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Sixth Order/Fourth Order P-Stable Methods for Second Order Initial Value Problems

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Abstract: A family of sixth order and fourth order P-stable methods for solving second order initial value problems is considered. The nonlinear algebraic systems, which results on applying one of the methods in this family to a nonlinear differential system, may be solved by using a modified Newton method. The local error estimation technique based on the derivation of suitable formula pairs is used in which we compute two approximations of the solution, one with a fourth order method and the other with a sixth order method. The error estimate is then obtained by subtracting our two approximations. The methods in each pair are chosen to have certain features in common. They have the same iteration matrix and some of the function evaluations are common to both methods. Finally numerical results are presented of these methods.

Key words: 2nd order initial value problem, oscillation problems, p-stable methods, combination of 6th and 4th order methods, modified Newton method

INTRODUCTION

We consider the class of direct hybrid methods proposed by Cash (1981) then extended by Khiyal (1991) for solving the second order initial value problem.

$$y'' = f(t, y),\ y(0),\ y'(0),\ \text{given} \tag{1}$$

The basic methods has the form

$$y_{n+1}-2y_n + y_{n-1} = h^2\{n_1 + n_2\} \tag{2}$$

Where:

$$n_1 = \beta_0\left(y''_{n+1} + y''_{n-1}\right) + \gamma y''_n$$

$$n_2 = \beta_1\left(y''_{n+\alpha_1} + y''_{n-\alpha_1}\right) + \beta_2\left(y''_{n+\alpha_2} + y''_{n-\alpha_2}\right)$$

The off-step values are defined by

$$y_{n\pm\alpha_1} = A_\pm y_{n+1} + B_\pm y_n + C_\pm y_{n-1} + h^2 m_1 \tag{3}$$

$$y_{n\pm\alpha_2} = R_\pm y_{n+1} + L_\pm y_n + T_\pm y_{n-1} + h^2 m_2 \tag{4}$$

$$m_1 = s_\pm y''_{n+1} + q_\pm y''_n + u_\pm y''_{n-1}$$

$$m_2 = Y_\pm y''_{n+1} + V_\pm y''_n + W_\pm y''_{n-1} + X_\pm y''_{n-\alpha_1} + Z_\pm y''_{n+\alpha_1}$$

and

$$y''_n = f(t_n, y_n)$$

$$y''_{n\pm1} = f(t_n \pm h, y_{n\pm1})$$

$$y_{n\pm\alpha_1} = f(t_n \pm \alpha_1 h, y_{n\pm\alpha_1})$$

$$y_{n\pm\alpha_2} = f(t_n \pm \alpha_2 h, y_{n\pm\alpha_2})$$

When the method (2-4) is applied to a nonlinear differential system (1), a nonlinear algebraic system must be solved at each step. This may be solved by using a modified Newton iteration scheme. The resulting iteration matrix involves J, J^2 and J^3, where J is an approximation for the Jacobian matrix of f with respect to y. Since matrix products are expensive, especially for large systems and any sparsity in J will be weakened in J^2 and J^3 and any ill-conditioning in J will be magnified in its powers, we wish to avoid the calculation of J^2 and J^3. Cash (1981), Thomas (1988), Khiyal (1991) and Khiyal (2007b) have derived sixth order P-stable (Lambert and Watson, 1976) methods of the form (2-4) for which the iteration matrix is a true perfect cube.

By taking $\beta_2 = 0$ in (2) and choosing the remaining parameters appropriately, several authors (Cash, 1981; Chawla, 1981; Chawla and Neta, 1986; Costabile and Costabile, 1982; Khiyal, 1991; Thomas, 1988; Khiyal, 2007a) have derived fourth order methods of the form (2-4) which are P-stable.

The present consider the fourth order method of Thomas (1987), Khiyal (1991) and Khiyal (2007a) and sixth order methods of Thomas (1988), Khiyal (1991) and Khiyal (2007b).

We give necessary and sufficient conditions for there to exit

- Fourth order, P-stable, two-evaluations, two-step methods with iteration matrix $(I-rh^2J)^2$
- Sixth order, P-stable, three-evaluations, two-step methods with iteration matrix $(I-rh^2J)^3$

The method may be chosen so that the value of r is the same for both the fourth and sixth order methods. We derive some formula pairs. (The formula pair technique discussed by Khiyal (1991) and Khiyal and Thomas (2006).

MATERIALS AND METHODS

The modified Newton iteration scheme for methods of the form (2-4) is given by:

$$F'(y_{n+1}^{(p-1)})\left(y_{n+1}^{(p)} - y_{n+1}^{(p-1)}\right) = -F\left(y_{n+1}^{(p-1)}\right), p = 1,2,... \quad (5)$$

For sixth order methods as defined by Khiyal (2007b)

$$F(y) = y - 2y_n + y_{n-1} - h^2(n_1 + n_2 + n_3 + n_4) \quad (6)$$

Where:

$$\begin{aligned}
n_1 &= \beta_0 f(t_{n+1}, y) + \beta_0 f(t_{n-1}, y_{n-1}) \\
n_2 &= \gamma f(t_n, y_n) \\
n_3 &= \beta_1 f(t_{n+\alpha_1}, y_{n+\alpha_1}) + \beta_1 f(t_{n-\alpha_1}, y_{n-\alpha_1}) \\
n_4 &= \beta_2 f(t_{n+\alpha_2}, y_{n+\alpha_2}) + \beta_2 f(t_{n-\alpha_2}, y_{n-\alpha_2}) \\
y_{n\pm\alpha_1} &\equiv A_\pm y + B_\pm y_n + C_\pm y_{n-1} + h^2 m_1 \\
y_{n\pm\alpha_2} &\equiv R_\pm y + L_\pm y_n + T_\pm y_{n-1} + h^2(m_2 + m_3) \\
m_1 &= s_\pm f(t_{n+1}, y) + q_\pm f(t_n, y_n) + u_\pm f(t_{n-1}, y_{n-1}) \\
m_2 &= Y_\pm f(t_{n+1}, y) + V_\pm f(t_n, y_n) + W_\pm f(t_{n-1}, y_{n-1}) \\
m_3 &= X_\pm f(t_{n-\alpha_1}, y_{n-\alpha_1}) + Z_\pm f(t_{n+\alpha_1}, y_{n+\alpha_1})
\end{aligned}$$

and

$$F'(y) = I - m_4 h^2 J - m_5 h^4 J^2 - m_6 h^6 J^3 \quad (7)$$

$$\begin{aligned}
m_4 &= \beta_0 + \beta_1(A_+ + A_-) + \beta_2(R_+ + R_-) \\
m_5 &= \beta_1(s_+ + s_-) + \beta_2\left[Y_+ + Y_- + A_-(X_+ + X_-) + A_+(Z_+ + Z_-)\right] \\
m_6 &= \beta_2\left[s_-(X_+ + X_-) + s_+(Z_+ + Z_-)\right]
\end{aligned}$$

To avoid the calculation of J^2 and J^3 Thomas (1988) and Khiyal (1991) has shown that the iteration matrix (5) may be factorized as a true perfect cube

$$(I - rh^2 J)^3$$

Where:

$$r = \frac{1}{3} m_4 \cdot$$

The necessary and sufficient conditions for this are

$$m_5 = -r^2 \quad (8a)$$

$$m_6 = r^3 \quad (8b)$$

The resulting methods are P-stable if and only if

$$\begin{aligned}
&1 + \left(3r - \frac{1}{4}\right)H^2 + \left(3r^2 - \frac{3}{4}r + \frac{1}{48}\right) \\
&H^4 + \left(r^3 - \frac{3}{4}r^2 + \frac{r}{16} - \frac{1}{1440}\right)H^6 \geq 0 \\
&1 + \left(3r - \frac{1}{12}\right)H^2 + \left(3r^2 - \frac{1}{4}r + \frac{1}{360}\right)H^4 \geq 0
\end{aligned} \quad (9)$$

hold for all H. These conditions are satisfied when r is greater than or equal to the largest real root of the polynomial equation

$$r^3 - \frac{3}{4}r^2 + \frac{r}{16} - \frac{1}{1440} = 0 \quad (10)$$

Note that the largest root of (10) is r = 0.6564 to four significant figure. Then the iteration scheme (5) has the form

$$\left(I - rh^2 J\right)^3 \left(y_{n+1}^{(p)} - y_{n+1}^{(p-1)}\right) = -F\left(y_{n+1}^{(p-1)}\right), p = 1,2,... \quad (11)$$

The Newton iteration scheme for fourth order methods of the form (2-4) with $\beta_2 = 0$ is given by

$$F(y) = y - 2y_n + y_{n-1} - h^2(n_1 + n_2 + n_3) \quad (12)$$

Where:

$$\begin{aligned}
n_1 &= \beta_0 f(t_{n+1}, y) + \beta_0 f(t_{n-1}, y_{n-1}) \\
n_2 &= \gamma f(t_n, y_n) \\
n_3 &= \beta_1 f(t_{n+\alpha_1}, y_{n+\alpha_1}) + \beta_1 f(t_{n-\alpha_1}, y_{n-\alpha_1}) \\
y_{n\pm\alpha_1} &\equiv A_\pm y + B_\pm y_n + C_\pm y_{n-1} + h^2 m_7 \\
m_7 &= s_\pm f(t_{n+1}, y) + q_\pm f(t_n, y_n) + u_\pm f(t_{n-1}, y_{n-1})
\end{aligned}$$

and

$$F'(y) = I - \{\beta_0 + \beta_1(A_+ + A_-)\}h^2 J - \beta_1(s_+ + s_-)h^4 J^2 \quad (13)$$

A true perfect square iteration matrix is

$$(I - rh^2 J)^2$$

Where:

$$r = \frac{1}{2}\left[\beta_0 + \beta_1\left(A_+ + A_-\right)\right]$$

For the fourth order methods, this is the case if and only if

$$4\beta_1\left(u_+ + u_-\right) = -\left[\beta_0 + \beta_1\left(A_+ + A_-\right)\right]^2 \quad (14)$$

and the resulting methods are then P-stable if and only if

$$\beta_0 + \beta_1\left(A_+ + A_-\right) \geq \frac{3+\sqrt{6}}{6}. \quad (15)$$

Derivation of formula pairs: The three evaluations P-stable, sixth order methods of the type given by Thomas (1988), Khiyal (1991), Khiyal and Thomas (2006) and Khiyal (2007b) may be combined in a variable step code with appropriate two evaluations P-stable fourth order methods of the type given by Thomas (1987), Khiyal (1991) and Khiyal (2007a). The iteration matrix for the sixth order methods is $\left(I - \bar{r}h^2J\right)^3$ where, to ensure P-stability $\bar{r} \geq R$ and R is the largest root of the polynomial (10). The iteration matrix for the fourth order method is $\left(I - \hat{r}h^2J\right)^2$ where

$$\hat{r} = \frac{3+\sqrt{6}}{6}$$

for P-stability. For common iteration matrix we choose r $\geq$R. The local error estimate is given by

$$Le_{n+1} = y_{n+1}^{[6]} - y_{n+1}^{[4]}$$

Where, $y_{n+1}^{[2m]}$ is the approximation for $y(t_{n+1})$ obtained by the method of order 2m, m = 2 and 3. The different formula pairs are:

SHK64A: The sixth order method is the method given by Khiyal (1991) and Khiyal (2007b) for which $y_{n-\alpha_1}$ is independent of y_{n+1} and the points $\left(t_{n+\alpha_2}, y_{n+\alpha_2}\right), \left(t_{n-\alpha_2}, y_{n-\alpha_2}\right)$ are coincident. For this method we must evaluate $f\left(t_{n-\alpha_1}, y_{n-\alpha_1}\right)$ once per step. With this sixth order method, we combine a fourth order method given by Khiyal (1991) and Khiyal (2007a) such that $y_{n-\alpha_1}$ is independent of y_{n+1} and $f\left(t_{n-\alpha_1}, y_{n-\alpha_1}\right)$ need only be evaluated once per step. Note that $f\left(t_{n-\alpha_1}, y_{n-\alpha_1}\right)$ has to be evaluated once per step for each of the fourth and sixth order methods. We choose α_1 to be the same for both methods and we also choose the coefficients in the expression (3) to ensure that $y_{n-\alpha_1}$ is the same for both methods. This means that $f\left(t_{n-\alpha_1}, y_{n-\alpha_1}\right)$ must be evaluated just once per step and may then be used for both the fourth and sixth order methods.

Sixth order method:

$$y_{n+1} - 2y_n + y_{n-1} = h^2\left\{\beta_0\left(y''_{n+1} + y''_{n-1}\right) + \gamma y''_n + m_8\right\} \quad (16)$$

$$\begin{aligned}
m_8 &= \beta_1 y''_{n-0.5} + \beta_1 y''_{n+0.5} + 2\beta_2 \bar{y}''_n \\
y_{n-0.5} &= \frac{1}{2}\left(y_n + y_{n-1}\right) - \frac{1}{16}h^2\left(y''_n + y''_{n-1}\right) \\
y_{n+0.5} &= A_+ y_{n+1} + \left(\frac{3}{2} - 2A_+\right)y_n + \left(A_+ - \frac{1}{2}\right)y_{n-1} + h^2 m_9 \\
\bar{y}_n &= R_+\left(y_{n+1} + y_{n-1}\right) + \left(1 - 2R_+\right)y_n + h^2 m_{10} \\
m_9 &= s_+ y''_{n+1} + q_+ y''_n + u_+ y''_{n-1} \\
m_{10} &= Y_+\left(y''_{n+1} + y''_{n-1}\right) + V_+ y''_n + Z_+\left(y''_{n-0.5} + y''_{n+0.5}\right)
\end{aligned}$$

Where:

$$\begin{aligned}
\beta_0 &= \frac{1}{60} \\
\beta_1 &= \frac{4}{15} \\
\gamma &= \frac{13}{30} - 2\beta_2 \\
Z_+ &= \frac{1}{2\beta_2}\left[-12r^3 + 3r^2 - \frac{r}{4} + \frac{1}{360}\right] \\
s_+ &= \frac{r^3}{2\beta_2 Z_+} \\
A_+ &= -\frac{1}{4} - 12s_+ \\
q_+ &= \frac{9}{16} + 10s_+ \\
u_+ &= s_+ + \frac{1}{16} \\
\beta_2 R_+ &= \frac{3}{2}r - \frac{1}{24} + 6s_+ \\
\beta_2 Y_+ &= \frac{1}{720} - \frac{2}{15}s_+ - \frac{r}{8} - \frac{1}{4}\beta_2 Z_+ \\
V_+ &= -R_+ - 2y_+ - 2z_+
\end{aligned}$$

β_2 is free parameter (we choose $\beta_2 = 1$).

Fourth order method

$$y_{n+1} - 2y_n + y_{n-1} = h^2\left\{\beta_0\left(y''_{n+1} + y''_{n-1}\right) + m_{11}\right\} \quad (17)$$

$$m_{11} = \gamma y''_n + \beta_1\left(y''_{n-0.5} + y''_{n+0.5}\right)$$
$$y_{n-0.5} = \frac{1}{2}\left(y_n + y_{n-1}\right) - \frac{1}{16}h^2\left(y''_n + y''_{n-1}\right)$$
$$y_{n+0.5} = A_+ y_{n+1} + \left(\frac{3}{2} - 2A_+\right)y_n + \left(A_+ - \frac{1}{2}\right)y_{n-1} + h^2 m_{12}$$
$$m_{12} = s_+ y''_{n+1} + q_+ y''_n + u_+ y''_{n-1}$$

Where:

$$\beta_0 = \frac{1}{12} - \frac{1}{4}\beta_1$$
$$\gamma = \frac{5}{6} - \frac{3}{2}\beta_1$$
$$A_+ = \frac{1}{4} + \frac{1}{\beta_1}\left(2r - \frac{1}{12}\right)$$
$$u_+ = -\frac{r^2}{\beta_1}$$
$$s_+ = u_+ - \frac{1}{16}$$
$$q_+ = \frac{1}{8} + \frac{1}{\beta_1}\left(r^2 - 2r + \frac{1}{12}\right) - s_+$$

β_1 is free parameter (we choose $\beta_1 = 1$).
r is greater than or equal to the largest root of the cubic Eq. 10 for both sixth and fourth order methods.

SHK64B: The sixth order method is the method given by Khiyal (1991) and Khiyal (2007b) for which $y''_{n-\alpha_1}$ is identically equal to y''_n and $y_{n-\alpha_2}$ is independent of y_{n+1}. For this method we must evaluate $f\left(t_{n-\alpha_2}, y_{n-\alpha_2}\right)$ once per step. We combine this method with a fourth order method given by Thomas (1987), Khiyal (1991) and Khiyal (2007a), for which $y''_{n-\alpha_1}$ is identically equal to y''_n.

Sixth order method

$$y_{n+1} - 2y_n + y_{n-1} = h^2\left\{\beta_0\left(y''_{n+1} + y''_{n-1}\right) + m_{13}\right\} \qquad (18)$$

$$m_{13} = \left(\gamma + \beta_1\right)y''_n + \beta_2\left(y''_{n-0.5} + y''_{n+0.5}\right) + \beta_1\bar{y}''_n$$
$$y_{n-0.5} = \frac{1}{2}\left(y_n + y_{n-1}\right) - \frac{1}{16}h^2\left(y''_n + y''_{n-1}\right)$$
$$y_{n+0.5} = R_+ y_{n+1} + \left(\frac{3}{2} - 2R_+\right)y_n + \left(R_+ - \frac{1}{2}\right)y_{n-1} + h^2 m_{14}$$
$$\bar{y}_n = A_+\left(y_{n+1} + y_{n-1}\right) + \left(1 - 2A_+\right)y_n + h^2 m_{15}$$
$$m_{15} = s_+\left(y''_{n+1} + y''_{n-1}\right) + q_+ y''_n$$
$$m_{14} = \left\{Y_+ y''_{n+1} + \left(V_+ + X_+\right)y''_n + W_+ y''_{n-1} + Z_+\bar{y}''_n\right\}$$

Where:

$$\beta_0 = \frac{1}{60}$$
$$\beta_2 = \frac{4}{15}$$
$$Z_+ = 540\beta_1\left(r^3 - \frac{1}{4}r^2 + \frac{1}{48}r - \frac{1}{4320}\right)$$
$$\gamma = \frac{13}{30} - 2\beta_1$$
$$A_+ = \frac{15}{4Z_+}\left(-3r^2 + \frac{r}{4} - \frac{1}{360}\right)$$
$$s_+ = \frac{1}{144\beta_1} - \frac{A_+}{12}$$
$$R_+ = \frac{15}{4}\left(3r - \frac{1}{60} - \beta_1 A_+\right)$$
$$q_+ = -A_+ - 2s_+$$
$$Y_+ = -\frac{1}{48} - \frac{1}{12}R_+$$
$$W_+ = Y_+ + \frac{1}{16}$$
$$V_+ + X_+ = \frac{17}{48} - \frac{5}{8}R_+ - Z_+$$

β_1 are free parameters.

Fourth order method

$$y_{n+1} - 2y_n + y_{n-1} = h^2\left\{\beta_0\left(y''_{n+1} + y''_{n-1}\right) + m_{16}\right\} \qquad (19)$$

$$\bar{y}_n = A_+\left(y_{n+1} + y_{n-1}\right) + \left(1 - 2A_+\right)y_n + h^2 m_{17}$$
$$m_{16} = \left(\gamma + \beta_1\right)y''_n + \beta_1\bar{y}''_n$$
$$m_{17} = s_+\left(y''_{n+1} + y''_{n-1}\right) + q_+ y''_n$$

Where:

$$\beta_0 = \frac{1}{12}$$
$$\gamma = \frac{5}{6} - 2\beta_1$$
$$A_+ = \frac{1}{\beta_1}\left(2r - \frac{1}{12}\right)$$
$$s_+ = -\frac{r^2}{\beta_1}$$
$$q_+ = \frac{1}{\beta_1}\left(2r^2 - 2r + \frac{1}{12}\right)$$

β_1 are free parameters.
r is greater than or equal to the largest root of the cubic Eq. 10 for both the sixth and fourth order methods.

SHK64C: Consider the sixth order method given by Khiyal (1991) and Khiyal (2007b) for which the points

$(t_{n+\alpha_1}, y_{n+\alpha_1}), (t_{n-\alpha_1}, y_{n-\alpha_1})$ are coincident and $y_{n-\alpha_2}$ is independent of y_{n+1}. For this method we must evaluate $f(t_{n-\alpha_2}, y_{n-\alpha_2})$ once per step. We combine the method given by Thomas (1987), Khiyal (1991) and Khiyal (2007a) for which the points $(t_{n+\alpha_1}, y_{n+\alpha_1}), (t_{n-\alpha_1}, y_{n-\alpha_1})$ are coincident.

Sixth order method

$$y_{n+1}-2y_n + y_{n-1} = h^2\{\beta_0(y''_{n+1} + y''_{n-1}) + \gamma y''_n + m_{18}\} \quad (20)$$

$$m_{18} = 2\beta_1\bar{y}''_n + \beta_2(y''_{n-0.5} + y''_{n+0.5})$$
$$y_{n-0.5} = \frac{1}{2}(y_n + y_{n-1}) - \frac{1}{16}h^2(y''_n + y''_{n-1})$$
$$y_{n+0.5} = R_+ y_{n+1} + \left(\frac{3}{2} - 2R_+\right)y_n + \left(R_+ - \frac{1}{2}\right)y_{n-1} + h^2 m_{19}$$
$$\bar{y}_n = A_+(y_{n+1} + y_{n-1}) + (1-2A_+)y_n + h^2\{s_+(y''_{n+1} + y''_{n-1}) + q_+ y''_n\}$$
$$m_{19} = \{Y_+ y''_{n+1} + V_+ y''_n + W_+ y''_{n-1} + 2(Z_+ + Z_-)\bar{y}''_n\}$$

Where:

$$\beta_0 = \frac{1}{60}$$
$$\beta_2 = \frac{4}{15}$$
$$\gamma = \frac{13}{30} - 2\beta_1$$
$$Z_+ + Z_- = 540\beta_1\left(r^3 - \frac{1}{4}r^2 + \frac{1}{48}r - \frac{1}{4320}\right)$$
$$A_+ = \frac{15}{8(Z_+ + Z_-)}\left(-3r^2 + \frac{r}{4} - \frac{1}{360}\right)$$
$$s_+ = \frac{1}{288\beta_1} - \frac{A_+}{12}$$
$$R_+ = \frac{15}{4}\left(3r - \frac{1}{60} - 2\beta_1 A_+\right)$$
$$q_+ = -A_+ - 2s_+$$
$$Y_+ = -\frac{1}{48} - \frac{1}{12}R_+$$
$$W_+ = Y_+ + \frac{1}{16}$$
$$V_+ = \frac{5}{16} - R_+ - Y_+ - 2(Z_+ + Z_-)$$

β_1 are free parameters.

Fourth order method

$$y_{n+1}-2y_n + y_{n-1} = h^2\{\beta_0(y''_{n+1} + y''_{n-1}) + \gamma y''_n + 2\beta_1\bar{y}''_n\} \quad (21)$$

$$\bar{y}_n = A_+(y_{n+1} + y_{n-1}) + (1-2A_+)y_n + h^2\{s_+(y''_{n+1} + y''_{n-1}) + q_+ y''_n\}$$

Where:

$$\beta_0 = \frac{1}{12}$$
$$\gamma = \frac{5}{6} - 2\beta_1$$
$$A_+ = \frac{1}{\beta_1}\left(r - \frac{1}{24}\right)$$
$$s_+ = -\frac{r^2}{2\beta_1}$$
$$q_+ = -A_+ - 2s_+$$

β_1 are free parameters.

r is greater than or equal to the largest root of the cubic Eq. 10 for both the sixth and fourth order methods.

SHK64D: Consider the sixth order given by Khiyal (1991) and Khiyal (2007b) for which $y_{n-\alpha_1}$ is independent of y_{n+1} and $y''_{n-\alpha_2} \equiv y''_{n-1}$. For this method we need to evaluate $f(t_{n-\alpha_1}, y_{n-\alpha_1})$ once per step. We combine fourth order method given by Khiyal (1991) and Khiyal (2007a) such that $y_{n-\alpha_1}$ is independent of y_{n+1}. This means that $f(t_{n-\alpha_1}, y_{n-\alpha_1})$ must be evaluated just once per step and may be used for both fourth and sixth order methods.

Sixth order method

$$y_{n+1}-2y_n + y_{n-1} = h^2\{\beta_0(y''_{n+1} + y''_{n-1}) + \gamma y''_n + m_{20}\} \quad (22)$$

$$m_{20} = \beta_1(y''_{n+0.5} + y''_{n-0.5}) + \beta_2(\bar{y}''_{n+1} + y''_{n-1})$$
$$y_{n-0.5} = \frac{1}{2}(y_n + y_{n-1}) - \frac{h^2}{16}(y''_n + y''_{n-1})$$
$$y_{n+0.5} = A_+ y_{n+1} + \left(\frac{3}{2} - 2A_+\right)y_n + \left(A_+ - \frac{1}{2}\right)y_{n-1} + h^2 m_{21}$$
$$\bar{y}_n = R_+ y_{n+1} + (2 - 2R_+)y_n + (R_+ - 1)y_{n-1} + h^2 m_{22}$$
$$m_{21} = s_+ y''_{n+1} + q_+ y''_n + u_+ y''_{n-1}$$
$$m_{22} = Y_+(y''_{n+1} + y''_{n-1}) + V_+ y''_n + Z_+(y''_{n+0.5} + y''_{n-0.5})$$

Where:

$$\beta_0 = \frac{1}{60} - \beta_2$$
$$\beta_1 = \frac{4}{15}$$
$$\gamma = \frac{13}{30} - 4\beta_2$$
$$s_+ = \frac{r^3}{\beta_2 Z_+},$$

$$\beta_2 Z_+ = 8\left(12r^3 - 3r^2 + \frac{r}{4} - \frac{1}{360}\right)$$

$$u_+ = s_+ + \frac{1}{16}$$

$$q_+ = -\frac{1}{4} + 10s_+$$

$$A_+ = \frac{3}{8} - 12s_+$$

$$\beta_2 R_+ = 3r - \frac{1}{60} + \beta_2 - \frac{4}{15}A_+$$

$$\beta_2 Y_+ = -\frac{1}{144} + \beta_2\left(1 - R_+ - 3Z_+\right)$$

$$V_+ = 1 - R_+ - 2Y_+ - 2Z_+$$

β_2 are free parameters.

Fourth order method

$$y_{n+1} - 2y_n + y_{n-1} = h^2\left\{\beta_0\left(y''_{n+1} + y''_{n-1}\right) + m_{23}\right\} \quad (23)$$

$$m_{23} = \gamma y''_n + \beta_1\left(y''_{n-0.5} + y''_{n+0.5}\right)$$

$$y_{n-0.5} = \frac{1}{2}\left(y_n + y_{n-1}\right) - \frac{1}{16}h^2\left(y''_n + y''_{n-1}\right)$$

$$y_{n+0.5} = A_+ y_{n+1} + \left(\frac{3}{2} - 2A_+\right)y_n + \left(A_+ - \frac{1}{2}\right)y_{n-1} + h^2 m_{24}$$

$$m_{24} = s_+ y''_{n+1} + q_+ y''_n + u_+ y''_{n-1}$$

Where:

$$\beta_0 = \frac{1}{12} - \frac{1}{4}\beta_1$$

$$\gamma = \frac{5}{6} - \frac{3}{2}\beta_1$$

$$A_+ = \frac{1}{4} + \frac{1}{\beta_1}\left(2r - \frac{1}{12}\right)$$

$$u_+ = -\frac{r^2}{\beta_1}$$

$$s_+ = u_+ - \frac{1}{16}$$

$$q_+ = \frac{1}{8} + \frac{1}{\beta_1}\left(r^2 - 2r + \frac{1}{12}\right) - s_+$$

β_1 is free parameter.
r is greater than or equal to the largest root of the cubic Eq. 10 for both sixth and fourth order methods.

SHK64E: Consider the sixth order method given by Khiyal (1991) and Khiyal (2007b) for which $y'_{n-\alpha_1}$ is identically equal to y''_{n-1} and $y_{n-\alpha_2}$ is independent of y_{n+1}. For this method we need to evaluate $f\left(t_{n-\alpha_2}, y_{n-\alpha_2}\right)$ once per step. We combine this method with a fourth order method given by Thomas (1987), Khiyal (1991) and Khiyal (2007a).

Sixth order method

$$y_{n+1} - 2y_n + y_{n-1} = h^2\left\{\beta_0\left(y''_{n+1} + y''_{n-1}\right) + \gamma y''_n + m_{24}\right\} \quad (24)$$

$$m_{24} = \beta_2\left(y''_{n+0.5} + y''_{n-0.5}\right) + \beta_1\left(\bar{y}''_{n+1} + y''_{n-1}\right)$$

$$y_{n-0.5} = \frac{1}{2}\left(y_n + y_{n-1}\right) - \frac{h^2}{16}\left(y''_n + y''_{n-1}\right)$$

$$y_{n+0.5} = R_+ y_{n+1} + \left(\frac{3}{2} - 2R_+\right)y_n + \left(R_+ - \frac{1}{2}\right)y_{n-1} + h^2 m_{25}$$

$$\bar{y}_{n+1} = A_+ y_{n+1} + \left(2 - 2A_+\right)y_n + \left(A_+ - 1\right)y_{n-1} + h^2 m_{26}$$

$$m_{25} = Y_+ y''_{n+1} + V_+ y''_n + \left(W_+ + X_+\right)y''_{n-1} + Z_+ \bar{y}''_{n+1}$$

$$m_{26} = s_+\left(y''_{n+1} + y''_{n-1}\right) + \left(1 - A_+ - 2s_+\right)y''_n$$

Where:

$$\beta_0 = \frac{1}{60} - \beta_1$$

$$\beta_2 = \frac{4}{15}$$

$$\gamma = \frac{13}{30} - 4\beta_1$$

$$s_+ = \frac{15r^3}{4Z_+}$$

$$Z_+ = 45\beta_1\left(12r^3 - 3r^2 + \frac{r}{4} - \frac{1}{360}\right)$$

$$A_+ = 1 - 12s_+ - \frac{1}{12\beta_1}$$

$$R_+ = \frac{15}{4}\left[3r - \frac{1}{60} + \beta_1\left(1 - A_+\right)\right]$$

$$Y_+ = \frac{1}{32} - \frac{1}{12}R_+ - Z_+$$

$$W_+ + X_+ = Y_+ + Z_+ + \frac{1}{16}$$

$$V_+ = \frac{5}{16} - R_+ - 2\left(Y_+ + Z_+\right)$$

β_1 are free parameter.

Fourth order method

$$y_{n+1} - 2y_n + y_{n-1} = h^2\left\{\beta_0\left(y''_{n+1} + y''_{n-1}\right) + m_{27}\right\} \quad (25)$$

$$m_{27} = \gamma y''_n + \beta_1\left(\bar{y}''_{n+1} + y''_{n-1}\right)$$

$$\bar{y}_{n+1} = A_+ y_{n+1} + \left(2 - 2A_+\right)y_n + \left(A_+ - 1\right)y_{n-1} + h^2 m_{28}$$

$$m_{28} = s_+\left(y''_{n+1} + y''_n\right) + u_+ y''_{n-1}$$

Where:

$$\beta_0 = \frac{1}{12} - \beta_1$$
$$\gamma = \frac{5}{6}$$
$$A_+ = 1 + \frac{1}{\beta_1}\left(2r - \frac{1}{12}\right)$$
$$s_+ = -\frac{r^2}{\beta_1}$$
$$q_+ = -\frac{5}{8} + \frac{1}{\beta_1}\left(2r^2 - 2r + \frac{1}{12}\right)$$

β_1 is free parameter.
r is greater than or equal to the largest root of the cubic Eq. 10 for both sixth and fourth order methods.

SHK64F: Consider the sixth order method given by Khiyal (1991) and Khiyal (2007b) for which $y_{n-\alpha_1}$ is independent of y_{n+1} and $y''_{n-\alpha_2} \equiv y''_n$. For this method we need to evaluate $f(t_{n-\alpha_1}, y_{n-\alpha_1})$ once per step. We combine this method with a fourth order method given by Khiyal (1991) and Khiyal (2007a) for which $y_{n-\alpha_1}$ is independent of y_{n+1}. This means that $f(t_{n-\alpha_1}, y_{n-\alpha_1})$ must be evaluated once per step and may be used for both fourth and sixth order methods.

Sixth order method

$$y_{n+1} - 2y_n + y_{n-1} = h^2\{\beta_0(y''_{n+1} + y''_{n-1}) + (\gamma + \beta_2)y''_n + m_{29}\} \quad (26)$$

$$m_{29} = \beta_1 y''_{n-0.5} + \beta_1 y''_{n+0.5} + \beta_2 \bar{y}''_n$$
$$y_{n-0.5} = \frac{1}{2}(y_n + y_{n-1}) - \frac{1}{16}h^2\{y''_n + y''_{n-1}\}$$
$$y_{n+0.5} = A_+ y_{n+1} + \left(\frac{3}{2} - 2A_+\right)y_n + \left(A_+ - \frac{1}{2}\right)y_{n-1} + h^2 m_{30}$$
$$m_{30} = s_+ y''_{n+1} + q_+ y''_n + u_+ y''_{n-1}$$
$$\bar{y}_n = R_+(y_{n+1} + y_{n-1}) + (1 - 2R_+)y_n + h^2 m_{31}$$
$$m_{31} = Y_+(y''_{n+1} + y''_{n-1}) + V_+ y''_n + Z_+(y''_{n-0.5} + y''_{n+0.5})$$

Where:

$$\beta_0 = \frac{1}{60}$$
$$\beta_1 = \frac{4}{15}$$
$$\gamma = \frac{13}{30} - 2\beta_2$$
$$s_+ = \frac{r^3}{\beta_2 Z_+}$$
$$\beta_2 Z_+ = -2\left(12r^3 - 3r^2 + \frac{r}{4} - \frac{1}{360}\right)$$
$$A_+ = -\frac{1}{2} - 12s_+$$
$$u_+ = s_+ + \frac{1}{16}$$
$$q_+ = \frac{9}{16} + 10s_+$$
$$\beta_2 R_+ = 3r - \frac{1}{60} - \frac{4}{15}A_+$$
$$\beta_2 Y_+ = \frac{1}{360} - \frac{4}{15}s_+ - \frac{r}{4} - \frac{1}{4}\beta_2 Z_+$$
$$V_+ = 1 - R_+ - 2Y_+ - 2Z_+$$

β_2 is free parameter.

Fourth order method

$$y_{n+1} - 2y_n + y_{n-1} = h^2\{\beta_0(y''_{n+1} + y''_{n-1}) + m_{32}\} \quad (27)$$

$$m_{32} = \gamma y''_n + \beta_1(y''_{n-0.5} + y''_{n+0.5})$$
$$y_{n-0.5} = \frac{1}{2}(y_n + y_{n-1}) - \frac{1}{16}h^2(y''_n + y''_{n-1})$$
$$y_{n+0.5} = A_+ y_{n+1} + \left(\frac{3}{2} - 2A_+\right)y_n + \left(A_+ - \frac{1}{2}\right)y_{n-1} + h^2 m_{33}$$
$$m_{33} = s_+ y''_{n+1} + q_+ y''_n + u_+ y''_{n-1}$$

Where:

$$\beta_0 = \frac{1}{12} - \frac{1}{4}\beta_1$$
$$\gamma = \frac{5}{6} - \frac{3}{2}\beta_1$$
$$A_+ = \frac{1}{4} + \frac{1}{\beta_1}\left(2r - \frac{1}{12}\right)$$
$$u_+ = -\frac{r^2}{\beta_1}$$
$$s_+ = u_+ - \frac{1}{16}$$
$$q_+ = \frac{1}{8} + \frac{1}{\beta_1}\left(r^2 - 2r + \frac{1}{12}\right) - s_+$$

β_1 is free parameter.
r is greater than or equal to the largest root of the cubic Eq. 10 for both sixth and fourth order methods.

RESULTS

We are mainly concerned with solving oscillatory stiff initial value problem. We have tried a number of explicit scalar (nonstiff) test problems of the form (1).

They give similar results and so we restrict our attention to one oscillatory example.

Example 1:

$$y'' + \sinh y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

This is a pure oscillation problem whose solution has maximum amplitude unity and period approximately six. We have calculated error as |Error at t = 6|

To verify that our techniques works for systems, we use a test problem a moderately stiff system of two equations

Example 2:

$$y_1'' + \sinh(y_1 + y_2) = 0, \quad y_1(0) = 1, \quad y_1'(0) = 0,$$
$$y_2'' + 10^4 y_2 = 0, \quad y_2(0) = 10^{-8}, \quad y_2'(0) = 0,$$

For this example we have deliberately introduced coupling from the stiff (linear) equation to the nonstiff (nonlinear) equation. For this example again we have calculated error as $\|\text{Error at } t = 6\|_\infty$.

In Table 1-6, we present the following statistics:

- No. of evaluation of the differential equation right hand side f, FCN
- No. of evaluation of the Jacobian $\frac{\partial f}{\partial y}$, JCB
- No. of iteration overall, NIT
- No. of iteration on steps where the iteration converges, NSIT
- No. of steps overall, NST
- No. of successful steps to complete the integration, NSST
- No. of steps where the stepsize is changes, NCST
- No. of failed steps, NFST
- No. of LU factorization of the iteration matrix, NFAC
- No. of function evaluation required on a per step basis, rather than on each iteration, NFPS

The results are given in Table 1-6. The cost of the starting technique is not included in the tables. Before we can employ this error estimation technique, we need a special starting technique to obtain the values y_1 and y_2. Thus we take two steps with fourth order, two stages Implicit Runge Kutta method. The technique used is given by Khiyal and Rashid (2005). After two steps we use the formula pair technique. After word at the end of each step, we form the error estimate Le_{n+1}. To estimate the local error we use predictor corrector approach. For predictor we use the polynomial of degree five given by Khiyal (1991).

$$y_{n+1} = 16(y_n + y_{n-2}) - 30y_{n-1} - y_{n-3} - 12h^2 y''_{n-1}$$

If the local error is satisfied, we use the sixth order direct hybrid method. At the each step, we form the error estimate;

$$Le_{n+1} = y_{n+1}^{[6]} - y_{n+1}^{[4]}$$

Where, $y_{n+1}^{[2m]}$ is the approximation for $y(t_{n+1})$ obtained by the method of order 2m, m = 2 and 3. The step size for the next step is given by:

$$\bar{h} = h\left[\frac{Tol}{2\|Le_{n+1}\|}\right]^{\frac{1}{5}}$$

Where, Tol is the local error tolerance and h is the current step size and $\bar{h}$ is the next predicted step size. We do not allow the step size to decrease by more than a

Table 1: Methods are compared for example 1 with $Tol = 10^{-2}$

Methods	ERROR	FCN	JCB	NIT	NSIT	NST	NSST	NCST	NFST	NFPS
SHK64A	1.931×10^{-3}	309	2	108	64	36	25	7	11	27
SHK64B	7.863×10^{-3}	310	2	108	65	36	25	7	11	27
SHK64C	6.395×10^{-3}	320	2	112	62	37	24	9	13	28
SHK64D	2.593×10^{-2}	289	2	100	65	35	26	4	9	26
SHK64E	1.062×10^{-2}	308	2	108	64	36	25	7	11	27
SHK64F	2.348×10^{-3}	307	2	107	63	36	25	7	11	27

Table 2: Methods are compared for example 1 with $Tol = 10^{-4}$

Methods	ERROR	FCN	JCB	NIT	NSIT	NST	NSST	NCST	NFST	NFPS
SHK64A	1.130×10^{-4}	584	2	210	147	64	51	10	13	56
SHK64B	4.809×10^{-5}	605	2	217	151	67	53	11	14	59
SHK64C	4.392×10^{-5}	605	2	217	151	67	53	11	14	59
SHK64D	5.225×10^{-4}	662	2	238	155	72	55	17	17	64
SHK64E	2.113×10^{-6}	622	2	223	152	68	54	12	14	60
SHK64F	8.478×10^{-5}	584	2	210	147	64	51	10	13	56

Table 3: Methods are compared for example 1 with Tol = 10^{-6}

Methods	ERROR	FCN	JCB	NIT	NSIT	NST	NSST	NCST	NFST	NFPS
SHK64A	5.236×10^{-5}	1125	2	411	330	127	111	10	16	116
SHK64B	5.056×10^{-5}	1152	2	421	336	130	113	11	17	119
SHK64C	5.068×10^{-5}	1152	2	421	336	130	113	11	17	119
SHK64D	5.466×10^{-5}	1158	2	423	338	131	114	11	17	120
SHK64E	5.041×10^{-5}	1160	2	424	339	131	114	11	17	120
SHK64F	5.231×10^{-5}	1125	2	411	330	127	111	10	16	116

Table 4: Methods are compared for example 2 with Tol = 10^{-2}

Methods	ERROR	FCN	JCB	NIT	NSIT	NST	NSST	NCST	NFST	NFAC	NFPS
SHK64A	4.263×10^{-2}	3659	7	1322	1186	486	452	31	34	43	477
SHK64B	4.531×10^{-2}	3439	5	1247	1162	457	436	14	21	23	447
SHK64C	4.534×10^{-2}	3437	5	1246	1161	457	436	14	21	23	447
SHK64D	2.407×10^{-2}	447	6	163	131	51	43	0	8	9	42
SHK64E	4.632×10^{-2}	588	8	218	172	73	58	3	15	18	56
SHK64F	2.830×10^{-1}	3612	7	1295	1171	491	460	27	31	39	482

Table 5: Methods are compared for example 2 with Tol = 10^{-4}

Methods	ERROR	FCN	JCB	NIT	NSIT	NST	NSST	NCST	NFST	NFAC	NFPS
SHK64A	3.148×10^{-4}	3561	8	1279	1116	462	427	30	35	42	448
SHK64B	9.400×10^{-5}	760	2	275	224	87	76	0	11	6	73
SHK64C	9.400×10^{-5}	760	2	275	224	87	76	0	11	6	73
SHK64D	2.431×10^{-3}	2555	4	934	801	312	282	23	30	32	298
SHK64E	9.400×10^{-5}	760	2	275	224	87	76	0	11	6	73
SHK64F	1.320×10^{-3}	3376	8	1226	1075	424	392	27	32	39	410

Table 6: Methods are compared for example 2 with Tol = 10^{-6}

Methods	ERROR	FCN	JCB	NIT	NSIT	NST	NSST	NCST	NFST	NFAC	NFPS
SHK64A	1.120×10^{-5}	2578	3	917	797	341	314	13	27	22	321
SHK64B	1.522×10^{-5}	2246	3	809	688	283	256	12	27	21	263
SHK64C	1.522×10^{-5}	2246	3	809	688	283	256	12	27	21	263
SHK64D	1.684×10^{-5}	2915	3	1044	896	379	345	23	34	32	359
SHK64E	1.525×10^{-5}	2303	3	828	695	291	261	16	30	25	271
SHK64F	1.120×10^{-5}	2578	3	917	797	341	314	13	27	22	321

factor ρ_1 or increase by more than a factor ρ_3. These restrictions help to avoid large fluctuation in the step size caused by local changes in the error estimate. Also, we do not increase the step size at all unless it can be increased by a factor of at least ρ_2, where $\rho_2<\rho_3$. This restriction is designed to avoid the extra function and Jacobian evaluation involved in changing the step size unless a worthwhile increase is predicted. Following Thomas (1987), in our tests we take $\rho_1 = 0.1$, $\rho_2 = 2$ and $\rho_3 = 10$.

CONCLUSIONS

In the study, we have derived some formula pairs, consisting of a fourth order and a sixth order direct hybrid method. The two methods in each pair have been chosen to have some features in common, so that the computational cost of using the formula pair is reduced. The formula pairs provide an estimate of the local error and this allows the stepsize to be varied so that the size of the local error is controlled.

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