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C-Fusion Frame

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Abstract: In this study, we shall generalized the concept of fusion frame, namely, c-fusion frames, which is continuous version of the fusion frames. We give characterization of c-fusion frames and show that many basic properties can be derived within this general context.

Key words: Operator, Hilbert space, bessel, frame, fusion frame

INTRODUCTION

Throughout this study H will be a Hilbert space and $\hat{H}$ will be the collection of all closed subspace of H, respectively. Also, (X, μ) will be a measure space and $v:X\to[0,\infty)$ a measurable mapping such that $v\neq 0$ a.e. We shall denote the unit closed ball of H by H_1.

Frames was first introduced at (Duffin and Schaeffer, 1952) in the context of nonharmonik Fourier series. Outside of signal processing, frames did not seem to generate much interest until the ground breaking work of Daubechies *et al*. (1986). Since then the theory of frames began to be more widely studied. During the last 20 years the theory of frames has been growing rapidly, several new applications have been developed. For example, besides traditional application as signal processing, image processing, data compression and sampling theory, frames are now used to mitigate the effect of losses in pocket- based communication systems and hence to improve the robustness of data transmission (Casazza and Kovacevic, 2003) and to design high-rate constellation with full diversity in multiple-antenna code design (Hassibi *et al*., 2001). In Bolcskel *et al*. (1998), Benedetto *et al*. (2004) and Candes and Donoho (2004) some applications have been developed.

The fusion frames were considered by Casazza *et al*. (2000) in connection with distributed processing and are related to the construction of global frames. The fusion frame theory is in fact more delicate due to complicated relations between the structure of the sequence of weighted subspaces and the local frames in the subspaces and due to the extreme sensitivity with respect to changes of the weights.

In this study, we shall extend the fusion frames to their continuous versions in measure spaces.

PRELIMINARIES AND METHODS

This topics can be found by Christensen (2002).

Definition 1: Let $\{f_i\}_{i\in I}$ be sequence of members of H. We say that $\{f_i\}_{i\in I}$ is a frame for H if there exist $0<A\leq B<\infty$ such that for all $h\in H$,

$$A\|h\|^2\leq\sum_{i\in I}|<f_i,h>|^2\leq B\|h\|^2$$

The constants A and B are called frame bounds. If A, B can be chosen so that A = B, we call this frame an A−tight frame and if A= B = 1 it is called a parseval frame. If we only have the upper bound, we call $\{f_i\}_{i\in I}$ a Bessel sequence. If $\{f_i\}_{i\in I}$ is a Bessel sequence then the following operators are bounded,

$$T:l^2(I)\to H,\ T(c_i)=\sum_{i\in I}c_if_i$$
(synthesis operator)
$$T^*:H\to l^2(I),\ T^*f=\{<f,f_i>\}_{i\in I},$$
(analysis operator)
$$S:H\to H,$$
$$Sf=TT^*f=\sum_{i\in I}<f,f_i>f_i$$
(frame operator)

Definition 2: For a countable index set I, let $\{W_i\}_{i\in I}$ be a family of closed subspace in H and let $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i>0$ for all $i\in I$. Then $\{(W_i,v_i)\}_{i\in I}$ is a fusion frame for H if there exist $0<C\leq D<\infty$ such that for all $h\in H$:

$$C\|f\|^2\leq\sum_{i\in I}v_i^2\|\pi_{W_i}(f)\|^2\leq D\|f\|^2,$$

where, π_{W_i} is the orthogonal projection onto the subspace W_i. We call C and D the fusion frame bounds.

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The family $\{(W_i,v_i)\}_{i\in I}$ is called a C−tight fusion frame, if in above inequality the constants C and D can be chosen so that C = D, a parseval fusion frame provided C = D = 1 and an orthonormal fusion basis if $H = \oplus_{i\in I} W_i$. If $\{(W_i,v_i)\}_{i\in I}$ possesses an upper fusion frame bound, but not necessarily a lower bound, we call it is a Bessel fusion sequence with Bessel fusion bound D.

The theory of frames has a continuous version as follows: Let (X, μ) be a measure space. Let f: X→H be weakly measurable (i.e., for all h ∈ H, the mapping $x \to < f(x), h >$ is measurable). Then f is called a continuous frame for H if there exist $0 < A \le B < \infty$ such that, for all h ∈ H,

$$A\|h\|^2 \le \int_X |< f(x),h >|^2 d\mu \le B\|h\|^2$$

The following lemmas can be found in operator theory text books (Pedersen and Gert, 1989; Rudin, 1973, 1986; Sakai, 1998) which we shall use then in the text.

Lemma 1: Let u : H→K be a bounded operator. Then:

- $\|u\| = \|u^*\|$ and $\|uu^*\| = \|u\|^2$.
- R_u is closed, if and only if, R_u is closed.
- u is subjective, if and only if, there exists c > 0 such that for each h ∈ H
$$c\|h\| \le \|u^*(h)\|$$

Lemma 2: Let u be a self-adjoint bounded operator on H. Let

$$m_u = \inf_{h\in H_1} < uh,h >$$

and

$$M_u = \sup_{h\in H_1} < uh,h >$$

Then, m_u, $M_u \in \sigma(u)$.

Theorem 1: Let u: K→H be a bounded operator with closed range R_u. Then there exists a bounded operator $u^\dagger$: H→K for which $uu^\dagger f = f$, $F \in R_u$.

Also, u^*: H → K has closed range and $(u^*)^\dagger = (u^\dagger)^*$. The operator $u^\dagger$ is called the pseudo-inverse of u.

Theorem 2: Let u: K→H be a bounded surjective operator. Given y ∈ H, the equation ux = y has a unique solution of minimal norm, namely, $x = u^\dagger y$.

Now we introduce the concept of c-fusion frame and shall show some its properties.

Definition 3: Let F: X→Ĥ be such that for each h ∈ H, the mapping x a $\pi_{F(X)}(h)$ is measurable (i.e., F is weakly measurable). We say that (F, v) is a c-fusion frame for H if there exist $0 < A \le B < \infty$ such that for all h ∈ H,

$$A\|h\|^2 \le \int_X v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu \le B\|h\|^2$$

(F, v) is called a tight c-fusion frame for H if A, B can be chosen so that A = B and parseval if A = B = 1. If just the right hand inequality satisfies then we say that (F, v) is a Bessel c-fusion mapping for H.

Definition 4: Let F: X→Ĥ. Let $L^2(X, H, F)$ be the class of all measurable mapping f: X→H. such that for each x ∈ X and f(x) ∈ F(x) and $\int_X \|f(x)\|^2 d\mu < \infty$.

It can be verified that $L^2(X, H, F)$ is a Hilbert space with inner product defined by:

$$< f,g >= \int_X < f(x),g(x) > d\mu,\ f,g \in L^2(X,H,F)$$

Remark 1: For brevity, we shall denote $L^2(X, H, F)$ by $L^2(X, F)$. Let (F, v) be a Bessel c-fusion mapping, $f \in L^2(X, F)$ and h ∈ H. Then:

$$\begin{aligned}
&\left|\int_X v(x) < f(x),h > d\mu\right| \\
&= \left|\int_X v(x) < \pi_{F(x)}(f(x)),h > d\mu\right| \\
&= \left|\int_X v(x) < f(x),\pi_{F(x)}(h) > d\mu\right| \\
&\le \int_X v(x)\|f(x)\|\cdot\|\pi_{F(x)}(h)\| d\mu \\
&\le \left(\int_X \|f(x)\|^2 d\mu\right)^{1/2} \\
&\times \left(\int_X v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu\right)^{1/2} \\
&\le B\|h\|\left(\int_X \|f(x)\|^2 d\mu\right)^{1/2}
\end{aligned}$$

So we may define:

Definition 5: Let (F, v) be a Bessel c-fusion mapping for H. We define the c-fusion pre-frame operator T_F: $L^2(X, F)$ by

$$\begin{aligned}
&< T_F(f),h > \\
&= \int_X v(x) < f(x),h > d\mu, \\
&f \in L^2(X,F), h \in H.
\end{aligned}$$

By the remark (5),

$$T_F: L^2(X, F) \to H$$

is a bounded linear mapping. Its adjoint T_F^*: H→$L^2(X, F)$ will be called c-fusion analysis operator and $S_F = T_F \circ T_F^*$ will be called c-fusion frame operator.

Remark 2: Let (F, v) be a Bessel c-fusion mapping for H. Then T_F: $L^2(X, F)$→H is indeed a vector-valued integral, which we shall denote by:

$$T_F(f) = \int_X vf d\mu,\ f \in L^2(X,F)$$

Where:

$$< \int_X v f d\mu, h > = \int_X v(x) < f(x), h > d\mu,$$

For each $h \in H$ and $f \in L^2(X, F)$ we have:

$$\begin{aligned} &< T_F^*(h), f >=< h, T_F(f) > \\ &= \int_X v(x) < h, f(x) > d\mu \\ &= \int_X v(x) < \pi_{F(x)}(h), f(x) > d\mu \\ &=< v\pi_F(h), f > . \end{aligned}$$

Hence for each $h \in H$, $T_F^*(h) = v\pi_F(h)$.
So $T_F^* = v\pi_F$.

Therefore, S_F: $H \to H$ is also a vector-valued integral which for each $h \in H$, we have

$$\begin{aligned} &S_F(h) = T_F T_F^*(h) \\ &= T_F(v\pi_F(h)) = \int_X v^2 \pi_F(h) d\mu. \end{aligned}$$

Definition 6: Let (F, v) and (G, v) are Bessel c-fusion mapping for H. We say (F, v) and (G, v) are weakly equal if $T_F^* = TG^*$, which is equivalent with

$$v\pi_F(h) = v\pi G(h), \text{ a.e.}$$

for all $h \in H$ Since, $v \neq 0$ a.e,. (F, v) and (G, v) are weakly equal if

$$\pi_F(h) = \pi_G(h), \quad \text{a.e.}$$

for all $h \in H$.

Remark 3: Let $T_F = 0$. Now, Let O: $X \to \hat{H}$ be defined by:

$$O(x) = \{0\},$$

for almost all $x \in X$. Then (O, v) is a Bessel c-fusion mapping and $T_O = 0$. Let $h \in H$. Since, $v\pi_F(h) \in L^2(X, F)$, so

$$\begin{aligned} &\int_X v(x)^2 < \pi_{F(x)}(h), \pi_{F(x)}(h) > d\mu \\ &= \int_X v(x) < v(x)\pi_{F(x)}(h), h > d\mu \\ &=< T_F(v\pi_F(h)), h >= 0. \end{aligned}$$

Thus,

$$\pi_{F(x)}(h) = 0, \quad \text{a.e.}$$

Therefore,

$$\pi_F(h) = \pi_O(h), \quad \text{a.e.}$$

Hence, (F, v) and (G, v) are weakly equal.

RESULTS AND DISCUSSION

Definition 7: For each Bessel c-fusion mapping F for H, we shall denote

$$\begin{aligned} &A_{F,v} = \inf_{h \in H_1} \| v\pi_F(h) \|^2, \\ &B_{F,v} = \sup_{h \in H_1} \| v\pi_F(h) \|^2 = \| v\pi_F \|^2 . \end{aligned}$$

Remark 4: Let F is a Bessel c-fusion mapping for H. Since, for each $h \in H$.

$$< T_F T_F^*(h), h > = \|v\pi_F(h)\|^2,$$

$A_{F,y}$ and $B_{F,y}$ are optimal scalars which satisfy

$$A_{F,y} \le T_F T_F^*(h) \le B_{F,y}$$

So (F, v) is a c-fusion frame for H if and only if $A_{F,y}>0$.

Lemma 3: Let (F, v) is a Bessel c-fusion mapping for H. Then F is c-fusion frame for H if and only if T_F is surjective.

Proof: Let $A_{F,y}>0$ Since, for each $h \in H$

$$\begin{aligned} &< T_F T_F^*(h), h > \\ &= \int_X v^2(x) \| \pi_{F(x)}(h) \|^2 d\mu \\ &= \| v\pi_F(h) \|^2 \ge A_{F,v} \| h \|^2 . \end{aligned}$$

Therefore,

$$T_F: L^2(X, F) \to H$$

is surjective.

Now let T_F be surjective. Let

$$T_F^\dagger: H \to L^2(X, F)$$

be its pseudo-inverse. Since, for each $h \in H$

$$\begin{aligned} \| h \| &= \| T_F^{\dagger *} T_F^*(h) \| \le \| T_F^{\dagger *} \| \| T_F^*(h) \| \\ &= \| T_F^{\dagger *} \| \| v\pi_F(h) \|, \end{aligned}$$

so

$$A_{F,v} \ge \| T_F^{\dagger *} \|^{-2} > 0.$$

Theorem 3: Let (F, v) be a Bessel c-fusion mapping for H, and K be a Hilbert space. Let u: $H \to K$ be a bounded bijective operator and (u oF, v) is a Bessel c-fusion mapping for K. Then:

(i) $u\, o L^2(X,H,F) = L^2(X,K,u oF)$.
(ii) For each $f \in L^2(X,F)$
$u\, o\int_X vfd\mu = \int_X vu\, ofd\mu$
(iii) F is a c-fusion frame for H if and only if (u oF, v) is a c-fusion frame for K.

Proof:
(i) It is straightforward.
(ii) For each k ∈ K, we have

$$\begin{aligned}&< u(\int_X vfd\mu), k > \\ &= < T_F(f), u^*(k) > \\ &= \int_X v(x) < f(x), u^*(k) > d\mu \\ &= \int_X v(x) < u(f(x)), k > d\mu \\ &= < \int_X vu\, ofd\mu, k >\end{aligned}$$

Hence

$$u\int_X vfd\mu = \int_X vu \circ fd\mu$$

(iii) It is clear from (ii) and Lemma 3.

Lemma 4: Let (F, v) be a Bessel c-fusion mapping for H. Then the frame operator $S_F = T_F T_F^*$ is invertible if and only if F is a c-fusion frame for H.

Proof: Let $S_F = T_F T_F^*$ be invertible. We have

$$A_{F,v} \le \inf_{h \in H_1} \| T_F^* \|^2 = \inf_{h \in H_1} < T_F T_F^*(h), h > \in \sigma(T_F T_F^*),$$

so, $A_{F,v}>0$. Now let $A_{F,v}>0$. So, by the Lemma 3, T_F is surjective. Then there exist A>0 such that

$$A \| h \| \le \| T_F^*(h) \|, h \in H.$$

Hence

$$A_{F,v} \ge A^2 > 0.$$

Theorem 4: Let $\{H_i\}_{i \in I}$ be a collection of Hilbert space and $H = \oplus_i H_i$. Let (F, v) be a Bessel c-fusion mapping for H such that for each i ∈ I there exist at most one x ∈ X such that $F(x) \subseteq H_i$. Let each finite subset of X be measurable. Then, for each h ∈ H

$$h = \sum_{x \in X} \pi_{F(x)}(h).$$

Proof: Let

$$K = \left\{ h \in H : h = \sum_{x \in X} \pi_{F(x)}(h) \right\}.$$

Let $\{f_n\}$ be a sequence of members of K which tends to f ∈ H. Given ε>0, we can find N>0 such that $\|f_N - f\| < \varepsilon$ There exists a finite Z⊆X such that for each finite Z⊆Y⊆Y,

$$\| f_N - \sum_{x \in Y} \pi_{F(x)} f_N \| < \varepsilon.$$

We have

$$\begin{aligned}&\| f - \sum_{x \in Y} \pi_{F(x)} f \| \\ &\le \| f - f_N \| + \| f_N - \sum_{x \in Y} \pi_{F(x)}(f_N) \| \\ &+ \| \sum_{x \in Y} \pi_{F(x)}(f) - \sum_{x \in Y} \pi_{F(x)}(f_N) \|.\end{aligned}$$

But

$$\begin{aligned}&A_{F,v} \| \sum_{x \in Y} \pi_{F(x)}(f) - \sum_{x \in Y} \pi_{F(x)}(f_N) \|^2 \\ &= A_{F,v} \| \sum_{x \in Y} \pi_{F(x)}(f_N - f) \|^2 \\ &\le \int_X v^2(t) \| \pi_{F(t)} \sum_{x \in Y} \pi_{F(x)}(f_N - f) \|^2 d\mu \\ &= \int_Y v^2(t) \| \pi_{F(t)} \sum_{x \in Y} \pi_{F(x)}(f_N - f) \|^2 d\mu \\ &= \int_Y v^2(t) \| \pi_{F(t)} \sum_{x \in Y} \pi_{F(x)}(f_N - f) \|^2 d\mu \\ &= \int_Y v^2(t) \| \pi_{F(t)}(f_N - f) \|^2 d\mu \\ &\le B_{F,v} \| f_N - f \|^2.\end{aligned}$$

So, K is a closed subspace of H. Now, let $h \in K^\perp$ Since, for each t ∈ X

$$\pi_{F(t)}(h) = \sum_{x \in X} \pi_{F(x)} \pi_{F(t)}(h),$$

$\pi_{F(t)}(h) \in K$. Since
and $A_{F,v}>0$. H = 0.

Theorem 5: Let (X, μ) and (Y, λ) be two σ−finite measure space and let f: X×Y→H, F: X→Ĥ be weakly measurable mappings. Let for each x ∈ X, f(x,..): Y→F(x) be measurable and for every x ∈ F(x), f(x,.) is a continuous frame for H. Let

$$\begin{aligned}&0 < A(x) \\ &= \inf_{x \in F(x)_1} \int_Y |< f(x,y), h >|^2 d\lambda \\ &\le \sup_{x \in F(x)_1} \int_Y |< f(x,y), h >|^2 d\lambda \\ &= B(x) < \infty\end{aligned}$$

and let

$$\begin{aligned}&0 < A = \inf_x A(x) \\ &\le \sup_x B(x) = B < \infty\end{aligned}$$

Then, (F, v) is a c-fusion frame for H if and only if

$$v.f : X\times Y \to H,$$
$$(x,y) \mapsto v(x)f(x,y)$$

is a continuous frame for H.

Proof: For each $h \in H$ we have

$$\begin{aligned} & A\| v\pi_F(h)\|^2 \\ &= A\int_X v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu \\ &\le \int_X A(x)v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu \\ &\le \int_X\int_Y |< v(x)\pi_{F(x)}(h), f(x,y)>|^2 d\lambda d\mu \\ &= \int_X\int_Y |< h, v(x)f(x,y)>|^2 d\lambda d\mu \\ &= \int_{X\times Y} |< v(x)f(x,y), h>|^2 d(\mu\times\lambda) \\ &= \int_X\int_Y |< \pi_{F(x)}(h), v(x)f(x,y)>|^2 d\lambda d\mu \\ &\le \int_X B(x)v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu \\ &\le B\int_X v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu, \end{aligned}$$

and the theorem is proved.

Theorem 6: Let (X, μ) be a $\sigma-$finite measure space and K be a Hilbert space. Let u: $H \to K$ be a bijective linear operator. Let F: $X \to \hat{H}$ and $u\, o F : X \to \hat{K}$ be weakly measurable. Then, (F, v) is a c-fusion frame for H if and only if (u oF, v) is a c-fusion frame for K.

Proof: Let F be a c-fusion frame for H. Let (Y, λ) be a σ-finite measure space and let

$$f: X\times Y \to H$$

be such that for each

$$x \in X,\ f(x,.) : Y \to F(x)$$

with

$$\begin{aligned} & 0 < A(x) \\ &= \inf_{x\in F(x)_1} \int_Y |< f(x,y), h>|^2 d\lambda \\ &\le \sup_{x\in F(x)_1} \int_Y |< f(x,y), h>|^2 d\lambda \\ &= B(x) < \infty, \end{aligned}$$

measurable and $0 < A = \inf_x A(x) \le \sup_x B(x) = B < \infty$. Choosing such mapping is always possible, because let $\{e_i^x\}_{i\in I_x}$ be an orthonormal basis for F(x). We can suppose that $\{I_x\}_{x\in X}$ is pairwise disjoint (we can consider $\{x\}\times I$). Let $Y = \bigcup_{x\in X} I_x$ and λ be the counting measure on Y. Then we can define f: $X\times Y \to H$ by

$$f(x,i) = e_i^x \text{ if } i \in I_x$$

and

f(x, i) = 0 otherwise

Then, for each $x \in X$

A(x) = B(x) = 1

By the Theorem 3

$$\begin{aligned} & 0 < \\ & \inf_{x\in H_1}\int_{X\times Y} |< v(x)f(x,y), h>|^2 d(\mu\times\lambda) \\ &\le \sup_{x\in H_1}\int_{X\times Y} |< v(x)f(x,y), h>|^2 d(\mu\times\lambda) \\ & < \infty \end{aligned}$$

Then, $u\, oF : X\times Y \to \hat{K}$ and for each $x \in X$,

$$u \circ f(x,.) : Y \to u(F(x))$$

Since, u is surjective, there is $C > 0$ such that

$$\begin{aligned} & (C^2\|h\|^2) \\ & \times(\int_Y |< f(x,y), u^*(h)/\|u^*(h)\|>|^2 d\lambda) \\ &\le (\int_Y |< f(x,y), u^*(h)/\|u^*(h)\|>|^2 d\lambda) \\ & \times(\|u^*(h)\|^2) \\ &= \int_Y |< u(f(x,y), h>|^2 d\lambda \\ &\le (\|u\|^2\|h\|^2) \\ & (\int_Y |< f(x,y), u^*(h)/\|u^*(h)\|>|^2 d\lambda) \end{aligned}$$

So,

$$\begin{aligned} & C^2 A(x) \\ &\le \inf_{h\in H_1}\int_Y |< u(f(x,y), h>|^2 d\lambda \\ &\le \sup_{h\in H_1}\int_Y |< u(f(x,y), h>|^2 d\lambda \\ &\le \|u\|^2 B(x) \end{aligned}$$

Similarly, we have

$$\begin{aligned} & C^2 \inf_{h\in H_1}\int_{X\times Y} |< v(x)f(x,y), h>|^2 d(\mu\times\lambda) \\ &\le \inf_{h\in H_1}\int_{X\times Y} |< v(x)u(f(x,y)), h>|^2 d(\mu\times\lambda) \\ &\le \sup_{h\in H_1}\int_{X\times Y} |< v(x)u(f(x,y)), h>|^2 d(\mu\times\lambda) \\ &\le (\|u\|^2) \end{aligned}$$

Therefore by the Theorem 3 be a c-fusion frame for (u oF, v). The proof of the converse is similar.

Theorem 7: Let (F, v) be a c-fusion frame for H. Let $h \in H$ and $SF = T_F T_F^*$. Then:

(i) We have the following retrieval formulas

$$h = T_{S_F^{-1}\circ F}(S_F^{-1} v\pi_F(h))$$

and

$$h = T_F(v\pi_F(S_F^{-1}(h)))$$

(ii) In the retrieval formula

$$h = T_F(v\pi_F(S_F^{-1}(h))),$$

$v\pi_F(S_F^{-1}(h))$ has least norm among all of the retrieval formulas.

(iii) For each $h \in H$,

$$T_F^{\dagger}(h) = v\pi_F(S_F^{-1}(h))$$

Proof:

(i) Since (F, v) is a c-fusion frame, S_F is an invertible operator. By the Theorem 4, we have

$$\begin{aligned} h &= S_F^{-1}S_F(h) = S_F^{-1}T_F(v\pi_F(h)) \\ &= S_F^{-1}\int_X v^2\pi_F(h)d\mu \\ &= \int_X vS_F^{-1}\circ v\pi_F(h)d\mu \\ &= T_{S_F^{-1}oF}(S_F^{-1}\circ v\pi_F(h)) \end{aligned}$$

Also, we have

$$h = S_F S_F^{-1}(h) = T_F(v\pi_F(S_F^{-1}(h))).$$

(ii) Let $f \in L^2(X, F)$ and

$$h = T_F(f)$$

Thus, for each $k \in H$ we have

$$\begin{aligned} &< h,k >=< T_F(f),k > \\ &= \int_X v(x) < f(x),k > d\mu, \\ &< h,k >=< T_F(v\pi_F(S_F^{-1}(h))),k > \\ &= \int_X v(x) < v(x)\pi_{F(x)}(S_F^{-1}(h)),k > d\mu. \end{aligned}$$

Therefore

$$< T_F(v\pi_F(S_F^{-1}(h)) - f),k > 0$$

So,

$$T_F(v\pi_F(S_F^{-1}(h))) = 0$$

Hence

$$v\pi_F(S_F^{-1}(h)) - f \in \ker T_F$$

Since, F is a c-fusion frame,

$$v\pi_F(S_F^{-1}(h)) \in \operatorname{ran} T_F^{*}$$

But,

$$L^2(X,F) = (\ker T_F) \oplus (\operatorname{ran} T_F^{*})$$

So,

$$\begin{aligned} \| f \|^2 &= \| v\pi_F(S_F^{-1}(h)) - f \|^2 \\ &+ \| v\pi_F(S_F^{-1}(h)) \|^2, \end{aligned}$$

and (ii) is proved.

(iii) Let $f \in L^2(X, F)$. Since, $T_F^{\dagger}$ is the unique solution of minimal norm of $T_F(f) = h$ so by (ii),

$$\int_X | f - v\pi_F(S_F^{-1}(h)) |^2 \, d\mu = 0$$

Therefore,

$$f = v\pi_F(S_F^{-1}(h)) = T_F^{\dagger}(h)$$

Theorem 8: Let (F, v) and (G, v) be Bessel c-fusion mapping for H. Then the following assertions are equivalent:

(i) For each $h \in H$,

$$h = \int_X v^2\pi_G\pi_F(h)d\mu$$

(ii) For each $h \in H$,

$$h = \int_X v^2\pi_F\pi_G(h)d\mu$$

(iii) For each $h, k \in H$,

$$< h,k >= \int_X v^2 < \pi_G(h),\pi_F(k) > d\mu.$$

(iv) For each $h \in H$,

$$\| h \|^2 = \int_X v^2 < \pi_G(h),\pi_F(h) > d\mu$$

(v) For each orthonormal bases

$$\{e_i\}_{i\in I} \text{ and } \{\lambda_j\}_{j\in J}$$

for H we have

$$\begin{aligned}&<e_i,\gamma_j>\\&=\int_X v^2<\pi_F(e_i),\pi_G(\gamma_j)>d\mu,\\&i\in I, j\in J\end{aligned}$$

(iv) For each orthonormal bases $\{e_i\}_{i\in I}$ for H and $i \in I$,

$$\int_X v^2 <\pi_F(e_i),\pi_G(\gamma_j)>d\mu = 1$$

Proof: (i) → (ii) Let h, k ∈ H. We have

$$\begin{aligned}&<h,k>=<T_F(v\pi_F\pi_G(h)),k>\\&=\int_X v<v\pi_F\pi_G(h),k>d\mu\\&=\int_X v<h,v\pi_G\pi_F(k)>d\mu\\&=<h,T_G(v\pi_G\pi_F(k))>\end{aligned}$$

Hence, $k = T_G(v\pi_G\pi_F(k))$
(ii) → (iii) It is evident by the proof of (i) → (ii).
(iii) → (i) For each h, k ∈ H, we have

$$\begin{aligned}<h,k>&=\int_X v^2<\pi_G(h),\pi_F(k)>d\mu\\&=<T_F(v\pi_F\pi_G(h)),k>\end{aligned}$$

Thus $h = T_F(v\pi_F\pi_G(h))$
(iv) → (i) Let L: H → H be defined by

$$L(h) = T_F(v\pi_F\pi_G(h))$$

It clear that L is linear. Since

$$\begin{aligned}&\|L(h)\|=\sup_{k\in H_1}|<L(h),k>|\\&=\sup_{k\in H_1}\left|\int_X v^2<\pi_F\pi_G(h),k>d\mu\right|\\&\le (v^2\int_X\|\pi_G(h)\|^2 d\mu)^{1/2}\\&\times(\sup_{k\in H_1}(v^2\int_X\|\pi_F(k)\|^2 d\mu)^{1/2})\\&\le(\sup_{k\in H_1}(v^2\int_X\|\pi_G(h)\|^2 d\mu)^{1/2})\\&(\sup_{k\in H_1}(v^2\int_X\|\pi_F(k)\|^2 d\mu)^{1/2}\|h\|)\\&\le B_{F,v}^{1/2}B_{G,v}^{1/2}\|h\|,\end{aligned}$$

that, L ∈ B(H). For each h ∈ H, we have

$$\begin{aligned}&<h,h>=\|h\|^2\\&=\int_X v^2<\pi_G(h),\pi_F(k)>d\mu\\&=<T_F(v\pi_F\pi_G(h)),h>\end{aligned}$$

Hence, for each h ∈ H,

$$h = T_F(v\pi_F\pi_G(h))$$

(iii) → (iv) is evident.
(v) → (iii) We have

$$\begin{aligned}&\int_X v^2<\pi_F(h),\pi_G(k)>d\mu\\&=<v\pi_F(h),v\pi_G(k)>\\&=<v\pi_F(\sum_i<h,e_i>e_i),v\pi_G(\sum_j<k,\gamma_j>\gamma_j)>\\&=\sum_{i,j}<<h,e_i>v\pi_F(e_i),<k,\gamma_j>v\pi_G(\gamma_j)>\\&=\sum_{i,j}<h,e_i><\gamma_j,k><v\pi_F(e_i),v\pi_G(\gamma_j)>\\&=\sum_{i,j}<h,e_i><\gamma_j,k><e_i,\gamma_j>\\&=<h,k>\end{aligned}$$

(vi) → (v) it is similar with the proof of (v) → (iii).

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