Journal of Applied Sciences

ISSN 1812-5654

Journal of Applied Sciences 8 (1): 120-126, 2008
ISSN 1812-5654

Variational Iteration Method and Homotopy-Perturbation Method for Solving Different Types of Wave Equations

[1]A. Barari, [2]Abdoul R. Ghotbi, [1]F. Farrokhzad and [3]D.D. Ganji
[1,3]Department of Civil and Mechanical Engineering,
Mazandaran University of Technology, P.O. Box 484, Babol, Iran
[2]Department of Civil Engineering, Shahid Bahonar University, Kerman, Iran

Abstract: Due to wide range of interest in use of wave equations to gain insight in to vibrating problems, Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM) are employed to approximate the solution of the three types of wave equations including one-dimensional wave equation, kinematic wave equation and non-linear homogeneous wave equation. The final results obtained by HPM and VIM are compared with those results obtained from the exact solution. The comparison shows a precise agreement between the results and introduces these new methods as the applicable methods which they need less computations and are much easier and more convenient than other approximate methods, so they can be widely used in engineering.

Key words: One-dimensional wave equation, kinematic wave equation, non-linear homogeneous wave equation, Homotopy perturbation method, variational iteration method

INTRODUCTION

In this study, we consider the linear and non-linear wave equation (Inc *et al*., 2004):

$$u_{tt} = u_{xx} - \varphi\,(u, u_t, u_x, uu_x, u^2) + F\,(x, t) \tag{1}$$

On the finite x-interval $[0, \pi]$ with Dirichlet boundary condition:

$$u\,(x, 0) = 0 = u\,(t, \pi),\ -\infty<t<\infty \tag{2}$$

and F (x,t) is a given function of x and t.

This equation describes the propagation of a wave (or disturbance) and it arises in a wide variety of physical problems. some of these problems include a vibrating string, vibrating membrane, longitudinal vibrations of an elastic rod or beam, shallow water waves, acoustic problems for the velocity potential for a fluid flow through which sound can be transmitted, transmission of electric signals along a cable, shock waves, chemical exchange processes in chromatography, sediment transport in rivers and waves in plasmas and both electric and magnetic fields in the absence of charge and dielectric (Debnath, 1997).

We will apply the Variational Iteration Method (VIM) (He, 1999a, 2000; Momani and Abuasad, 2006; Ganji *et al*., 2007; Sweilam and Khader, 2007; Bildik and Konuralp, 2006) and Homotopy Perturbation Method (HPM) (He, 1999b, 2006; Zhang and He, 2006; Ganji and Sadighi, 2007) for the three types of partial differential wave equations.

Variational Iteration Method (VIM) and Homotopy-Perturbation Method (HPM) are the most effective and convenient ones for both linear and nonlinear equations. The VIM is to construct correction functional using general Lagrange multipliers identified optimally via the variational theory and the initial approximation can be freely chosen with unknown constants. The HPM deforms a difficult problem in to a simple problem which can be easily solved using ordinary methods.

Extensive studies have been done regarding this matter. Such as investigation of Helmholtz equation and fifth-order KdV equation or Nonlinear Coupled Systems of reaction-diffusion Equations using Homotopy perturbation method (Ganji and Sadighi, 2006).

BASIC IDEA OF HOMOTOPY-PERTURBATION METHOD

Linear and Nonlinear phenomena are of fundamental importance in various fields of science and engineering. Most models of real-life problems are still very difficult to solve. Therefore, approximate analytical solutions such as Homotopy Perturbation Method (HPM) were introduced. This method is the most effective and convenient ones for both linear and nonlinear equations.

Corresponding Author: Abdoul R. Ghotbi, Department of Civil Engineering, Shahid Bahonar University, Kerman, Iran
Tel: +98 2133939450 Fax: +98 2133939450

Perturbation method is based on assuming a small parameter. The majority of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all and the approximate solutions obtained by the perturbation methods, in most cases, are valid only for small values of the small parameter.

Generally, the perturbation solutions are uniformly valid as long as a scientific system parameter is small. However, we cannot rely fully on the approximations, because there is no criterion on which the small parameter should exists. Thus, it is essential to check the validity of the approximations numerically and/or experimentally. To overcome these difficulties, HPM have been proposed recently.

To explain this method, let us consider the following function:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{3}$$

With the boundary conditions of:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \tag{4}$$

Where, A,B, f (r) and Γ are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively.

Generally speaking the operator can be divided in to a linear part L and a nonlinear part N (u). Equation 3 can therefore, be written as:

$$L(u) + N(u) - f(r) = 0 \tag{5}$$

By the homotopy technique, we construct a homotopy v (r, p): $\Omega \times [0,1] \to R$ Which satisfies:

$$H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0, \quad p \in [0,1], r \in O, \tag{6}$$

Or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r)\right] = 0 \tag{7}$$

Where, $p \in [0,1]$ is an embedding parameter, while u_0 is an initial approximation of Eq. 3, which satisfies the boundary conditions. Obviously, from Eq. 6 and 7 we will have:

$$H(v,0) = L(v) - L(u_0) = 0, \tag{8}$$

$$H(v,1) = A(v) - f(r) = 0, \tag{9}$$

The changing process of p from zero to unity is just that of v (r, p) from u_0 to u (r). In topology, this is called deformation, while L (v)-L (u_0) and A (v) -f (r) are called homotopy.

According to the HPM, we can first use the embedding parameter p as a small parameter and assume that the solutions of Eq. 6 and 7 can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \ldots \tag{10}$$

Setting p = 1 yields in the approximate solution of Eq. 3 to:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \tag{11}$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantage.

The series (11) is convergent for most cases. However, the convergent rate depends on the nonlinear operator A (v). Moreover, He (1999b) made the following suggestions:

- The second derivative of N (v) with respect to v must be small because the parameter may be relatively large, i.e., $p \to 1$.
- The norm of $L^{-1}\frac{\partial N}{\partial v}$ must be smaller than one so that the series converges.

BASIC IDEA OF VARIATIONAL ITERATION METHOD

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(t) \tag{12}$$

Where:
L = A linear operator
N = A nonlinear operator
g (t) = A homogeneous term

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\right)d\tau \quad (13)$$

Where, λ is a general lagrangian multiplier which can be identified optimally via the variational theory. The subscript n indicates the nth approximation and u_n is considered as a restricted variation, i.e., $\delta\,\tilde{u}_n = 0$.

Example 1: We consider a homogeneous linear wave equation:

$$\begin{aligned} &u_{tt} = u_{xx} - \varphi(u, u_t, u_x, uu_x, u^2) + F(x,t) \\ &\varphi = 0,\ F(x,t) = 0. \end{aligned} \quad (14)$$

With the initial and boundary conditions posed are:

$$\begin{aligned} &u(x, 0) = \sin x, \\ &u_t(x, 0) = u(0, t) = u(\pi, t) = 0 \end{aligned} \quad (15)$$

Exact solution of this equation is:

$$u(x, t) = \cos t \sin x \quad (16)$$

APPLICATION OF HOMOTOPY-PERTURBATION METHOD

To solve Eq. 14 by means of HPM, we consider the following process after separating the linear and nonlinear parts of the equation; we apply Homotopy-Perturbation to Eq. 6.

A homotopy-perturbation method can be constructed as follows:

$$H(v,p) = (1-p)\left(\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v_0}{\partial t^2}\right) + p\left(\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2}\right) = 0 \quad (17)$$

Substituting Eq. 10 into 17 and rearranging the resultant equation based on powers of p-terms, one has:

$$p^0 : \frac{\partial^2}{\partial t^2} v_0(x,t) = 0 \quad (18)$$

$$p^1 : \frac{\partial^2}{\partial t^2} v_1(x,t) - \frac{\partial^2}{\partial x^2} v_0(x,t) = 0 \quad (19)$$

$$p^2 : \frac{\partial^2}{\partial t^2} v_2(x,t) - \frac{\partial^2}{\partial x^2} v_1(x,t) = 0 \quad (20)$$

With the following conditions:

$$\begin{aligned} &v_0(x,0) = \sin x, \quad \frac{d}{dt} v_0(x,0) = v_0(0,t) = v_0(\pi,t) = 0 \\ &v_i(x,0) = 0, \quad \frac{d}{dt} v_i(x,0) = v_i(0,t) = v_i(\pi,t) = 0,\ i = 1,2,... \end{aligned} \quad (21)$$

With the effective initial approximation for v_0 from the conditions (21) and solutions of Eq. 18-20 may be written as follows:

$$v_0(x, t) = \sin x \quad (22)$$

$$v_1(x,t) = -\frac{1}{2}(\sin x)t^2 \quad (23)$$

$$v_2(x,t) = \frac{1}{24}(\sin x)t^4 \quad (24)$$

In the same manner, the rest of components were obtained using the maple package.

According to the HPM, we can conclude that:

$$u(x,t) = \lim_{p \to 1} v(x,t) = v_0(x,t) + v_1(x,t) + ... \quad (25)$$

Therefore, substituting the values of v_0 (x, t), v_1 (x, t) and v_2 (x, t) from Eq. 22-24 into 25 yields:

$$u(x,t) = \sin x\left(1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 + ...\right) = \sin x \cos t \quad (26)$$

APPLICATION OF VARIATIONAL ITERATION METHOD

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\left(\frac{\partial^2 u_n(x,\tau)}{\partial \tau^2} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2}\right)d\tau \quad (27)$$

Its stationary conditions can be obtained as follows:

$$1 - \lambda'\big|_{\tau=t} = 0,\ \lambda\big|_{\tau=t} = 0,\ \lambda''\big|_{\tau=t} = 0, \quad (28)$$

We obtain the lagrangian multiplier:

$$\lambda = \tau - t \quad (29)$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\tau - t)\left[\frac{\partial^2 u_n(x,\tau)}{\partial \tau^2} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2}\right]d\tau \quad (30)$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$u_0(x, t) = \sin x \tag{31}$$

Using the above variational formula (30), we have

$$u_1(x,t) = u_0(x,t) + \int_0^t (\tau - t)\left[\frac{\partial^2 u_0(x,\tau)}{\partial \tau^2} - \frac{\partial^2 u_0(x,\tau)}{\partial x^2}\right] d\tau \tag{32}$$

Substituting Eq. 31 into 32 and after simplifications, we have:

$$u_1(x,t) = -\frac{1}{2}\sin(x)(-2 + t^2) \tag{33}$$

In the same way, we obtain $u_2(x, t)$ as follows:

$$u_2(x,t) = u(x,t) = \sin x(1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 + ...) = \sin x \cos t \tag{34}$$

And so on. In the same way the rest of the components of the iteration formula can be obtained.

As it can be seen, using these two approximate methods lead to the exact solution (Fig. 1) using more iteration.

Example 2: We consider the first-order non-linear wave equation. The equation of the form:

$$\begin{aligned} &u_{tt} = u_{xx} - \varphi(u, u_t, u_x, uu_x, u^2) + F(x,t), \\ &u(x,0) = x, \varphi = uu_x, F(x,t) = 0 \end{aligned} \tag{35}$$

With the exact solution of:

$$u(x,t) = \frac{x}{1+t} \tag{36}$$

APPLICATION OF HOMOTOPY-PERTURBATION METHOD

A homotopy-perturbation method can be constructed as follows:

$$H(v,p) = (1-p)(\frac{\partial^2 v}{\partial t} - \frac{\partial^2 v_0}{\partial t}) + p\left(\frac{\partial}{\partial t}v(x,t) + v(x,t)(\frac{\partial}{\partial x}v(x,t)\right) = 0 \tag{37}$$

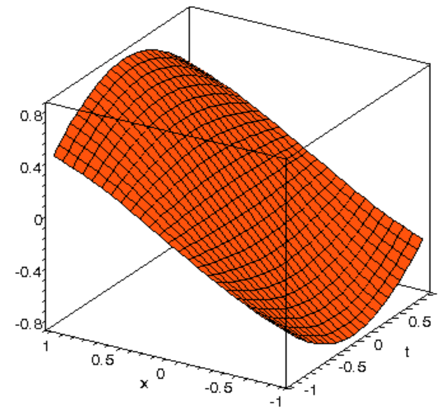

Fig. 1: 3D obtained results of u(x,t) by the HPM, VIM and the exact solution

Substituting Eq. 10 into 37 and rearranging the resultant equation based on powers of p-terms, one has:

$$p^0 : \frac{\partial^2}{\partial t^2} v_0(x,t) = 0 \tag{38}$$

$$p^1 : \frac{\partial}{\partial t} v_1(x,t) + v_0(x,t)\left(\frac{\partial}{\partial x} v_0(x,t)\right) = 0 \tag{39}$$

$$p^2 : \frac{\partial}{\partial t} v_2(x,t) + v_0(x,t)\left(\frac{\partial}{\partial x} v_1(x,t)\right) + v_1(x,t)\left(\frac{\partial}{\partial x} v_0(x,t)\right) = 0 \tag{40}$$

$$\begin{aligned} &p^3 : u_2(x,t)\left(\frac{\partial}{\partial x} v_0(x,t)\right) + v_0(x,t)\left(\frac{\partial}{\partial x} v_2(x,t)\right) + \\ &v_1(x,t)\left(\frac{\partial}{\partial x} v_1(x,t)\right) + (\frac{\partial}{\partial t} v_3(x,t)) = 0 \end{aligned} \tag{41}$$

With the following conditions:

$$\begin{aligned} &v_0(x,0) = x \\ &v_i(x,0) = 0, \qquad i = 1,2,... \end{aligned} \tag{42}$$

With the effective initial approximation for from the conditions (42) and solutions of Eq. 38-41 may be written as follows:

$$v_0(x, t) = x \tag{43}$$

$$v_1(x, t) = -xt \tag{44}$$

$$v_2(x, t) = xt^2 \tag{45}$$

$$v_3(x,t) = -xt^3 \tag{46}$$

In the same manner, the rest of components were obtained using the maple package.

According to the HPM, we can conclude that:

$$u(x,t) = \lim_{p \to 1} v(x,t) = v_0(x,t) + v_1(x,t) + \ldots \tag{47}$$

Therefore, substituting the values of $v_0(x,t)$, $v_1(x,t)$, $v_2(x,t)$ and $v_3(x,t)$ from Eq. 43-46 into 47 yields:

$$u(x,t) = x(1 - t + t^2 - t^3 + \ldots) = \frac{x}{1+t} \tag{48}$$

APPLICATION OF VARIATIONAL ITERATION METHOD

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\left(\frac{\partial u_n(x,\tau)}{\partial \tau} + u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial x}\right)d\tau \tag{49}$$

Its stationary conditions can be obtained as follows:

$$\begin{aligned} 1 + \lambda\big|_{\tau=t} &= 0 \\ \lambda'\big|_{\tau=t} &= 0 \end{aligned} \tag{50}$$

The lagrangian multiplier can therefore be identified as:

$$\lambda = -1 \tag{51}$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (-1)\left(\frac{\partial u_n(x,\tau)}{\partial \tau} + u_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial x}\right)d\tau \tag{52}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$u_0(x,t) = x \tag{53}$$

Using the above variational formula (52), we have

$$u_1(x,t) = u_0(x,t) + \int_0^t (-1)\left(\frac{\partial u_0(x,\tau)}{\partial \tau} + u_0(x,\tau)\frac{\partial u_0(x,\tau)}{\partial x}\right)d\tau \tag{54}$$

Substituting Eq. 53 into 54 and after simplification, we have:

$$u_1(x,t) = x - xt \tag{55}$$

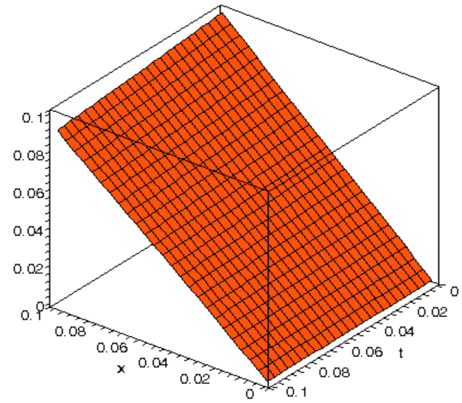

Fig. 2: 3D obtained results of u(x,t) by the HPM, VIM and the exact solution

In the same way, we obtain $u_2(x,t)$, $u_3(x,t)$ as follows:

$$u_2(x,t) = x - xt - \frac{1}{3}xt^3 + xt^2. \tag{56}$$

$$u_3(x,t) = u(x,t) = x(1 - t + t^2 - t^3 + \ldots) = \frac{x}{1+t} \tag{57}$$

And so on. In the same way the rest of the components of the iteration formula can be obtained.

As it can be seen, using these two approximate methods in solving this equation also lead to the exact solution (Fig. 2) using more iteration.

Example 3: We consider a second-order non-linear wave equation:

$$\begin{aligned} &u_{tt} = u_{xx} - \varphi(u, u_t, u_x, uu_x, u^2) + F(x,t) \\ &\varphi = u^2 - uu_x,\ F(x,t) = 0 \end{aligned} \tag{58}$$

With the initial and boundary conditions posed are:

$$\begin{aligned} &u(x,0) = u_t(x,0) = e^x, 0 < x < 1, \\ &u(0,t) = e^t, t > 0, \end{aligned} \tag{59}$$

Exact solution of this equation is:

$$u(x,t) = e^{x+t} \tag{60}$$

APPLICATION OF HOMOTOPY-PERTURBATION METHOD

A homotopy can be constructed as follows:

$$H(v,p) = (1-p)(\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v_0}{\partial t^2}) + p(\frac{\partial^2}{\partial t^2} v(x,t) - \frac{\partial^2}{\partial x^2} v(x,t) + v(x,t)^2 - v(x,t)(\frac{\partial}{\partial x} v(x,t)) = 0 \tag{61}$$

Substituting Eq. 10 into 61 and rearranging the resultant equation based on powers of p-terms, one has:

$$p^0 : \frac{\partial^2}{\partial t^2} v_0(x,t) = 0 \tag{62}$$

$$p^1 : (\frac{\partial^2}{\partial t^2} v_1(x,t)) - (\frac{\partial^2}{\partial x^2} v_0(x,t)) - v_0(x,t) (\frac{\partial}{\partial x} v_0(x,t)) + v_0(x,t)^2 = 0 \tag{63}$$

$$p^2 : \frac{\partial^2}{\partial t^2} v_2(x,t) + 2v_0(x,t)v_1(x,t) - v_1(x,t) (\frac{\partial}{\partial x} v_0(x,t)) - v_0(x,t)(\frac{\partial}{\partial x} v_1(x,t)) - (\frac{\partial^2}{\partial x^2} v_1(x,t)) = 0 \tag{64}$$

With the following conditions:

$$v_0(x,0) = \frac{\partial}{\partial t} v_0(x,0) = e^{3x},\ v_0(0,t) = e^t \quad 0 < x < 1,\ t > 0$$
$$v_i(x,0) = \frac{\partial}{\partial t} v_i(x,0) = 0,\ v_i(0,t) = 0\ \ i = 1,2,\ldots\ldots \tag{65}$$

With the effective initial approximation for v_0 from the conditions (65) and solutions of Eq. 62-64 may be written as follows:

$$v_0\ (x,\ t)\ e^x\ (t+1) \tag{66}$$

$$v_1(x,t) = \frac{1}{6} e^x t^3 + \frac{1}{2} e^x t^2 \tag{67}$$

$$v_2(x,t) = \frac{1}{6} e^x (\frac{1}{20} t^5 + \frac{1}{4} t^4) \tag{68}$$

In the same manner, the rest of components were obtained using the maple package.

According to the HPM, we can conclude that:

$$u(x,t) = \lim_{p \to 1} v(x,t) = v_0(x,t) + v_1(x,t) + \ldots \tag{69}$$

Therefore, substituting the values of v_0 (x, t), v_1 (x, t) and v_2 (x, t) from Eq. 66-69 yields:

$$u(x,t) = e^x(1 + t + \frac{1}{2}t^2 + \frac{1}{6}e^x t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \ldots.) = e^{x+t} \tag{70}$$

Application of variational iteration method

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\frac{\partial^2 u_n(x,\tau)}{\partial \tau^2} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2} + u_n(x,\tau)^2 - u(x,\tau)(\frac{\partial u_n(x,\tau)}{\partial x}))d\tau. \tag{71}$$

Its stationary conditions can be obtained as follows:

$$1 - \lambda'\big|_{\tau=t} = 0,\ \ \lambda\big|_{\tau=t} = 0,\ \ \lambda''\big|_{\tau=t} = 0, \tag{72}$$

The lagrangian multiplier can therefore be identified as:

$$\lambda = \tau\text{-}t \tag{73}$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\tau - t)\left(\frac{\partial^2 u_n(x,\tau)}{\partial \tau^2} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2} + u_n(x,\tau)^2 - u_n(x,\tau)(\frac{\partial u_n(x,\tau)}{\partial x})\right)d\tau \tag{74}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$u_0(x,t) = e^x(t+1). \tag{75}$$

Using the above variational formula (74), we have:

$$u_1(x,t) = u_0(x,t) + \int_0^t (\tau - t)\left(\frac{\partial^2 u_0(x,\tau)}{\partial \tau^2} - \frac{\partial^2 u_0(x,\tau)}{\partial x^2} + u_0(x,\tau)^2 - u_0(x,\tau)(\frac{\partial u_0(x,\tau)}{\partial x})\right)d\tau \tag{76}$$

Substituting Eq. 75-76 and after simplification, we have:

$$u_1(x,t) = \frac{1}{6} e^x (6t + 6 + t^3 + 3t^2) \tag{77}$$

In the same way, we obtain u_2 (x, t), u_3 (x, t) as follows:

$$u_2(x,t) = \frac{1}{120} e^x (120t + 120 + 120t^3 + 60t^2 + t^5 + 5t^4) \tag{78}$$

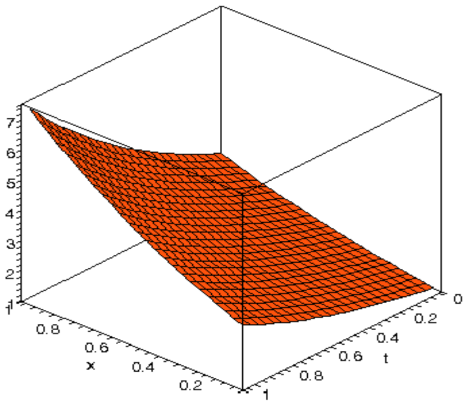

Fig. 3: 3D obtained results of u(x,t) by the HPM, VIM and the exact solution

$$u_3(x,t) = u(x,t) = e^x(1 + t + \frac{1}{2}t^2 + \frac{1}{6}e^x t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + ...) = e^{x+t} \quad (79)$$

And so on. In the same way the rest of the components of the iteration formula can be obtained.

As it can be seen, using these two approximate methods in solving this equation also lead to the exact solution (Fig. 3) using more iteration.

CONCLUSION

The homotopy perturbation method and variational iteration method have been successfully used to study three types of partial differential wave equations. These equations describe the propagation of a wave (disturbance) and it arises in a wide variety of physical problems. The results obtained here were compared with the exact solutions. It can be easily seen that applying these two methods to the mentioned equations were led to the exact solution in all examples, so the results revealed that the homotopy perturbation method and variational iteration method are powerful mathematical tools for solutions of differential equations in terms of accuracy and efficiency.

REFERENCES

Bildik, N. and A. Konuralp, 2006. The use of variational iteration method, differential transform method and adomian decomposition method for solving different types of nonlinear partial differential equations. Int. J. Nonl. Sci., 7 (1): 65-70.

Debnath, L., 1997. Nonlinear partial differential equations for scientists and engineering. Birkhauser Boston, 18 (2): 141-230.

Ganji, D.D. and A. Sadighi, 2006. Application of he's Homotopy-Prturbation method to nonlinear coupled systems of reaction-diffusion equations. Int. J. Nonl. Sci. Num. Simu, 7 (4): 411-418.

Ganji, D.D. and A. Sadighi, 2007. Application of Homotopy-Perturbation and variational iteration methods to nonlinear heat transfer and porous media equations. J. Comput. Applied Math., 207 (1): 24-34.

Ganji, D.D., M. Jannatabadi and E. Mohseni, 2007. Application of He's variational iteration method to nonlinear Jaulent-Miodek equations and comparing it with ADM. J. Comput. Applied Math., 207 (1): 35-45.

He, J.H., 1999a. Variational iteration method: A kind of nonlinear analytical technique: Some examples. Int. Nonl. Mech., 344: 699-708.

He, J.H., 1999b. Homotopy perturbation technique. Comput. Meth. Applied Mech. Eng., 178 (3-4): 257-262.

He, J.H., 2000. Variational iteration method for autonomous ordinary differential systems. Applied Math. Comput., 114 (2-3): 115-123.

He, J.H., 2006. New interpretation of homotopy perturbation method. Int. J. Mod. Phys. B., 20 (18): 2561-2568.

Inc, M., Y. Cherrualt and K. Abbaoui, 2004. A computational approach to the wave equations: An application of the decomposition method. Kybernates, pp: 80-97.

Momani, S. and S. Abuasad, 2006. Application of He's variational iteration method to Helmholtz equation. Chaos. Solitons Fract., 27 (5): 1119-1123.

Sweilam, N.H. and M.M. Khader, 2007. Variational iteration method for one dimensional nonlinear thermoelasticity. Chaos. Solitons Fract., 32 (1): 145-149.

Zhang, L.N. and J.H. He, 2006. Homotopy perturbation method for the solution of the electrostatic potential differential equation. Mathematical Problems Engineering Art. No. 83878.