 of (Carnal and Felman, 2000) implies that P can be extended to a density on Z.

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