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Analytical Solution of Time-Dependent Non-Linear Partial Differential Equations Using HAM, HPM and VIM

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Abstract: This study deals with analytical solution of time-dependent partial differential equations. The analyses are carried out by the means of Homotopy Analysis Method (HAM), Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM). The results have been compared and depicted graphically. It is shown that the presented approaches are very effective, straightforward and capable to the analytical solutions of the large classes of linear or nonlinear time-dependent partial differential equations while this set of problems is widely spread in Engineering.

Key words: Time-dependent partial differential equations, Variational Iteration Method (VIM), Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM)

INTRODUCTION

One of the most frequent problems in the physical sciences is to obtain the time solution of a (linear or nonlinear) partial differential equation which satisfies a set of boundary conditions on a rectangular boundary. For instance, let us consider the following problem (García-Olivares, 2002a):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - R\, u\frac{\partial u}{\partial x} + f(x, y) \tag{1}$$

With the following boundary conditions defined on a rectangle (García-Olivares, 2002a):

$$u(x, y, t = 0) = e(x, y) \tag{2}$$

$$u(x = 0, y, t) = f_1(y, t) \tag{3}$$

$$u(x = x_l, y, t) = f_2(y, t) \tag{4}$$

$$u(x, y = -y_l, t) = g_1(x, t) \tag{5}$$

$$u(x, y = y_l, t) = g_2(x, t) \tag{6}$$

For instance, if have R = 1 and f (x, y) = 0 in Eq. 1 subject to the following initial and boundary conditions (García-Olivares, 2002a):

$$e(x, y) = \frac{(y_1^2 - y^2)(x_1 - x)}{y_1^2 x_1} \tag{7}$$

$$f_1(y, t) = \frac{y_1^2 - y^2}{y_1^2} \tag{8}$$

$$f_2(y,t) = 0 \tag{9}$$

$$g_1(x,t) = 0 \tag{10}$$

$$g_2(x,t) = 0 \tag{11}$$

This kind of Partial Differential Equations (PDE) appears frequently coupled with others. For example, in the incompressible fluid flow problem, the equation above would be slightly completed to become the first component of the Navier-Stokes equations which should be solved in parallel with a poisson equation for the pressure.

Some methods to obtain an analytical solution of PDE with boundary conditions by means of power series have been explored in Garcia-Olivares (2002b) and Mahmodi *et al*. (2008). Those works are based on the method proposed by George Adomian called, decomposition method (Adomian, 1998) that uses analytic functions to approximate the problem solution.

In order to develop these efforts, He's variational iteration method and homotopy perturbation method

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(He, 2000, 2004; Abdou and Soliman, 2005; Ganji and Rafei, 2006; Tolou *et al*., 2008) also, homotopy analysis method (Khatami *et al*., 2008) have been used to conduct an analytical investigation on the solution of time-dependent partial differential equations. In order to assess benefits of the methods, firstly, fundamentals of the proposed method have been presented and some illustrating examples have been used. Afterwards, the results obtained by aforementioned methods have been shown and compared graphically. Finally, conclude with some discussion.

MATERIALS AND METHODS

Variational Iteration Method (VIM)

Fundamentals: To illustrate the basic concepts of variational iteration method, consider the following deferential:

$$Lu+Nu = g\,(x) \tag{12}$$

where, L is a linear operator, N a nonlinear operator and g (x) a heterogeneous term. According to VIM, can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\{L\,u_n(t) + N\,\tilde{u}_n(t) - g(t)\}dt \tag{13}$$

where, λ is a general Lagrangian multiplier (He, 1998a, 2005), which can be identified, optimally via the variational theory (He, 1998b), the subscript n indicates the nth order approximation, $\tilde{u}_n$ which is considered as a restricted variation, i.e., $\delta\tilde{u}_n = 0$.

Application: Considering time-dependent partial differential equations as (García-Olivares, 2002a):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - R\,u\frac{\partial u}{\partial x} + f(x,y) \tag{14}$$

Subject to the following initial condition (García-Olivares, 2002a):

$$u(x,y,t=0) = \frac{(y_1^2 - y^2)(x_1 - x)}{y_1^2 x_1} \tag{15}$$

Solve Eq. 13 and 14 using VIM, have the correction functional as:

$$u_{n+1}(x,y,t) = u_n(x,y,t) + \lambda\int_0^t \Big(\frac{-\partial u_n(x,y,\tau)}{\partial\tau} + \frac{\partial^2 u_n(x,y,\tau)}{\partial x^2} + \frac{\partial^2 u_n(x,y,\tau)}{\partial y^2} - Ru_n(x,y,\tau)\Big(\frac{\partial u_n(x,y,\tau)}{\partial x}\Big) + f(x,y)\Big)d\tau \tag{16}$$

where, $u_n(x,y,\tau)(\partial u_n(x,y,\tau)/\partial x)$ indicates the restricted variations; i.e., $\delta(u_n(x,y,\tau)(\partial u_n(x,y,\tau)/\partial x)) = 0$

Making the above correction functional stationary, obtain the following stationary conditions:

$$1+\lambda|_{\tau=t} = 0 \tag{17}$$

$$\lambda' = -1 \tag{18}$$

The Lagrangian multiplier can therefore be identified as:

$$\lambda = -1 \tag{19}$$

Substituting Eq. 19 into the correction functional equation system (16) results in the following iteration formula:

$$u_{n+1}(x,y,t) = u_n(x,y,t) - \int_0^t \Big(\frac{-\partial u_n(x,y,\tau)}{\partial\tau} + \frac{\partial^2 u_n(x,y,\tau)}{\partial x^2} + \frac{\partial^2 u_n(x,y,\tau)}{\partial y^2} - Ru_n(x,y,\tau)\Big(\frac{\partial u_n(x,y,\tau)}{\partial x}\Big) + f(x,y)\Big)d\tau \tag{20}$$

Each result obtained from Eq. (20) is u (x, y, t) with its own error relative to the exact solution, but higher number iterations leads to better approximation, even to the exact solution. Using the iteration formula (20) and the initial condition as u_0, two iterations were made as follows:

The first iteration results in:

$$u_1(x,y,t) = \frac{(y_1^2 - y^2)(x_1 - x)}{y_1^2 x_1} - \frac{2\,(x_1 - x)t}{y_1^2 x_1} + \frac{R\,(y_1^2 - y^2)^2(x_1 - x)\,t}{y_1^4 x_1^2} \tag{21}$$

The second iteration results in:

$$\begin{aligned}u_2(x,y,t) = {} & \frac{(y_1^2 - y^2)(x_1 - x)}{y_1^2 x_1} - \frac{2(x_1 - x)t}{y_1^2 x_1} + \frac{R\,(y_1^2 - y^2)^2(x_1 - x)t}{y_1^4 x_1^2} \\ & - \frac{1}{3}R\Big(\frac{-2(x_1 - x)}{y_1^2 x_1} + \frac{R\,(y_1^2 - y^2)^2\,(x_1 - x)}{y_1^4 x_1^2}\Big)\Big(\frac{2}{y_1^2 x_1} - \frac{R\,(y_1^2 - y^2)^2}{y_1^4 x_1^2}\Big)t^3 \\ & + \frac{1}{2}\Big(\frac{8R\,y^2\,(x_1 - x)}{y_1^4 x_1^2} - \frac{4R\,(y_1^2 - y^2)^2(x_1 - x)}{y_1^4 x_1^2} \\ & - \frac{R\,(y_1^2 - y^2)\,(x_1 - x)\Big(\frac{2}{y_1^2 x_1} - \frac{R\,(y_1^2 - y^2)^2}{y_1^4 x_1^2}\Big)}{y_1^2 x_1} \\ & + \frac{R\Big(\frac{-2(x_1 - x)}{y_1^2 x_1} + \frac{R\,(y_1^2 - y^2)^2(x_1 - x)}{y_1^4 x_1^2}\Big)(y_1^2 - y^2)}{y_1^2 x}t^2\end{aligned} \tag{22}$$

In the same manner the rest of the component of the iteration formula can be obtained.

Homotopy Perturbation Method (HPM)

Fundamentals: To clarify the basic ideas of HPM, consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{23}$$

Considering the boundary conditions of:

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma \tag{24}$$

where, A is a general differential operator, B a boundary operator, f(r) a known analytical function and Γ is the boundary of the domain Ω.

The operator A can be divided into two parts of L and N, where L is the linear part, while N is a nonlinear one. Eq. 23 can, therefore, be rewritten as:

$$L(u) + N(u) - f(r) = 0 \tag{25}$$

By the homotopy technique, construct a homotopy as which satisfies:

$$H(v, p) = (1-p) = [L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad P \in [0, 1], r \in \Omega \tag{26}$$

where, $p \in [0, 1]$ is an embedding parameter and u_0 is an initial approximation of Eq. 26 which satisfies the boundary conditions. Obviously, considering Eq. 26 will have:

$$\begin{aligned} H(v, 0) &= L(v) - L(u_0) = 0 \\ H(v, 1) &= A(v) - f(r) = 0 \end{aligned} \tag{27}$$

The changing process of P from zero to unity is just that of v (r, p) from u_0 (r) to u (r). In topology, this is called deformation and $L(v)-L(u_0)$ and $A(v)-f(r)$ are called homotopy. According to HPM, can first use the embedding parameter P as small parameter and assume that the solution of Eq. 26 can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \ldots \tag{28}$$

Setting p = 1 result in the approximate solution of Eq. 26:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \tag{29}$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the limitations of the traditional perturbation methods while it has full advantages of the traditional perturbation techniques.

The series (29) is convergent for most cases. However, the convergence rate depends on the nonlinear operator A (v). The following opinions are suggested by He (2004):

- The second derivative of N (v) with respect to v must be small because the parameter p may be relatively large, i.e., $p \to 1$.
- The norm of $L^{-1}\,\partial N/\partial v$ must be smaller than one so that the series converges.

Application: With the same first example as mentioned previously, the equation is as:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - Ru\frac{\partial u}{\partial x} + f(x, y) \tag{30}$$

With the initial condition of:

$$u(x, y, t=0) = \frac{(y_1^2 - y^2)(x_1 - x)}{y_1^2 x_1} \tag{31}$$

Substituting Eq. 29 into 26 and then substituting v from Eq. 28 and rearranging based on power series of P, have an equation system including n+1 equations to be simultaneously solved; n is the order of P in Eq. 28. Assuming 3, the system is as follows Eq. 32.

Now try to obtain a solution for equation system (32), in the form of (33):

$$\begin{cases} \dfrac{\partial u_0(x,y,t)}{\partial t} = 0 \\[2ex] \dfrac{\partial u_1(x,y,t)}{\partial t} - \dfrac{\partial^2 u_0(x,y,t)}{\partial x^2} - \dfrac{\partial u_0(x,y,t)}{\partial t} \\ \qquad + R\,u_0(x,y,t)(\dfrac{\partial u_0(x,y,t)}{\partial x}) - \dfrac{\partial^2 u_0(x,y,t)}{\partial y^2} = 0 \\[2ex] \dfrac{\partial u_2(x,y,t)}{\partial t} - \dfrac{\partial^2 u_1(x,y,t)}{\partial x^2} - \dfrac{\partial u_1(x,y,t)}{\partial t} \\ + R\,u_0(x,y,t)(\dfrac{\partial u_1(x,y,t)}{\partial x}) - \dfrac{\partial^2 u_1(x,y,t)}{\partial y^2} + R\,u_1(x,y,t)(\dfrac{\partial u_0(x,y,t)}{\partial x}) = 0 \\[2ex] R\,u_0(x,y,t)(\dfrac{\partial u_2(x,y,t)}{\partial x}) + R\,u_2(x,y,t)(\dfrac{\partial u_0(x,y,t)}{\partial x}) \\ \qquad - \dfrac{\partial u_2(x,y,t)}{\partial t} - \dfrac{\partial^2 u_2(x,y,t)}{\partial x^2} - \dfrac{\partial^2 u_2(x,y,t)}{\partial y^2} \\ \qquad + \dfrac{\partial u_3(x,y,t)}{\partial t} + R\,u_1(x,y,t)(\dfrac{\partial u_1(x,y,t)}{\partial x}) = 0 \end{cases} \tag{32}$$

$$\begin{cases} u_0(x,y,t) = \dfrac{(y_1^2 - y^2)(x_1 - x)}{y_1^2 x_1} \\ u_1(x,y,t) = \dfrac{(-2Ry_1^2y^2 + Ry_1^4 - 2y_1^2x_1 + Ry^4)(-x_1 + x)\,t}{y_1^4 x_1^2} \\ u_2(x,y,t) = \dfrac{R\,t^2\,(-Ry_1^6 + Ry^6 + 3Ry_1^4y^2 + 4y_1^4x_1 - 3Ry_1^2y^4 - 8y^2y_1^2x_1)(-x_1 + x)}{y_1^6 x_1^3} \\ \qquad + (\dfrac{2Rxy^2}{y_1^2x_1^2} + \dfrac{R}{x_1} - \dfrac{2Ry^2}{y_1^2x_1} - \dfrac{Rx}{x_1^2} + \dfrac{2x}{y_1^2x_1} + \dfrac{Ry^4}{y_1^4x_1} - \dfrac{Ry^4x}{y_1^4x_1^2} - \dfrac{2}{y_1^2})\,t \\ u_3(x,y,t) = \dfrac{-1}{y_1^8x_1^4}(R(-x_1 + x) \\ \qquad \times (\dfrac{3R^2y_1^8 + 3R^2y^8 - 12R^2y^2y_1^6 + 18R^2y^2y_1^4 - 18Rx_1y_1^6 + 68y_1^4Rx_1y^2 - 12R^2y^6y_1^2 - 50y^4Ry_1^2x_1 + 20y_1^4x_1^2}{3})\,t^3 \\ \qquad + \dfrac{32y^2y_1^4x_1^2 - 16y_1^6x_1^2 + 4y_1^8x_1R - 12y_1^6x_1Ry^2 + 12y_1^4x_1Ry^4 - 4y^6y_1^2x_1R}{2})t^2)) \\ \qquad - (\dfrac{R}{x_1} + \dfrac{2Ry^2}{y_1^2x_1} - \dfrac{Rx}{x_1^2} + \dfrac{2Rxy^2}{y_1^2x_1^2} + \dfrac{Ry^4}{y_1^4x_1} - \dfrac{Ry^4x}{y_1^4x_1^2} - \dfrac{2}{y_1^2} + \dfrac{2x}{y_1^2x_1})t \end{cases} \tag{33}$$

Finally, $u(x, y, t) = u_0(x, y, t)+u_1(x, y, t)+u_2(x, y, t)+u_3(x, y, t)$

Homotopyz Analysis Method (HAM)

Fundamentals: Consider the following differential equation:

$$N[u(\tau)] = 0 \tag{34}$$

where, N is a nonlinear operator, τ denotes an independent variable, $u(\tau)$ is an unknown function. For simplicity, ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao (2003) constructed the so-called zero-order deformation equation as:

$$(1-p)L[\phi(\tau;p) - u_0(\tau)] = p\hbar H(\tau)N[\phi(\tau;p)] \tag{35}$$

where, $p \in [0, 1]$ is the embedding parameter, $\hbar \neq 0$ a nonzero auxiliary parameter, $H(\tau) \neq 0$ an auxiliary function, L an auxiliary linear operator, $u_0(\tau)$ an initial guess of $u(\tau)$ and $\phi(\tau; p)$ is an unknown function. It is important to have enough freedom to choose auxiliary unknowns in HAM. Obviously, when p = 0 and p = 1, it holds: $\phi(\tau; 0) = u_0(\tau)$ and $\phi(\tau; 1) = u(\tau)$.

Thus, as p increases from 0 to 1, the solution $\phi(\tau; p)$ varies from the initial guess, $u_0(\tau)$ to the solution $u(\tau)$. Expanding $\phi(\tau; p)$ in Taylor series with respect to p, have:

$$\phi(\tau;p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)p^m \tag{36}$$

Where:

$$u_m(\tau) = \frac{1}{m!}\frac{\partial^m \phi(\tau;p)}{\partial p^m}\bigg|_{p=0} \tag{37}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter ħ and the auxiliary function are quite properly chosen, the series (37) converges at p = 1 then have:

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau) \tag{38}$$

This must be one of the solutions of the original nonlinear equation, as proved by Liao (2003). As $\hbar = -1$ and $H(\tau) = 1$, Eq. 35 becomes:

$$(1-p)L[\phi(\tau;p) - ul0(\tau)] + pN[\phi(\tau;p)] = 0 \tag{39}$$

This is mostly used in HPM, whereas the solution can be obtained directly without using Taylor series. According to the Eq. 35, the governing equation can be deduced from the zero-order deformation Eq. 40. The vector is defined as:

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \ldots, u_n(\tau)\} \tag{40}$$

Differentiating Eq. 35 for m times with respect to the embedding parameter p and then setting p = 0 and finally dividing them by m!, will have the so-called mth-order deformation equation as:

$$L[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar H(\tau) R_m(\vec{u}_{m-1}) \tag{41}$$

Where:

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(\tau;p)]}{\partial p^{m-1}}\bigg|_{p=0} \tag{42}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases} \tag{43}$$

It should be emphasized that $u_m(\tau)$ for $m \ge 1$ is governed by the linear Eq. 41 with the linear boundary conditions coming from the original problem, which can be easily solved using symbolic computation software.

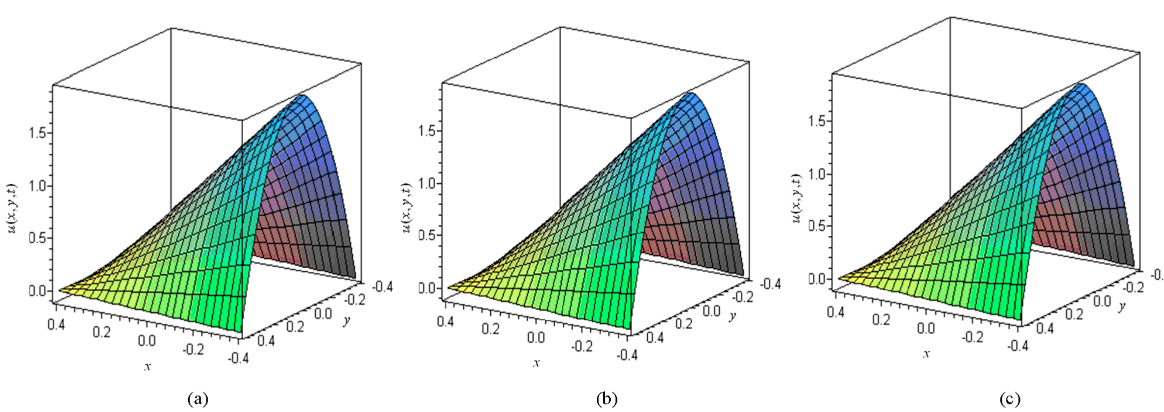
Application: Consider Eq. 30, 31 and let us solve them through HAM with proper assignment of $H(\tau) = 1$ subject to the initial condition and assuming m = 2.

$$\begin{cases} u_0(x,y,t) = \dfrac{(y_1^2 - y^2)(x_1 - x)}{y_1^2 x_1} \\ u_1(x,y,t) = \dfrac{(-2Ry_1^2y^2 + Ry_1^4 - 2y_1^2x_1 + Ry^4)(-x_1 + x)\hbar t}{y_1^4 x_1^2} \\ u_2(x,y,t) = \dfrac{2\hbar t(x_1 - x)}{y_1^2 x_1} - \dfrac{\hbar Rt(y_1^2 - y^2)^2(x_1 - x)}{y_1^4 x_1^2} \\ -\dfrac{(\hbar^3(-2y_1^2x_1^2 + 2xy_1^2x_1 + Ry_1^4x_1 - Rxy_1^4 - 2Ryy_1^2x_1 + 2Ry^2xx_1^2 + Ry^4x_1 - Ry^4x)^2R(y_1^2 - y^2)t^2)}{2y_1^{10}x_1^5} \\ +\dfrac{(\hbar(-1-\hbar)(-2y_1^2x_1^2 + 2xy_1^2x_1 + Ry_1^4x_1 - Rxy_1^4 - 2Ry^2y_1^2x_1 + 2Ry^2xy_1^2 + Ry^4x_1 - Ry^4x)t)}{y_1^4x_1^2} - \\ \dfrac{(\hbar^2R(y_1^2 - y^2)(x_1 - x)(2y_1^2x_1 - Ry_1^4 + 2Ry^2y_1^2 - Ry^4)t^2)}{2y_1^6x_1^3} \\ +\dfrac{\hbar^2(-4y_1^4x_1R + 4y_1^2xR + 12Ry^2x_1 - 12Ry^2x)t^2}{2y_1^4x_1^2} \end{cases} \tag{44}$$

Finally, $u(x, y, t) = u_0(x, y, t) + u_1(x, y, t) + u_2(x, y, t)$.

RESULTS AND DISCUTION

In this study, new kind of analytical methods, VIM, HPM, HAM, have been used in order to obtain the solution of time-dependent non-linear partial differential equations. Figure 1a-c shows the behavior of u(x, y, t) versus x and y from VIM, HPM HAM respectively for t = 0.001. This figure clearly shows the well agreement between results of these methods. For further verification, the cross-section of u(x, y, t) is shown in Fig. 2a ,b fort = 0.001 and t = 0.002 while y assumed to be constant at value o f zero. Figure 3 a, b shows the cross-section of u(x, y, t) for t = 0.001 and t = 0.003 while the constant value of x is zero. Figure 1 as well as Fig. 2 and 3 are obtained for R = 1, $y_1 = 0.4$, $x_1 = 0.4$. Figure 2, 3 approve once more an excellent agreement between methods.



(a) (b) (c)

Fig. 1: The behavior of u (x, y, t) versus *x* and *y* evaluates by VIM (a), HPM (b), HAM (c) at t = 0.001: R = 1, $y_1 = 0.4$, $x_1 = 0.4$

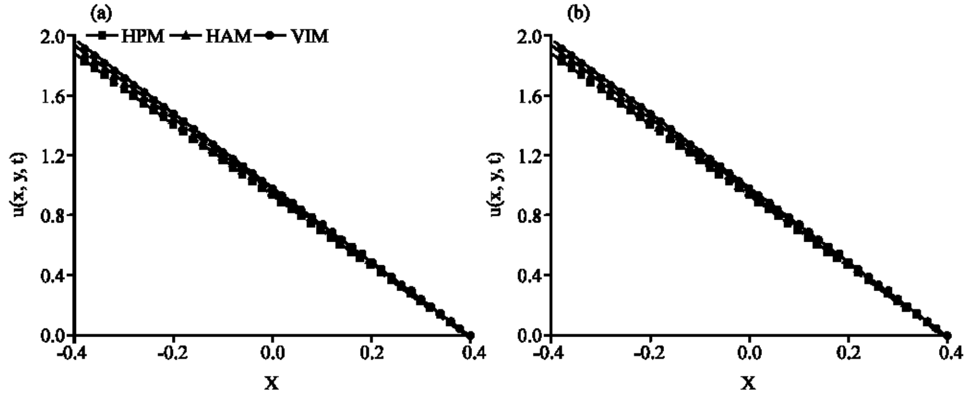


Fig. 2: Cross-section of u(x, y, t) at t = 0.001 (a), t = 0.002 (b), y = 0: R = 1, y_1 = 0.4, x_1 = 0.4

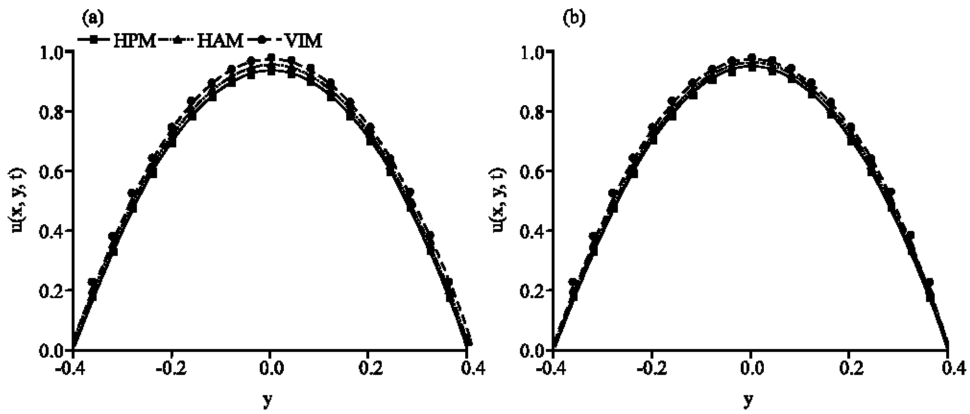


Fig. 3: Cross-section of u(x, y, t) at t = 0.001 (a), t = 0.003 (b), x = 0: R = 1, y_1 = 0.4, x_1 = 0.4

CONCLUSION

In this survey, VIM, HPM and VIM have been successfully applied to obtain the analytical solution of nonlinear time-depended partial differential equations. As a clear conclusion, these methods provide successive rapidly convergent approximations without any restrictive assumptions or transformations causing changes in the physical properties of the problem. Also adding up the number of iterations leads to the explicit solution for the problem. Moreover, the VIM, HPM and HAM do not require small parameters in the equation so that it overcomes the limitations that have arisen in traditional perturbation methods. The approximations are valid not only for small parameters but also for larger ones. The VIM, HPM and HAM are all efficient and powerful mathematical method to overcome this kind of problem and can be appropriate substitutions for each other. However, since the HPM has got shorter equations, the related results converge more rapidly. VIM including internalization calculations, takes a longer time and more difficulty arising in calculations. HAM is a new method that can be use for wide rang of nonlinear equation and the auxiliary parameter, h, provides a convenient way to adjust and control convergence region and rate of solution series so, it may leads to obtain the solution for fewer approximations. Moreover the solution of given nonlinear problem can be expressed by many base function and thus, can be more efficiently approximated by letter set of base function. All the aforementioned methods give rapidly convergent successive approximations of the exact solution if such a solution exists, otherwise approximations can be used for numerical purposes.

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