Journal of Applied Sciences

ISSN 1812-5654

Journal of Applied Sciences 8 (1): 181-184, 2008
ISSN 1812-5654

Probabilistic Aspects of Lagrange 3D-Interpolational

Kamal Al-Dawoud
Department of Mathematics and Statistics, Mutah University, 61710 Mutah, Jordan

Abstract: In this study, the problem of polynomial 3D interpolation on finite elements is studied and probabilistic aspects of finite-element approximation on three-dimensional models is presented. The theorems for new probabilistic properties of basis functions are proved.

Key words: Geometrical probability, Lagrange polynomial, finite-element method, tetrahedron, regular hexahedron, three-linear interpolation

INTRODUCTION

An application of geometrical probability (Kamal Al-Dawoud and Khomchenko, 2007) for constructing polynomials basic functions essentially simplifies problems of approximation in finite-elements method (Norrie and de Vries, 1978; Oden, 1972). In this paper the probabilistic aspects of finite-element approximation on three-dimensional models are presented. Special attention is given to a simplex (a tetrahedron, 4 units) and multiplex (a cube, 8 units). Usually nodal parameters are more favorably to choose on vertexes of an element, as vertexes are the general for more number of elements, than units on edges or lateral sides. Such choice reduces the general number of central parameters of system elements and reduces the size of a global matrix of the linear algebraic equations system (Norrie and de Vries, 1978).

Simplex models (Oden, 1972) are concerned with using linear polynomials in finite-elements. They were among the first used in (FEM) in 1956 (Turner, Clough, Martin and Topp), in 1957 (Synge), in 1962 (Gallagher, Padlog and Bijlaard). Finite-elements in the form of a cube have quickly won popularity in three-dimensional problems, where one cube took the same volume, as 6 tetrahedrons. Let's notice, that irregular splitting of the area into tetrahedrons is difficult for carrying out even with the help of a Computer (Strang and Fix, 1973). Three-linear approximation on a cubic element for the first time was used in 1963. (Melosh), then in 1966 (Key), in 1967 (Zienkiewicz and Cheung), in 1969 (Oden).

Kolmogorov's model of random wanderings on a three-dimensional grid allows schematizing random wanderings with random start and absorbing units in vertexes of a finite-element. Computer experiments give the basis to assume, that the transitive probabilities have properties of stability and are independent of the form of the trajectory and the number of steps. The establishment of the specified properties allows ignoring a history of random wanderings and stimulates searches of the simplified single-step scheme, which appreciably accelerate calculations in Monte Carlo methods. The economical schemes of random transitions are the result of minimizing the number of steps. In this study, it is theoretically proved that the transitive probability invariant concerning the form of a route depends only on coordinates of a starting-point and vertex of an element (finish-point). In the optimum scheme, the particle for one step on a start-line route reaches vertex of an element. Interpolations function of three arguments is a mathematical expectation of nodal values. From the mechanical point of view, transitive probabilities show, how to distribute a single mass on vertexes of an element, which barycentric appeared in a reselected point.

FORMULATION AND SOLUTION

On Fig. 1, the three-dimensional simplex-a tetrahedron with 4 nodes is represented. This element has equipment with 4 basic functions. In research problems of scalar fields in each node, there is one degree of freedom (for example, temperature). Traditional algebraic procedure of designing polynomial a interpolation is reduced to the definition of 4 parameters α_i in a general view polynomial:

$$P(x, y, z) = \alpha_1 + \alpha_{2x} + \alpha_{3y} + \alpha_{4z} \quad (1)$$

The source information contains 16 numbers: coordinates of nodes $\Pi_k\ (k = \overline{1,4})$ and nodal temperatures $f_k\ (k = \overline{1,4})$. For determining α_i using systems of linear algebraic equations 4×4, where k-th equation of system is given by:

$$\alpha_1 + \alpha_2 x_k + \alpha_3 y_k + \alpha_4 z_k = f_k, k = \overline{1,4} \quad (2)$$

The system (2) has the unique solution as its determinant Δ is not zero:

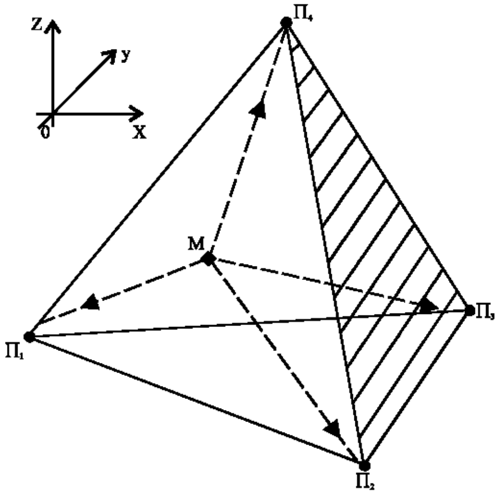

Fig. 1: Tetrahedron (4 nodes)

$$\Delta = \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} = 6V, \qquad (3)$$

Where, V is the volume of tetrahedron.

Procedure of designing of a polynomial actually ends of substitution of parameters α_i in Eq. 1. However, in most cases a polynomial (1) can be written in the form of Lagrange. For this element in (1) it is necessary to rearrange so that each element contains a multiplier f_k:

$$P(x, y, z) = \sum_{k=1}^{4} N_k(x,y,z) \cdot f_k, \qquad (4)$$

Where:

$$N_k(x, y, z) = \frac{\Delta_k}{\Delta}, \quad k = \overline{1, 4}$$

The determinant Δ_k produced from a determinant (3) replacement in k-th row of coordinates of vertex Π_k by coordinates of the current point M(x, y, z). It is easy to notice, that basis Lagrange $\{N_k\}$ will consist of barycentric coordinates of a three-dimensional simplex, which have the following properties:

$$0 \le N_k \le 1, \quad \sum_{k=1}^{4} N_k = 1, \quad N_k(x_i, y_i, z_i) = \delta_{ki}, \qquad (5)$$

Where:
δ_{ki} = Kronecker's symbol

Property (5) represents special interest, as it has a precise probabilistic sense. To each nodal value of function f_k is matched a corresponding probability N_k. Thus, we can write the law of distribution of probabilities for the function of a random point M (x, y, z):

F	f_1	f_2	f_3	f_4
p_i	N_1	N_2	N_3	N_4

Now it is clear that interpolation polynomials value (4) in any point of a simplex is determined by the formula of expectation. Feature of the resulted table, where selective values are fixed and a random factor is present at the second row. Functions of a random point N_k (x, y, z) are interpreted as transitive probabilities of a wandering particle, from a random point M (x, y, z), to vertexes of a tetrahedron Π_k.

On Fig. 1, arrows are shown the routes of random transitions. Thus, in a tetrahedron the single-step 4-routing scheme of random transitions with random start and absorption in vertexes is realized. In terms of Monte Carlo method, formula (4) is the average compensation for an output of particles in vertex. It means construction of interpolation polynomials is reduced to the definition of transitive probabilities. On a simplex N_k are easily defined geometrically through relations of volumes of two tetrahedrons with the general side. For example:

$$N_1(x,y,z) = \frac{mes(M\Pi_2\Pi_3\Pi_4)}{mes(\Pi_1\Pi_2\Pi_3\Pi_4)} = \frac{1}{\Delta}\begin{vmatrix} 1 & x & y & z \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix}.$$

On Fig. 1, the side bound is hatched.

Interpolation on a cube: On Fig. 2, the standard cube with the sizes 2×2×2 is represented. The origin coordinates coincides with barycentric cube. Algebraic procedure of constructing interpolation polynomials begins with the general expression containing 8 parameters (by number of nodes):

$$P(x, y, z) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 xy + \alpha_6 yz + \alpha_7 xz + \alpha_8 xyz \qquad (6)$$

Coefficients α_i $(i = \overline{1,8})$ are defined from the system of linear equations 8×8 using 32 numbers (coordinate of vertexes Π_k and a degree of freedom $f_k, k = \overline{1,8}$). Now the k-th system equation is given by:

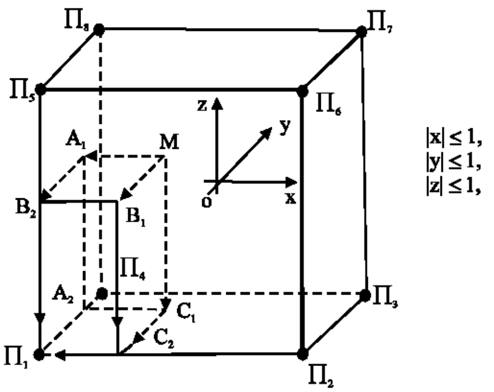

Fig. 2: Regular hexahedron (cube, 8 nodes)

$$\alpha_1 + \alpha_2 x_k + \alpha_3 y_k + \alpha_4 z_k + \alpha_5 x_k y_k + \alpha_6 y_k z_k + \alpha_7 x_k z_k + \alpha_8 x_k y_k z_k = f_k, (k = \overline{1,8}) \quad (7)$$

The determinant of system Eq. 7 is not zero, which provides uniqueness of the solution. Substitution of α_i in Eq. 6 and corresponding transformations results in a polynomial Lagrange form:

$$P(x,y,z) = \sum_{k=1}^{8} N_k(x,y,z)\cdot f_k, \text{ where}$$
$$N_k(x,y,z) = \frac{1}{8}(1+x_k x)(1+y_k y)(1+z_k z), \quad (8)$$
$$x_k, y_k, z_k = \pm 1$$

In the special literature on (FEM) (Norrie and de Vries, 1978; Strang and Fix, 1973; Oden, 1972), local coordinates on a standard cube are designated through ξ, η, ς. The mean-values of nodes (expectation) in formula (8) and the law of distribution of probabilities for the function of a random point M is:

F	f_1	f_2	...	f_8
p_i	N_1	N_2	...	N_8

In multiplex Fig. 2, a single-step 8-routing scheme of random transitions with random start point M and absorbing node in vertexes $П_k$ is realized.

In multiplex transition probabilities are also defined geometrically. It excludes necessity of drawing up and the resolution of the system of equations 8×8. Firstly, through the current point M (x, y, z) it is necessary to carry out three planes, parallel to coordinate planes. Thus the cube is divided into 8 rectangular parallelepipeds. Now, for defining N_k, it is necessary to find relative volume of a parallelepiped, opposite to node k. For example:

$$N_1(x,y,z) = \frac{1}{8}(1-x)(1-y)(1-z) \quad (9)$$

Other functions N_k are defined similarly or from N_1 using consecutive transformation of parallel route carried on 2 units along one of coordinate directions (Fig. 2). Properties (5) are easy to check up, as in this model.

A lot of interesting properties of posteriori transitive probabilities n_k/n are found out in computer experiments with random wanderings in multiplex on nodes of an orthogonal spaces grid. Here n-the general number of particles, starting from control node, n_k - (number of the particles absorbing vertex $П_k$). In the experiments, the Kolmogorov's classical model with a 6-routing pattern and equally probability transitions on each step is used. An output of a particle on side multiplex wanderings turns into two-dimensional, at an output on an edge-in one-dimensional and come to an end in one of two vertexes $П_k$, belonging to the given edge. First of all it is necessary to specify convergence in probability:

$$n_k/n \to N_k \text{ at } n \to \infty$$

Experiments have confirmed independence of transitive probability of the form of a route and number of steps from start to finish. There are no bases to doubt about the result of experiments. However we shall try to prove the following theorem theoretically using probabilistic representations.

Theorem 1: For a particle, starting from any point of M multiplex, the probability of absorbing vertex $П_k$ is invariantly concerning the form of a trajectory and also coincides with corresponding function N_k-three-linear interpolation.

Proof: On Fig. 2, different routes from a point M (x, y, z) in vertex $П_1$ (-1;-1;-1) are demonstrated. Each broken line consists of three straight-line segments, parallel to coordinate axes. On any route, the particle for 3 steps reaches vertex $П_1$. On the first step, the particle goes to one of three edges containing vertex $П_1$, on the second step, the particle goes to one of two edges containing vertex $П_1$, on the third step, the particle is absorbed by vertex $П_1$. For the particles absorbed by other vertexes, the situation is similar. To prove this theorem, it is enough to consider one of six possible routes from M to $П_1$, for example:

$$M \to A_1 \to B \to П_1$$

The probability of transition $M \to A_1$ is defined geometrically and is:

$$p(M \to A_1) = \frac{1}{2}(1-x)$$

The probability of transition M→B_2 provided, that a route passes through A_1, is:

$$p(M \to A_1 \to B_2) = \frac{1}{2}(1-x)\cdot\frac{1}{2}(1-y)$$

Finally, the probability of transition M→Π_1 through points A_1 and B_2, is:

$$p(M \to \Pi_1) = \frac{1}{2}(1-x)\cdot\frac{1}{2}(1-y)\cdot\frac{1}{2}(1-z) = \frac{1}{8}(1-x)(1-y)(1-z),$$

that coincides with the formula (9).

By changing trajectory from M to Π_1 in the formula (9) and by changing only the order of factors all 3! Variants give the same results.

Remarks:

- Each side of a cube (Fig. 2) is two-dimensional multiplex with bilinear basis (Kamal Al-Dawoud and Khomchenko, 2007). Therefore in view of results (Kamal Al-Dawoud and Khomchenko, 2007) it is possible to offer the two-step-by-step scheme of wanderings with the same transitive probability (9). For methods of Monte Carlo: the more shorter a history of wanderings, the more effective is computing algorithm. This explained increased interest in single-step scheme.
- Simplicity and content of models Kolmogorov with orthogonal trajectories allow constructing the three-dimensional scheme of wanderings as a superposition of three one-dimensional wanderings. However, multi-step wanderings on a grid are long-term and practically useless.

Theorem 2: The probability of transition of a particle from random start in vertex Π_k multiplex is a harmonic function both on Laplace test and on Privalov's test.

Proof: The differential test of harmonic function, suggested by Laplace, is:

$$\Delta n_k(x, y, z) = 0$$

Where:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{– Laplace operator}$$

The integral test of harmonic, suggested by Privalov, is the mean-value integral by element:

$$\frac{1}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} N_k(x,y,z)dxdydz = N_k(0,0,0) = \frac{1}{8}. \tag{10}$$

Simple consideration shows, that N_k (x, y, z) satisfies Laplace equation and the rules mean-value. Notice, that in formula (10) multiplier 1/8 before triple integral is a density of uniform distribution of a random point in multiplex. Therefore, Privalov's test gives expectation of function of a random point. This result has exactly probability meaning and is formulated as the following theorem.

Theorem 3: Expectation of transitive probability N_k (x, y, z) on all random trajectories in multiplex is equal to probability of transition of a particle from barycentric to vertex.

For proof it is enough to refer to formula (10).

Remark: Surprisingly, the function N_k (x, y, z), containing members of the second and third degree, supposes exact integration using a simplified approached formula with a unique node in barycentric of an element.

CONCLUSION

New probability properties of basic functions Lagrange 3D-interpolation are established. It stimulates attempts to distribute probability approaches on polynomials of the higher orders in one-dimensional, two-dimensional and three-dimensional finite elements. Special interest is represented with penal routes with negative transitive probabilities. Such generalization of models of random wanderings will need correct and grounded formulations.

REFERENCES

Kamal Al-Dawoud and A.N. Khomchenko, 2007. Construction of Lagrange interpolational polynomials using geometrical probability. Umm Al-Qura Univ. J. Sci. Med. Eng., 19 (2): 153-163.

Norrie, D.H. and G. de Vries, 1978. An Introduction to Finite Element Analysis. Academic Press, New York.

Oden, J.T., 1972. Finite Elements of Nonlinear Continua. McGraw-Hill, New York.

Strang, G. and G.J. Fix, 1973. An Analysis of the Finite Element Method. Englewood Cliffs, Prentice-Hall, N.Y.