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Analytical Investigation of a Fourth-order Boundary Value Problem in Deformation of Beams and Plate Deflection Theory

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Abstract: In this research, variational iteration method and homotopy perturbation method are applied to solve a nonlinear fourth order boundary value problem. These problems used as mathematical models in viscoelastic inelastic flows and deformation of beams and plate deflection theory. Comparison is made between the exact solutions and the results of the Variational Iteration Method (VIM) and Homotopy Perturbation Method (HPM). The results reveal that these methods are very effective and simple. In this survey, it will be shown that these methods can also be used for solving nonlinear boundary value problems.

Key words: Deformation of elastic beams, plate deflection theory, variational iteration method, homotopy perturbation method, boundary-value problems, exact solution

INTRODUCTION

According to the classical beam theory, if u = u(x) denotes the configuration of the deformed beam, then the bending moment satisfies the relation M = -Eiu″ where, E is the Young modulus of elasticity and I is the inertial moment. Now if the deformation is caused by a load f = f(x), one deduces, from a free body diagram, that f = -v′ and v = M′ = -Eiu‴, where, v denotes the shear force. Suppose that u represents an elastic beam of length L = 1, which is clamped at its left side x = 0 and resting on a kind of elastic bearing at its right side x =1, Along its length, a load f is added to cause deformations (Fig. 1).

For simplicity it can be assumed EI = 1, then from above remarks, it can be got the following boundary value problem:

$$u^{(iv)}(x) = f(x, u(x)), \quad 0 < x < 1, \tag{1}$$

$$u(0) = u'(0) = 0, \tag{2}$$

$$u''(1) = 0 \text{ and } u'''(1) = g(u(1)), \tag{3}$$

where, $f \in C([0,1] \times \mathbb{R})$ and $g \in C(\mathbb{R})$ are real functions. In fact u‴(1) represents the shear force at x = 1, the second condition in (3) means that the vertical force is equal to g(u(l)) which denotes a relation, possibly nonlinear, between the vertical force and the displacement u(l) Furthermore, since u″(1) = 0 indicates that there is no bending moment at x = 1, the beam is resting on the bearing g (Ma and Silva, 2004).

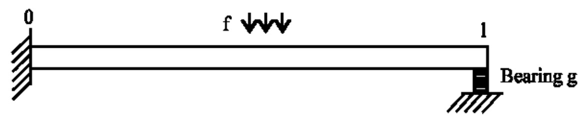

Fig. 1: Beam on elastic bearing

Chawla and Katti (1979) investigated finite difference scheme for the nonlinear differential equation of order 2n:

$$u^{(2n)} + f(x, u) = 0, \quad n \geq 2 \tag{4}$$

Subject to the boundary conditions:

$$u^{(2j)}(a) = A_{2j}, \; u^{(2j)}(b) = B_{2j}, \; j = 0(1)n-1, \tag{5}$$

$-\infty < a \leq x \leq b < \infty$, A_{2j}, B_{2j}, $j = 0(1)n-1$, are finite constants. Fourth order linear boundary value problems of Equation 4 are encountered frequently in plate-deflection theory.

In this study, we consider a general fourth-order boundary value problem of the form:

$$u^{(4)}(x) = f(x, u, u', u'', u''') \tag{6}$$

With the boundary conditions:

$$\begin{aligned} u(a) &= \alpha_1, \quad u'(a) = \alpha_2, \\ u(b) &= \beta_1, \quad u'(b) = \beta_2, \end{aligned} \tag{7}$$

where, f is a continuous function on [a, b] and the parameters α_i and β_i, i = 1, 2 are finite real arbitrary

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constants. Such type of systems are not only regarded as a general boundary value problem, but also used as mathematical models in viscoelastic and inelastic flows (Momani, 1991), deformation of beams (Ma and Silva, 2004) and plate deflection theory (Chawla and Katti, 1979). With the rapid development of nonlinear science, many different methods were proposed to solve various boundary-value problems (BVPS), such as the homotopy perturbation method (Rafei and Ganji, 2006; Ganji and Sadighi, 2006; He, 1999a, 2000, 2003; Zhang and He, 2006; Choobbasti *et al.*, 2008, Barari *et al.*, 2008) and the variational iteration method (VIM) (He, 1999b; He and Wu, 2006; Tari *et al.*, 2007; Momani and Abuasad, 2006; Odibat and Momani, 2006). In this letter, we apply the homotopy-perturbation method and variational iteration method to the discussed problem.

BASIC IDEA OF HOMOTOPY-PERTURBATION METHOD

Linear and Nonlinear phenomena are of fundamental importance in various fields of science and engineering. Most models of real-life problems are still very difficult to solve. Therefore, approximate analytical solutions such as Homotopy-Perturbation Method (HPM) were introduced. This method is the most effective and convenient ones for both linear and nonlinear equations.

Perturbation method is based on assuming a small parameter. The majority of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all and the approximate solutions obtained by the perturbation methods, in most cases, are valid only for small values of the small parameter.

Generally, the perturbation solutions are uniformly valid as long as a scientific system parameter is small. However, we cannot rely fully on the approximations, because there is no criterion on which the small parameter should exists. Thus, it is essential to check the validity of the approximations numerically and/or experimentally. To overcome these difficulties, HPM have been proposed recently.

To explain this method, let us consider the following function:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{8}$$

With the boundary conditions of:

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma, \tag{9}$$

where, A, B, f(r) and Γ are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively.

Generally speaking the operator A can be divided into a linear part Land a nonlinear part N (u). Equation 8 can, therefore, be written as:

$$L(u) + N(u) - f(r) = 0 \tag{10}$$

By the homotopy technique, we construct a homotopy $v(r,p): \Omega \times [0,1] \to R$ which satisfies:

$$H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0, \quad p \in [0,1], r \in \Omega, \tag{11}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r)\right] = 0 \tag{12}$$

where, $p \in [0,1]$ is an embedding parameter, while u_0 is an initial approximation of Eq. 8, which satisfies the boundary conditions. Obviously, from Eq. 11 and 12 we will have:

$$H(v,0) = L(v) - L(u_0) = 0, \tag{13}$$

$$H(v,1) = A(v) - f(r) = 0, \tag{14}$$

The changing process of p from zero to unity is just that of v (r, p) from u_0 to u (r). In topology, this is called deformation, while L(v) -L(u_0) and A(v)-f(r) are called homotopy.

According to the HPM, we can first use the embedding parameter p as a small parameter and assume that the solutions of Eq. 11 and 12 can be written as a power series in p:

$$H(v,1) = A(v) - f(r) = 0, \tag{15}$$

Setting p = 1 yields in the approximate solution of Equation 15 to:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{16}$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages.

The series (16) is convergent for most cases. However, the convergent rate depends on the nonlinear operator A(v). Moreover, He (1999a) made the following suggestions:

- The second derivative of N(v) with respect to v must be small because the parameter may be relatively large, i.e., $p \to 1$.

- The norm of $L^{-1}\frac{\partial N}{\partial v}$ must be smaller than one so that the series converges.

BASIC IDEA OF VARIATIONAL ITERATION METHOD

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(t), \quad (17)$$

where, L is a linear operator, N is a nonlinear operator and g (t) is a homogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\right)d\tau \quad (18)$$

where, λ is a general lagrangian multiplier which can be identified optimally via the variational theory. The subscript n indicates the nth approximation and u_n is considered as a restricted variation, i.e., $\delta\,\tilde{u}_n = 0$.

APPLICATION OF VARIATIONAL ITERATION METHOD

Consider the following nonlinear boundary value problem:

$$u^{(4)}(x) = u^2(x) + g(x), \quad 0 < x < 1, \quad (19)$$

Subject to the boundary conditions:

$$u(0) = 0, \quad u'(0) = 0, \quad u(1) = 1, \quad u'(1) = 1. \quad (20)$$

where, $g(x) = -x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48.$ (21)

The exact solution for this problem is:

$$u(x) = x^5 - 2x^4 + 2x^2. \quad (22)$$

According to Eq. 18, we have the following iteration formulation:

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{1}{6}(\tau - x)^3 \left\{u_n^{(4)}(\tau) - u_n^2(\tau) - g(\tau)\right\} d\tau. \quad (23)$$

Now we assume that an initial approximation has the form:

$$u_0(x) = ax^3 + bx^2 + cx + d. \quad (24)$$

where, a, b, c and d are unknown constants to be further determined.

By the iteration formula (23), we have the following first-order approximation:

$$\begin{aligned} u_1(x) &= u_0(x) + \int_0^x \frac{1}{6}(\tau - x)^3 \begin{Bmatrix} u_0^{(4)}(\tau) - u_0^2(\tau) + \tau^{10} - 4\tau^9 + 4\tau^8 \\ +4\tau^7 - 8\tau^6 + 4\tau^4 - 120\tau + 48 \end{Bmatrix} d\tau \\ &= -\frac{1}{24024}x^{14} + \frac{1}{4290}x^{13} - \frac{1}{2970}x^{12} - \frac{1}{1980}x^{11} + \\ &\left(\frac{1}{5040}a^2 + \frac{1}{630}\right)x^{10} + \frac{1}{1512}abx^9 + \\ &\left(-\frac{1}{420} + \frac{1}{1680}b^2 + \frac{1}{840}ac\right)x^8 + \\ &\left(\frac{1}{420}bc + \frac{1}{420}ad\right)x^7 + \left(\frac{1}{180}bd + \frac{1}{360}c^2\right)x^6 + \\ &\left(1 + \frac{1}{60}cd\right)x^5 + \left(\frac{1}{24}d^2 - 2\right)x^4 + ax^3 + bx^2 + cx + d, \end{aligned} \quad (25)$$

Incorporating the boundary conditions, Eq. 20, into $u_1(x)$, we obtain:

$$a = -0.006871650809, \; b = 2.005929593, \; c = 0, \; d = 0 \quad (26)$$

We therefore, obtain the following first-order approximate solution:

$$\begin{aligned} u_1(x) = &-4.162504162\times10^{-5}x^{14} + 2.331002331\times10^{-4}x^{13} \\ &-3.367003367\times10^{-4}x^{12} - 5.050505050\times10^{-4}x^{11} \\ &+1.587310956\times10^{-3}x^{10} - 9.116433669\times10^{-6}x^9 \\ &+1.4139007\times10^{-5}x^8 + x^5 - 2x^4 - 6.871650809\times10^{-3}x^3 \\ &+2.005929593x^2 \end{aligned} \quad (27)$$

APPLICATION OF HOMOTOPY-PERTURBATION METHOD

To solve Eq. 19 by means of HPM, we consider the following process after separating the linear and nonlinear parts of the equation.

A homotopy can be constructed as follows:

$$\begin{aligned} H(v,p) = (1-p)\left(\frac{d^4}{dx^4}v(x) - \frac{d^4}{dx^4}v_0(x)\right) + \\ p\left(\frac{d^4}{dx^4}v(x) - v(x,t)^2 + x^{10} - 4x^9 + \right. \\ \left. 4x^8 + 4x^7 - 8x^6 + 4x^4 - 120x + 48\right) = 0, \end{aligned} \quad (28)$$

Substituting $v = v_0 + pv_1 + \ldots$ in to Eq. 28 and rearranging the resultant equation based on powers of p-terms, one has:

Table 1: Comparison of the approximate solutions with exact solution

X	Exact solution	VIM	HPM	Error of VIM	Error of HPM
0.0	0.0000000000	0.0000000000	0.0000000000	0.0000000E+000	0.0000000E+000
0.1	0.0198100000	0.0198624243	0.01986242428	5.2424300E-005	5.2424300E-005
0.2	0.0771200000	0.0773022107	0.07730221055	1.8221070E-004	1.8221055E-004
0.3	0.1662300000	0.1665781379	0.1665781293	3.4813790E-004	3.481293E-004
0.4	0.2790400000	0.2795490972	0.2795489516	5.0909720E-004	5.089516E-004
0.5	0.4062500000	0.4068747265	0.4068734205	6.2472650E-004	6.2342050E-004
0.6	0.5385600000	0.5392178270	0.5392100485	6.5782700E-004	6.5004850E-004
0.7	0.6678700000	0.6684511385	0.6684162347	5.8113850E-004	5.4623470E-004
0.8	0.7884800000	0.7888727023	0.7887454961	3.9270230E-004	2.6549610E-004
0.9	0.8982900000	0.8984356964	0.8980403393	1.4569640E-004	-2.496607E-004
1.0	1.0000000000	1.0000000000	0.9989169551	0.0000000E+000	-1.083044E-003

$$p^0 : \frac{d^4}{dx^4} v_0(x) = 0, \tag{29}$$

$$p^1 : (\frac{d^4}{dx^4} v_1(x)) - 120x - v_0(x)^2 + 4x^7 + x^{10} - 4x^9 + 4x^8 + 48 - 8x^6 + 4x^4 = 0, \tag{30}$$

$$p^2 : (\frac{d^4}{dx^4} v_2(x)) - 2v_0(x)v_1(x) = 0, \tag{31}$$

With the following conditions:

$$\begin{aligned} &v_0(0) = 0, \quad \frac{d}{dx} v_0(0) = 0, \quad v_0(1) = 1, \quad \frac{d}{dx} v_0(1) = 1 \\ &v_i(0) = 0, \quad \frac{d}{dx} v_i(0) = 0, \quad v_i(1) = 0, \quad \frac{\partial}{\partial x} v_i(1) = 0, \\ &i = 1,2,.. \end{aligned} \tag{32}$$

With the effective initial approximation for v_0 from the conditions (32) and solutions of Eq. 29-31 may be written as follows:

$$v_0(x) = -0.00687165x^3 + 2.00592x^2 \tag{33}$$

$$\begin{aligned} v_1(x) = &-0.000041625x^{14} + 0.0002331x^{13} - 0.0003367x^{12} \\ &- 0.00050505x^{11} + 0.001587310956x^{10} \\ &- 0.000009116433671x^9 + 0.00001413900719x^8 \\ &+ x^5 - 2x^4 + \frac{1}{6} \end{aligned} \tag{34}$$

$$\begin{aligned} v_2(x) = &3.982633681 \times 10^{-12} x^{21} - 1.46368569 \times 10^{-9} x^{20} + \\ &1.010269055 \times 10^{-8} x^{19} - 1.829865574 \times 10^{-8} x^{18} - \\ &3.58544538 \times 10^{-8} x^{17} + 1.457919719 \times 10^{-7} x^{16} - \\ &1.122349353 \times 10^{-10} x^{15} + 2.361126618 \times 10^{-9} x^{14} - \\ &0.0000011568x^{12} + 0.00051002x^{11} - 0.0015920x^{10} \end{aligned} \tag{35}$$

In the same manner, the rest of components were obtained using the maple package.

According to the HPM, we can conclude that:

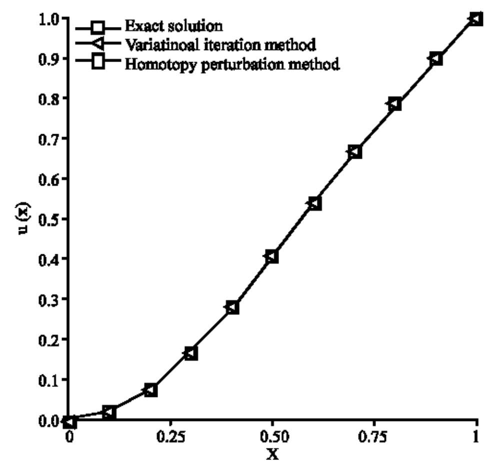

Fig. 2: Comparison between different solutions

$$u(x) = \lim_{p \to 1} v(x) = v_0(x) + v_1(x) + \ldots, \tag{36}$$

Therefore, substituting the values of $v_0(x)$, $v_1(x)$ and $v_2(x)$ from Eq. 33-35 into Eq. 36 yields:

$$\begin{aligned} u(x) = &-0.00687165x^3 + 2.00592x^2 - 0.000041625x^{14} + \\ &0.0002331x^{13} - 0.0003367x^{12} - 0.00050505x^{11} + \\ &0.001587310956x^{10} - 0.000009116433671x^9 + \\ &0.00001413900719x^8 + x^5 - 2x^4 + \frac{1}{6} + \\ &3.982633681 \times 10^{-12} x^{21} - 1.46368569 \times \\ &10^{-9} x^{20} + 1.010269055 \times 10^{-8} x^{19} - \\ &1.829865574 \times 10^{-8} x^{18} - 3.58544538 \times 10^{-8} x^{17} + \\ &1.457919719 \times 10^{-7} x^{16} - 1.122349353 \times 10^{-10} x^{15} + \\ &2.361126618 \times 10^{-9} x^{14} - 0.0000011568x^{12} + \\ &0.00051002x^{11} - 0.0015920x^{10} \end{aligned} \tag{37}$$

Comparison of the approximate solutions with exact solution is shown in Table 1 and Fig. 2 showing a remarkable agreement. Of course we can obtain even higher accurate solutions without any difficulty.

CONCLUSIONS

The homotopy perturbation method and variational iteration method are employed successfully to study a fourth order boundary value problem in structural engineering and fluid mechanic. The results revealed that The variational iteration method and homotopy perturbation method are remarkably effective for solving boundary value problems. Comparison between the approximate and exact solutions shows that the one iteration of variational iteration method is enough. These methods are very promoting method, which will be certainly found widely applications.

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