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Application of Homotopy Perturbation Method to Singularly Perturbed Volterra Integral Equations

[1]M.H. Alnasr and [2]Shaher Momani
[1]Department of Mathematics and Physics, Qatar University, Doha, Qatar
[2]Department of Mathematics, Mutah University, Al-Karak, Jordan

Abstract: In this study, the Homotopy-Perturbation Method (HPM) proposed by J.-H. He is adopted for solving singularly perturbed Volterra integral equations (SVIEs). Three numerical examples are given to demonstrate the effectiveness of the present method. Results show that the numerical scheme is very effective and convenient for solving a large number of singularly perturbed problems with high accuracy.

Key words: Homotopy perturbation method, singularly perturbed problems, Volterra integral equations

INTRODUCTION

In recent years, interest has substantially increased in the solution of singularly perturbed Volterra integral equations. In this study we continue this trend and consider a new analytical technique, the homotopy perturbation method (HPM) (He, 1999, 2000), for solving singularly perturbed Volterra integral equations (SVIEs) of the form:

$$\varepsilon y(x) = g(x) + \int_0^x K(x,t,y(t))dt, \quad t \in I := [0,X]. \qquad (1)$$

Here, $\varepsilon > 0$ is a small parameter that gives rise to singularly perturbed nature of the problem (Alnasr, 2000, 1997; Lange and Smith, 1988; Angell and Olmstead, 1987). The kernel K and the data function g(x) are given smooth functions. Under appropriate condition on g and K, for every $\varepsilon > 0$, Eq. 1 has a unique continuous solution on [0, X] (Alnasr, 1997; Brunner and Van Der Houwen, 1986) and references cited therein.

The singularly perturbed nature of (1) arises when the properties of the solution with $\varepsilon > 0$ are incompatible with those when $\varepsilon = 0$. For $\varepsilon > 0$, (1) is an integral equation of the second kind which typically is well posed whenever K is sufficiently well behaved. When $\varepsilon = 0$, (1) reduced to an integral equation of the first kind whose solution may well be incompatible with the case for $\varepsilon > 0$. The interest here is in those problems which do imply such an incompatibility in the behavior of y near x = 0. This suggests the existence of boundary layer near the origin where the solution undergoes a rapid transition (Alnasr, 1997; Lange and Smith, 1988; Brunner and Van Der Houwen, 1986).

Lange and Smith (1988) and Angell and Olmstead (1987) developed a formal methodology to obtain asymptotic solution for Eq. 1. Alnasr (1997, 2000) applied a multi-step method to solve the singular perturbation problem in Volterra integral equation. The author studied the stability of the multi-step method for the following SVIEs:

$$\varepsilon y(x) = \varepsilon - \int_0^x (\xi + \eta(x-t)y(t)dt, \quad x \ge 0 \qquad (2)$$

or

$$y(x) = 1 - \int_0^x (\lambda + \mu(x-t)y(t)dt, \quad \lambda = \frac{\xi}{\varepsilon}, \quad \mu = \frac{\eta}{\varepsilon}, \qquad (3)$$

of which the basic test equation ($\mu = 0$) is special case. Here ξ and η are positive real constants.

He's homotopy perturbation technique (He, 1999, 2000) has recently been used to solve singular boundary and initial value problems (Al-Khaled, 2007; Ramos, 2006). It has been claimed that this techniques is valid for nonlinear problems regardless of the presence or absence of a small parameter in the problem. In particular, He's homotopy perturbation method (He, 1999, 2000) for singular linear boundary-value problems (Al-Khaled, 2007) has been implemented by first reducing a second-order ordinary differential equation to a Volterra integral equation, introducing a homotopy parameter, expanding the solution as a series of this parameter and then setting this parameter to unity. In this present study we employ He's homotopy method to obtain explicit and numerical solutions of Eq. 1.

The homotopy perturbation method is a new approach which searches for an analytical approximate solution of linear and nonlinear problems. The homotopy

Corresponding Author: M.H. Alnasr, Department of Mathematics and Physics, Qatar University, Doha, Qatar

perturbation method has been applied to Volterra's integro-differential equation (El-Shahed, 2005), to nonlinear oscillators (He, 2004a), bifurcation of nonlinear problems (He, 2005a), bifurcation of delay-differential equations (He, 2005b), nonlinear wave equations (He, 2005c) and boundary value problems (He, 2006a) and to other fields (He, 2003, 2004b, 2006b; Abbasbandy, 2006; Siddiqui *et al*., 2006). Recently, the application of the method has been extended to ordinary and partial differential equations of fractional order (Odibat and Momani, 2008, 2007; Momani and Odibat, 2007a, b; Abdulaziz *et al*., 2007).

HOMOTOPY PERTURBATION METHOD

The homotopy perturbation method (HPM) was first proposed by Chinese mathematician J.-H. (He, 1999, 2000). The essential idea of this method is to introduce a homotopy parameter, say p, which takes the values from 0 to 1. When p = 0, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As p gradually increases to 1, the system goes through a sequence of deformation, the solution of each of which is close to that at the previous stage of deformation. Eventually at p = 1, the system takes the original form of the equation and the final stage of 'deformation' gives the desired solution.

For convenience of the reader, we will present a review of the HPM (He, 1999, 2000), then we apply the method to solve the nonlinear problem (1). To achieve our goal, we consider the nonlinear integral equation:

$$L(u) + N(u) = f(r), \quad r \in \Omega, \tag{4}$$

with boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \tag{5}$$

where, L is a linear operator, while N is nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ω and f(r) is a known analytic function.

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)], \tag{6}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)], \tag{7}$$

where, $r \in \Omega$ and $p \in [0,1]$ is an impeding parameter, u_0 is an initial approximation of Eq. 4 which satisfies the boundary conditions. Obviously, from Eq. 6 and 7, we have

$$H(v,0) = L(v) - L(u_0), \tag{8}$$

$$H(v,1) = L(v) + N(v) - f(r), \tag{9}$$

The changing process of p from zero to unity is just that of v(r, p) from $u_0(r)$ to u(r). In topology, this called deformation, L(v)-L(u_0) and L(v)+N(v)-f(r), are called homotopic. The basic assumption is that the solution of Eq. 6 and 7 can be expressed as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + ..., \tag{10}$$

The approximate solution of Eq. 4, therefore, can be readily obtained:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ... \tag{11}$$

The convergence of the series (11) has been proved in He (1999, 2000).

In view of the homotopy perturbation method, we can construct the following homotopy for Eq. 1:

$$\varepsilon y(x) = g(x) + p\int_0^x K(x,t,y(t))dt, \tag{12}$$

where, $p \in [0,1]$. Substituting $y = y_0 + py_1 + p^2 y_2 + ...$, into (12) and equating coefficients of like power of yields the following equations:

$$p^0 : \varepsilon y_0(x) = g(x) \Rightarrow y_0(x) = \frac{1}{\varepsilon} g(x),$$

$$p^1 : \varepsilon y_1(x) - \int_0^x K(x,t,y_0(t))dt = 0 \Rightarrow y_1(x) = \frac{1}{\varepsilon}\int_0^x K(x,t,y_0(t))dt,$$

$$p^2 : \varepsilon y_2(x) - \int_0^x K(x,t,y_1(t))dt = 0 \Rightarrow y_2(x) = \frac{1}{\varepsilon}\int_0^x K(x,t,y_1(t))dt,$$

$$\vdots$$

Finally, the approximate solution for Eq. 1 is given by

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{13}$$

APPLICATIONS AND NUMERICAL RESULTS

Example 1: Consider the following linear problem

$$\varepsilon y(x) = \int_0^x [1 + t - y(t)]dt \tag{14}$$

which has the exact solution

$$y(x) = x + 1 - \exp(-x/\varepsilon) - \varepsilon(1 - \exp(-x/\varepsilon)). \tag{15}$$

Table 1: Numerical results compared to the exact solution for Example 1

	ε = 1.0		ε = 0.75		ε = 0.5		ε = 0.25	
X	$y(x)_{Aprox.}$	$y(x)_{Exact}$	$y(x)_{Aprox.}$	$y(x)_{Exact}$	$y(x)_{Aprox.}$	$y(x)_{Exact}$	$y(x)_{Aprox.}$	$y(x)_{Exact}$
0.0	0.0	0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.1	0.1	0.131207	0.131207	0.190635	0.190635	0.347260	0.347260
0.2	0.2	0.2	0.258518	0.258518	0.364840	0.364840	0.613003	0.613003
0.3	0.3	0.3	0.382420	0.382420	0.525594	0.525594	0.824104	0.824104
0.4	0.4	0.4	0.503338	0.503338	0.675336	0.675336	0.998574	0.998574
0.5	0.5	0.5	0.621646	0.621646	0.816060	0.816060	1.148497	1.148497
0.6	0.6	0.6	0.737668	0.737668	0.949403	0.949403	1.281941	1.281941
0.7	0.7	0.7	0.851690	0.851690	1.076702	1.076702	1.404244	1.404244
0.8	0.8	0.8	0.963962	0.963962	1.199052	1.199052	1.518590	1.518590
0.9	0.9	0.9	1.074710	1.074710	1.317350	1.317351	1.625671	1.625671
1.0	1.0	1.0	1.184102	1.184102	1.432330	1.432330	1.721314	1.736263

and the asymptotic solution

$$y(x)=x+1-\exp(-x/\varepsilon)+O(\varepsilon). \tag{16}$$

In the view of the homotopy perturbation method; the homotopy for Eq. 14 can be constructed as

$$H(y,p)=\varepsilon y(x)-x-\frac{1}{2}x^2+p\int_0^x y(t)dt=0, \tag{17}$$

Substituting (10) into (17) and equating the terms with identical powers of p, we have

$$p^0:\varepsilon y_0(x)=g(x)\Rightarrow y_0(x)=\frac{1}{\varepsilon}(x+\frac{1}{2}x^2)$$
$$p^1:\varepsilon y_1(x)=-\int_0^x y_0(t)dt\Rightarrow y_1(x)=-\frac{1}{\varepsilon^2}(\frac{1}{2}x^2+\frac{1}{6}x^3)$$
$$p^2:\varepsilon y_2(x)=-\int_0^x y_1(t)dt\Rightarrow y_2(x)=\frac{1}{\varepsilon^3}(\frac{1}{6}x^3+\frac{1}{24}x^4)$$
$$p^3:\varepsilon y_3(x)=-\int_0^x y_2(t)dt\Rightarrow y_3(x)=-\frac{1}{\varepsilon^4}(\frac{1}{24}x^4+\frac{1}{120}x^5)$$
$$p^4:\varepsilon y_4(x)=-\int_0^x y_3(t)dt\Rightarrow y_4(x)=\frac{1}{\varepsilon^5}(\frac{1}{120}x^5+\frac{1}{720}x^6)$$
$$\vdots$$

and so on, in this manner the rest of components of the homotopy perturbation solution can be obtained. The twentieth-term approximate solution for Eq. 14 is given by

$$y(x)=\frac{1}{\varepsilon}(x+\frac{1}{2}x^2)-\frac{1}{\varepsilon^2}(\frac{1}{2}x^2+\frac{1}{6}x^3)+\frac{1}{\varepsilon^3}(\frac{1}{6}x^3+\frac{1}{24}x^4)-\frac{1}{\varepsilon^4}(\frac{1}{24}x^4+\frac{1}{120}x^5)+\cdots \tag{18}$$

As the number of terms involved increase, one can observe that the series solution obtained using the homotopy perturbation method converges to the series expansion of the exact solution. Comparison of numerical results with the exact solution (15) for ε = 1.0, 0.75, 0.5 and 0.25 are shown in Table 1.

Example 2: We second consider the following linear problem

$$\varepsilon y(x)=\int_0^x (1+x-t)[1+t-y(t)]ds \tag{19}$$

The exact solution for this problem is

$$y(x)=x+1+\frac{1}{\gamma_1-\gamma_2}[(\gamma_2-1+\frac{1}{\varepsilon})\exp(\gamma_1 x)-(\gamma_1-1+\frac{1}{\varepsilon})\exp(\gamma_2 x)], \tag{20}$$

where, the parameters γ_1 and γ_2 are defined as:

$$\gamma_1:=\frac{1}{2\varepsilon}(-1+\sqrt{1-4\varepsilon}),\quad \gamma_2:=\frac{1}{2\varepsilon}(-1-\sqrt{1-4\varepsilon})$$

and the asymptotic solution is given by

$$y(x)=x+1-\exp(-x/\varepsilon)-t.\exp(-x/\varepsilon)+O(\varepsilon^2).$$

In view of homotopy technique, we can construct the following homotopy

$$H(y,p)=\varepsilon y(x)-x-x^2-\frac{1}{6}x^3+p\int_0^x y(t)dt=0, \tag{21}$$

Substituting (10) into (21) and equating the terms with identical powers of p, we have

$$p^0:\varepsilon y_0(x)=g(x)\Rightarrow y_0(x)=\frac{1}{\varepsilon}(x+x^2+\frac{1}{6}x^3)$$
$$p^1:\varepsilon y_1(x)=-\int_0^x y_0(t)dt\Rightarrow y_1(x)=-\frac{1}{\varepsilon^2}(\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{3}{8}x^4+\frac{1}{120}x^5)$$
$$p^2:\varepsilon y_2(x)=-\int_0^x y_1(t)dt\Rightarrow y_2(x)=\frac{1}{\varepsilon^3}(\frac{1}{6}x^3+\frac{1}{6}x^4+\frac{1}{20}x^5+\frac{1}{180}x^6+\frac{1}{5040}x^7)$$
$$p^3:\varepsilon y_3(x)=-\int_0^x y_2(t)dt\Rightarrow y_3(x)=-\frac{1}{\varepsilon^3}(\frac{1}{24}x^4+\frac{1}{24}x^5+\frac{1}{72}x^6+\frac{1}{504}x^7+\frac{1}{8064}x^8+\frac{1}{362880}x^9)$$
$$\vdots$$

Table 2: Numerical results compared to the exact solution for Example 2

	$\varepsilon = 1.0$		$\varepsilon = 0.75$		$\varepsilon = 0.5$		$\varepsilon = 0.25$	
X	$y(x)_{Aprox.}$	$y(x)_{Exact}$	$y(x)_{Aprox.}$	$y(x)_{Exact}$	$y(x)_{Aprox.}$	$y(x)_{Exact}$	$y(x)_{Aprox.}$	$y(x)_{Exact}$
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	0.104833	0.104679	0.137510	0.137309	0.199683	0.199396	0.363142	0.362142
0.2	0.218669	0.217523	0.282340	0.280886	0.397589	0.395616	0.663744	0.660805
0.3	0.340519	0.336936	0.433015	0.428584	0.592269	0.586545	0.915832	0.908448
0.4	0.469413	0.461552	0.588175	0.578695	0.782594	0.770939	1.130400	1.117340
0.5	0.604405	0.590204	0.746572	0.729874	0.967719	0.948181	1.316060	1.297000
0.6	0.744584	0.721901	0.907074	0.881074	1.147050	1.118090	1.479500	1.454850
0.7	0.889072	0.855805	1.068660	1.031490	1.320190	1.280790	1.625850	1.596620
0.8	1.037040	0.991208	1.230440	1.180520	1.486940	1.436590	1.758590	1.726630
0.9	1.118769	1.127520	1.391600	1.327730	1.647240	1.585930	1.878380	1.848080
1.0	1.340300	1.264240	1.551470	1.472810	1.801120	1.729330	1.978620	1.963370

So, the approximate solution for Eq. 19 is given by

$$y(x) = \frac{1}{\varepsilon}(x + x^2 + \frac{1}{6}x^3) - \frac{1}{\varepsilon^2}(\frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{3}{8}x^3 + \frac{1}{120}x^5) + \frac{1}{\varepsilon^3}(\frac{1}{6}x^3 + \frac{1}{6}x^4 + \cdots) + \cdots \tag{22}$$

Table 2 shows the approximate solutions for Eq. 19 obtained for different values of using the homotopy perturbation method. From the numerical results in Table 2, it is clear that the approximate solutions are in high agreement with the exact solutions and the solutions continuously depend on the parameter ε. It is to be noted that only the twentieth-order term of the homotopy perturbation solution were used in evaluating the approximate solutions for Table 2. It is evident that the efficiency of this approach can be dramatically enhanced by computing further terms or further components of y(x).

Example 3: Consider the following nonlinear integral equation:

$$\varepsilon y(x) = \int_0^x e^{x-t}(y^2(t) - 1)dt, \tag{23}$$

which has the exact solution

$$y(x) = \frac{2(1 - e^{\gamma x})}{\varepsilon(\gamma - 1)e^{\gamma x} + \gamma + 1}, \quad \gamma = \frac{1}{\varepsilon}\sqrt{4 + \varepsilon^2}, \tag{24}$$

and the asymptotic solution is given by

$$y(x) = [-\tanh(x/\varepsilon)] + \varepsilon[-\frac{1}{\varepsilon}\tanh^2(x/\varepsilon)] + O(\varepsilon^2).$$

In view of homotopy technique, we can construct the following homotopy for Eq. 23

$$H(y,p) = \varepsilon y(x) - 1 + e^x - p\int_0^x e^x y^2(t)dt = 0. \tag{25}$$

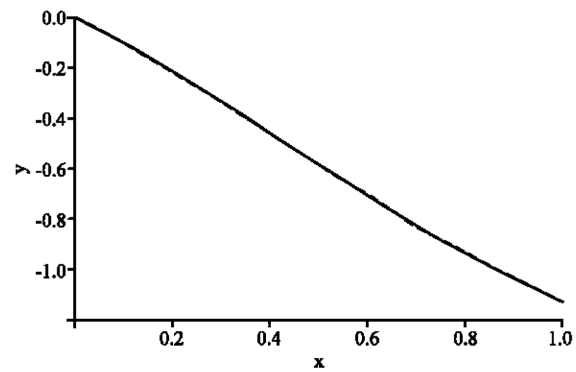

Fig. 1: Plots of Eq. 23 when $\varepsilon = 1$. Exact solution (——); HPM solution (– – –)

Substituting (10) into (25) and equating the terms with identical powers of p, we have

$$p^0 : \varepsilon y_0(x) = 1 - e^x \Rightarrow y_0(x) = \frac{1}{\varepsilon}(1 - e^x)$$

$$p^1 : \varepsilon y_1(x) = \frac{1}{\varepsilon}\int_0^x e^{x-t}y_0^2(t)dt \Rightarrow y_1(x) = \frac{1}{\varepsilon^3}(2e^x \sinh(x) - 2xe^x)$$

$$p^2 : \varepsilon y_2(x) = \frac{1}{\varepsilon}\int_0^x e^{x-t}(2y_0(t)y_1(t))dt \Rightarrow y_2(x) = \frac{1}{\varepsilon^5}(2 + e^x - 2e^{2x} - e^{3x} + 2xe^x + 4xe^{2x} - 2x^2e^x),$$

$$\vdots$$

and so on, in this manner the rest of components of the homotopy perturbation solution can be obtained. The tenth-term approximate solution for Eq. 23 is given by:

$$y(x) = \frac{1}{\varepsilon}(1 - e^x) - \frac{1}{\varepsilon^3}(2e^x \sinh(x) - 2xe^x) + \frac{1}{\varepsilon^5}(2 + e^x - 2e^{2x} - e^{3x} + 2xe^x + 4xe^{2x} - 2x^2e^x) + \cdots \tag{26}$$

The evolution results for the exact solution (24) and the approximate solution obtained using the homotopy perturbation method, for different values ε, are shown in

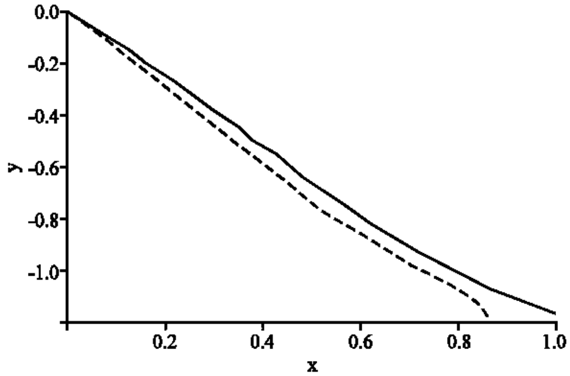

Fig. 2: Plots of Eq. 23 when ε = 0.75. Exact solution (——); HPM solution (– – –)

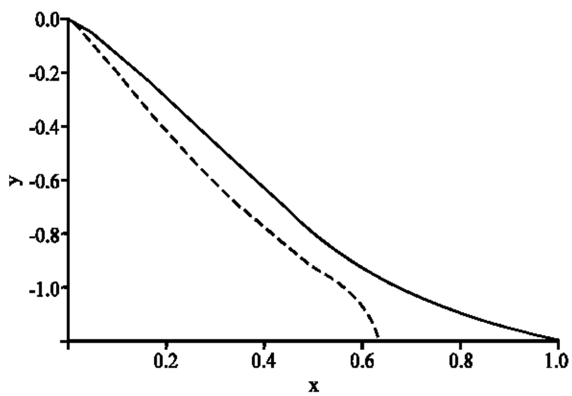

Fig. 3: Plots of Eq. 23 when ε = 0.5. Exact solution (——); HPM solution (– – –)

Fig. 1-3. It can be seen from Fig. 1-3 that the solution obtained by the present method is nearly identical with the exact solution when ε = 1 and the error increases as approaches zero. It is evident that the efficiency of this approach can be dramatically enhanced by computing further terms of y(x) when the homotopy perturbation method is used.

CONCLUSIONS

The homotopy perturbation method was employed successfully for solving singularly perturbed Volterra integral equations. The work emphasized our belief that the method is a reliable technique to handle linear and nonlinear singularly perturbed Volterra integral equations. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, discretization or restrictive assumptions.

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