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Assessment of He's Homotopy Perturbation Method in Burgers and Coupled Burgers' Equations

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Abstract: Various methods for finding explicit solution to nonlinear evolution equations have been proposed in this letter Homotopy Perturbation Method (HPM) is employed for solving Korteweg-de Vries-Burgeres (KdVB) equation and coupled Burgers' equations which both of them are very applicable in mathematics, physics and engineering. The final results obtained by means of HPM are compared with those results obtained from the exact solution and the Adomian Decomposition Method (ADM). The comparison shows a precise agreement between the results and introduces this new method as an applicable one which it needs less computations and is much easier and more convenient than others, so it can be widely used in engineering too.

Key words: Korteweg-de Vries-Burgeres (KdVB) equation, coupled Burgers' equations, HPM, ADM

INTRODUCTION

In various fields of science and engineering, many physical problems can be described by linear and nonlinear parabolic equations. In this study we investigate solutions of KdV-Burgers equation and coupled Burgers' equations. Burgers' equation has been found to describe various kinds of phenomena such as mathematical model of turbulence and the approximate theory of flow through a Shock wave traveling in viscous fluid. It is well known that many physical phenomena can be described by the Korteweg-de Vries-Burgeres (KdVB) equation. Typical examples are provided by the behavior of long waves in shallow water and waves in plasma. The KdV-Burgers' equation is a one-dimension generalization of the model description of the density and velocity fields that takes into account pressure forces as well as the viscosity and the dispersion. It may be a more flexible tool for physicists than the Burgers' equation. The coupled Burgers' system was derived by Esipov. It is simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity. Several studies in the literature, employing a large variety of methods, have been conducted to derive explicit solution s for KdV-Burgers and coupled Burgers' equations. Gard and Hu used a steadt-state version of Eq. 1 to describe a weak shock profile in plasma. They studied the same problem using a similar method to that used by Johnson and the related problem was studied by Jeffery. A numerical investigation of the problem was carried out by Canosa and Gaxdag. Bona and Schonbek studied the existence and uniqueness of bounded traveling wave solution to Eq. 1 which tend to constant states at plus and minus infinity. A comprehensive account of the traveling wave solution of the KdVB equation can also be found in the review paper by Jeffrey and Kakutani. Also, several researches have been implemented for solving coupled Burgers' equations. Using the Hop-Cole transformation, Fletcher gave an analytical solution for the system of two dimensional Burgers' equations. Several numerical methods for solving this equation have been given as algorithms based on the cubic alpine function technique. The explicit-implicit method and the implicit finite element scheme. Soliman used the reductions for the partial differential equations to develop a scheme for solving Burgers' equation. The variational iteration method was used to solve the on dimensional (1D) Burgers' and coupled equations. Recently, an extended tanh-function method and symbolic computation have been suggested for solving the new coupled modified KDV equations to obtain four soliton solutions. ADM has been previously implemented to obtain exact solutions of this system. Then variational iteration method proposed by He was used to solve different types such as on dimensional Burgers' equations and coupled Burgers' equations. In this research, homotopy perturbation method (He, 1999;

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2003; Ganji and Rajabi, 2006; Aabbasbandy, 2006) is employed to compute an approximation to the solution of these equations in comparison to the exact solution (Helal and Mehanna, 2006; Dehghan *et al*., 2007) and ADM.

The general form of Burgers' equation can be mentioned as fallows:

$$u_t + \mu_1 u u_x + \mu_2 u_{xx} + \mu_3 u_{xxx} = 0, \quad x \in R, \qquad (1)$$

Where, μ_1, μ_2 and μ_3 are constant coefficients, with the initial and boundary conditions:

$$u(x,0 = f(x) \qquad (2)$$

$$u0,t) = g(t) \qquad (3)$$

Where:
u = u(x,t) = Sufficiently smooth function
f(x) = Bounded

In addition we shall assume that the solution u(x,t), along with it's derivatives, tends to zero as $|x| \to \infty$.

The well-known KdV-Burgers' equation that involves both dispersion term u_{xxx} and dissipation term u_{xx}, is:

$$u_t + 2(u^3)_x - u_{xxx} + u_{xx} = 0 \qquad (4)$$

Subject to initial condition:

$$u(x,t) = \frac{1}{6}\left(1 + \tanh\left(\frac{1}{6}x\right)\right). \qquad (5)$$

This nonlinear partial differential equation has an exact special solution. We replace the independent parameters x and t by one composed parameter

$$(x - \frac{2}{9}t).$$

This transformation leads to the traveling wave solution

$$u(x,t) = \frac{1}{6}\left(1 + \tanh\left(\frac{1}{6}\left(x - \frac{2}{9}t\right)\right)\right). \qquad (6)$$

It is worth noting that this exact solution is a special one (Helal and Mehanna, 2006).

And also we will consider the systems of Burgers' equations in the form:

$$\begin{cases} u_t - u_{xx} - 2uu_x + u_x v + v_x u = 0, \\ v_t - v_{xx} - 2vv_x + u_x v + v_x u = 0. \end{cases} \qquad (7)$$

The solutions of which are to be determined subject to the initial conditions:

$$u(x,0) = \sin(x), \quad v(x,0) = \sin(x) \qquad (8)$$

Clearly, the exact solutions of this system are (Dehghan *et al*., 2007):

$$u(x,t) = \exp(-t)\sin(x), \quad v(x,t) = \exp(-t)\sin(x) \qquad (9)$$

BASIC IDEA OF HOMOTOPY PERTURBATION METHOD (HPM)

To explain this method, let us consider the following function:

$$A(u) - f(r) = 0, \quad r \in \Omega \qquad (10)$$

With the boundary conditions of

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma \qquad (11)$$

Where, A, B, f(r) and Γ are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively.

Generally speaking, the operator A can be divided in to a linear part L and a nonlinear part N. Eq. 10 can therefore, be written as:

$$L(u) + N(u) - f(r) = 0 \qquad (12)$$

By the homotopy technique, we construct a homotopy $v(r,p): \Omega \times [0,1] \to R$ which satisfies:

$$\begin{aligned} &H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + \\ &p\left[A(v) - f(r)\right] = 0, \\ &p \in [0,1], r \in \Omega, \end{aligned} \qquad (13)$$

Or

$$\begin{aligned} &H(v,p) = L(v) - L(u_0) + pL(u_0) + \\ &p[N(v) - f(r)] = 0, \end{aligned} \qquad (14)$$

Where, $p \in [0,1]$ is an embedding parameter, while u_0 is an initial approximation of Eq. 10, which satisfies the boundary conditions. Obviously, from Eq. 13 and 14 we will have:

$$H(v,0) = L(v) - L(u_0) = 0 \qquad (15)$$

$$H(v,1) = A(v) - f(r) = 0 \tag{16}$$

The changing process of p from zero to unity is just that of v(r,p) from u_0 to u(r). In topology, this is called deformation, while $L(v)-L(u_0)$ and $A(v)-f(r)$ are called homotopy.

According to the HPM, we can first use the embedding parameter p as a small parameter and assume that the solutions of Eq. 13 and 14 can be written as a power series in p:

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + ... \tag{17}$$

Setting p = 1 results in the approximate solution of Eq. 10:

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + ... \tag{18}$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages. The series (18) is convergent for most cases. However, the convergent rate depends on the nonlinear operator A(v). Moreover, the following suggestions were made by He (1999):

- The second derivative of N(v) with respect to v must be small because the parameter may be relatively large, i.e., $p \to 1$.
- The norm of $L^{-1}\frac{\partial N}{\partial v}$ must be smaller than one so that the series converges.

APPLICATION OF HOMOTOPY-PERTURBATION METHOD

To investigate Eq. 4, we first construct a homotopy by separating the linear and nonlinear parts of the equation; we apply homotopy-perturbation to Eq. 4 using Eq. 13 as fallows:

$$(1-p)\left(\frac{\partial}{\partial t}v(x,t)\right) + p\left(\begin{array}{l}\left(\frac{\partial}{\partial t}v(x,t)\right)+6v^2(x,t)\left(\frac{\partial}{\partial x}v(x,t)\right)- \\ \left(\frac{\partial^3}{\partial x^3}v(x,t)\right)+\left(\frac{\partial^2}{\partial x^2}v(x,t)\right)\end{array}\right)=0 \tag{19}$$

Substituting Eq. 17 into 19 and rearranging the resultant equation based on powers of p-terms, one has:

$$p^0 : \frac{\partial}{\partial t}v_0(x,t) = 0, \tag{20}$$

$$p^1 : \left(\frac{\partial}{\partial t}v_1(x,t)\right)+\left(\frac{\partial^2}{\partial x^2}v_0(x,t)\right)-\left(\frac{\partial^3}{\partial x^3}v_0(x,t)\right)+ 6{v_0}^2(x,t)\left(\frac{\partial}{\partial x}v_0(x,t)\right)=0 \tag{21}$$

$$p^2 : 12v_0(x,t)v_1(x,t)\left(\frac{\partial}{\partial x}v_0(x,t)\right)+\left(\frac{\partial}{\partial t}v_2(x,t)\right)+ 6{v_0}^2(x,t)\left(\frac{\partial}{\partial x}v_1(x,t)\right)+\left(\frac{\partial^2}{\partial x^2}v_1(x,t)\right)- \left(\frac{\partial^3}{\partial x^3}v_1(x,t)\right)=0 \tag{22}$$

$$p^3 : 12v_0(x,t)v_1(x,t)\left(\frac{\partial}{\partial x}v_1(x,t)\right)+ 12v_0(x,t)v_2(x,t)\left(\frac{\partial}{\partial x}v_0(x,t)\right)+ \left(\frac{\partial}{\partial t}v_3(x,t)\right)+6{v_0}^2(x,t)\left(\frac{\partial}{\partial x}v_2(x,t)\right)+ 6{v_1}^2(x,t)\left(\frac{\partial}{\partial x}v_0(x,t)\right)+\left(\frac{\partial^2}{\partial x^2}v_2(x,t)\right)- \left(\frac{\partial^3}{\partial x^3}v_2(x,t)\right)=0 \tag{23}$$

With the following initial conditions:

$$\begin{cases} u_0(x,t) = u(x,0) = v_0(x,t) = \frac{1}{6}\left(1+\tanh\left(\frac{1}{6}x\right)\right). \\ v_i(x,0) = 0, i = 1,2,..... \end{cases} \tag{24}$$

In order to obtain the unknowns, we should solve Eq. 20 through (23), considering the initial conditions Eq. 24 and having the initial approximations of Eq. 17. So we have:

$$v_0(x,t) = \frac{1}{6}\left(1+\tanh\left(\frac{1}{6}x\right)\right). \tag{25}$$

$$v_1(x,t) = -\frac{1}{162}t + \frac{1}{162}t.\tanh\left(\frac{1}{6}x\right)^2, \tag{26}$$

$$v_2(x,t)=\frac{1}{4374}\tanh\left(\frac{1}{6}x\right)t^2\left(-1+\tanh\left(\frac{1}{6}x\right)^2\right), \tag{27}$$

$$v_3(x,t)=\frac{1}{354294}t^3\left(3\tanh\left(\frac{1}{6}x\right)^2-1\right) \times\left(-1+\tanh\left(\frac{1}{6}x\right)^2\right), \tag{28}$$

In the same manner, the rest of components can be obtained using the maple package.

According to the HPM, we can conclude that:

$$u(x,t)=\lim_{p\to 1} v(x,t)=v_0(x,t)+v_1(x,t)+ v_2(x,t)+v_3(x,t)+..... \tag{29}$$

Therefore, substituting the values of $v_0(x,t)$ through $v_3(x,t)$ from Eq. 25 through 28 into Eq. 29 yields:

$$\begin{aligned} u(x,t)=&\frac{1}{6}\left(1+\tanh\left(\frac{1}{6}x\right)\right)- \\ &\frac{1}{162}t+\frac{1}{162}t.\tanh\left(\frac{1}{6}x\right)^2+ \\ &\frac{1}{4374}\tanh\left(\frac{1}{6}x\right)t^2\left(-1+\tanh\left(\frac{1}{6}x\right)^2\right)+ \\ &\frac{1}{354294}t^3\left(3\tanh\left(\frac{1}{6}x\right)^2-1\right)\times\left(-1+\tanh\left(\frac{1}{6}x\right)^2\right). \end{aligned} \tag{30}$$

The numerical comparison between the HPM and the ADM (Helal and Mehanna, 2006) and the exact solution are shown in Fig. 1.

To investigate Eq. 7 we first construct a homotopy as fallows:

$$(1-p)\left(\frac{\partial}{\partial t}v_1(x,t)\right)+ p\left(\begin{array}{l}\left(\frac{\partial}{\partial t}v_1(x,t)\right)-\left(\frac{\partial^2}{\partial x^2}v_1(x,t)\right)- \\ 2v_1(x,t)\left(\frac{\partial}{\partial x}v_1(x,t)\right)+\left(\frac{\partial}{\partial x}v_1(x,t)\right)v_2(x,t)+ \\ \left(\frac{\partial}{\partial x}v_2(x,t)\right)v_1(x,t)\end{array}\right)=0 \tag{31}$$

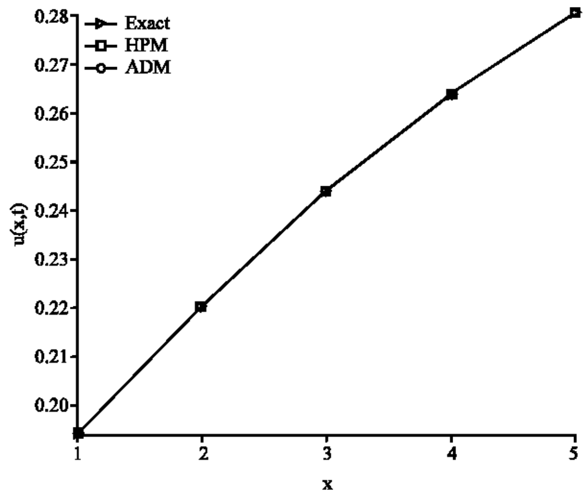

Fig. 1: Numerical comparison of the results obtained by the HPM, the ADM and the exact solution for t = 0.001 and 1<x<5 for Eq. 4

$$(1-p)\left(\frac{\partial}{\partial t}v_2(x,t)\right)+ p\left(\begin{array}{l}\left(\frac{\partial}{\partial t}v_2(x,t)\right)-\left(\frac{\partial^2}{\partial x^2}v_2(x,t)\right)- \\ 2v_2(x,t)\left(\frac{\partial}{\partial x}v_2(x,t)\right)+\left(\frac{\partial}{\partial x}v_2(x,t)\right)v_1(x,t)+ \\ \left(\frac{\partial}{\partial x}v_1(x,t)\right)v_2(x,t)\end{array}\right)=0 \tag{32}$$

And the initial approximations are as fallows:

$$\begin{cases} v_{1,0}(x,t)=v_1(x,0)=u_0(x,t)=\sin(x) \\ v_{2,0}(x,t)=v_2(x,0)=v_0(x,t)=\sin(x) \\ v_{1,i}(x,0)=v_{2,i}(x,0)=0, \\ i=1,2,3,.... \end{cases} \tag{33}$$

And

$$\begin{cases} v_1(x,t)=v_{1,0}(x,t)+pv_{1,1}(x,t)+ \\ p^2v_{1,2}(x,t)+p^3v_{1,3}(x,t)+..... \\ v_2(x,t)=v_{2,0}(x,t)+pv_{2,1}(x,t)+ \\ p^2v_{2,2}(x,t)+p^3v_{2,3}(x,t)+..... \end{cases} \tag{34}$$

Substituting Eq. 34 into Eq. 31 and 32 and rearranging the coefficients of p powers, we have:

$$p^0:\frac{\partial}{\partial t}v_{1,0}(x,t)=0 \tag{35}$$

$$p^1:\left(\begin{array}{l}\left(\frac{\partial}{\partial t}v_{1,1}(x,t)\right)+\left(\frac{\partial}{\partial x}v_{2,0}(x,t)\right)v_{1,0}(x,t)-\\ 2v_{1,0}(x,t)\left(\frac{\partial}{\partial x}v_{1,0}(x,t)\right)+\\ \left(\frac{\partial}{\partial x}v_{1,0}(x,t)\right)v_{2,0}(x,t)-\\ \left(\frac{\partial^2}{\partial x^2}v_{1,0}(x,t)\right)\end{array}\right)=0 \tag{36}$$

$$p^2:\left(\begin{array}{l}\left(\frac{\partial}{\partial t}v_{1,2}(x,t)\right)+\left(\frac{\partial}{\partial x}v_{1,1}(x,t)\right)v_{2,0}(x,t)+\\ \left(\frac{\partial}{\partial x}v_{2,0}(x,t)\right)v_{1,1}(x,t)+\\ \left(\frac{\partial}{\partial x}v_{2,1}(x,t)\right)v_{1,0}(x,t)-\\ \left(\frac{\partial^2}{\partial x^2}v_{1,1}(x,t)\right)-2v_{1,0}(x,t)\left(\frac{\partial}{\partial x}v_{1,1}(x,t)\right)-\\ 2v_{1,1}(x,t)\left(\frac{\partial}{\partial x}v_{1,0}(x,t)\right)+\\ \left(\frac{\partial}{\partial x}v_{1,0}(x,t)\right)v_{2,1}(x,t)\end{array}\right)=0 \tag{37}$$

$$p^3:\left(\begin{array}{l}\left(\frac{\partial}{\partial t}v_{1,3}(x,t)\right)-\left(\frac{\partial^2}{\partial x^2}v_{1,2}(x,t)\right)-\\ 2v_{1,0}(x,t)\left(\frac{\partial}{\partial x}v_{1,2}(x,t)\right)-\\ 2v_{1,1}(x,t)\left(\frac{\partial}{\partial x}v_{1,1}(x,t)\right)-2v_{1,2}(x,t)\left(\frac{\partial}{\partial x}v_{1,0}(x,t)\right)+\\ v_{1,2}(x,t)\left(\frac{\partial}{\partial x}v_{2,0}(x,t)\right)+\\ v_{2,2}(x,t)\left(\frac{\partial}{\partial x}v_{1,0}(x,t)\right)+v_{2,1}(x,t)\left(\frac{\partial}{\partial x}v_{1,1}(x,t)\right)+\\ v_{2,0}(x,t)\left(\frac{\partial}{\partial x}v_{1,2}(x,t)\right)+\\ v_{1,1}(x,t)\left(\frac{\partial}{\partial x}v_{2,1}(x,t)\right)+v_{1,0}(x,t)\left(\frac{\partial}{\partial x}v_{2,2}(x,t)\right)\end{array}\right) \tag{38}$$

$$p^0:\frac{\partial}{\partial t}v_{2,0}(x,t)=0 \tag{39}$$

$$p^1:\left(\begin{array}{l}\left(\frac{\partial}{\partial t}v_{2,1}(x,t)\right)+\left(\frac{\partial}{\partial x}v_{2,0}(x,t)\right)v_{1,0}(x,t)+\\ v_{2,0}(x,t)\left(\frac{\partial}{\partial x}v_{1,0}(x,t)\right)-\\ 2\left(\frac{\partial}{\partial x}v_{2,0}(x,t)\right)v_{2,0}(x,t)-\\ \left(\frac{\partial^2}{\partial x^2}v_{2,0}(x,t)\right)\end{array}\right)=0 \tag{40}$$

$$p^2:\left(\begin{array}{l}\left(\frac{\partial}{\partial t}v_{2,2}(x,t)\right)-\left(\frac{\partial^2}{\partial x^2}v_{2,1}(x,t)\right)-\\ 2v_{2,1}(x,t)\left(\frac{\partial}{\partial x}v_{2,0}(x,t)\right)-\\ 2v_{2,0}(x,t)\left(\frac{\partial}{\partial x}v_{2,1}(x,t)\right)+\\ \left(\frac{\partial}{\partial x}v_{1,1}(x,t)\right)v_{2,0}(x,t)+\\ \left(\frac{\partial}{\partial x}v_{1,0}(x,t)\right)v_{2,1}(x,t)+\\ \left(\frac{\partial}{\partial x}v_{2,1}(x,t)\right)v_{1,0}(x,t)+\\ \left(\frac{\partial}{\partial x}v_{2,0}(x,t)\right)v_{1,1}(x,t)\end{array}\right)=0 \tag{41}$$

$$p^3:\left(\begin{array}{l}\left(\frac{\partial}{\partial t}v_{2,3}(x,t)\right)-\left(\frac{\partial^2}{\partial x^2}v_{2,2}(x,t)\right)-\\ 2v_{2,1}(x,t)\left(\frac{\partial}{\partial x}v_{2,1}(x,t)\right)-\\ 2v_{2,2}(x,t)\left(\frac{\partial}{\partial x}v_{2,0}(x,t)\right)-2v_{2,0}(x,t)\left(\frac{\partial}{\partial x}v_{2,2}(x,t)\right)+\\ v_{1,2}(x,t)\left(\frac{\partial}{\partial x}v_{2,0}(x,t)\right)+\\ v_{2,2}(x,t)\left(\frac{\partial}{\partial x}v_{1,0}(x,t)\right)+v_{2,1}(x,t)\left(\frac{\partial}{\partial x}v_{1,1}(x,t)\right)+\\ v_{2,0}(x,t)\left(\frac{\partial}{\partial x}v_{1,2}(x,t)\right)+\\ v_{1,1}(x,t)\left(\frac{\partial}{\partial x}v_{2,1}(x,t)\right)+v_{1,0}(x,t)\left(\frac{\partial}{\partial x}v_{2,2}(x,t)\right)\end{array}\right)=0 \tag{42}$$

Solving Eq. 35 through 42 and using the initial conditions Eq. 33, one can find the fallowing results:

$$v_{1,0}(x,t)=\sin(x), \tag{43}$$

$$v_{1,1}(x,t) = -t\sin(x), \tag{44}$$

$$v_{1,2}(x,t) = \frac{1}{2}t^2\sin(x), \tag{45}$$

$$v_{1,3}(x,t) = -\frac{1}{6}t^3\sin(x). \tag{46}$$

And

$$v_{2,0}(x,t) = \sin(x), \tag{47}$$

$$v_{2,1}(x,t) = -t\sin(x), \tag{48}$$

$$v_{2,2}(x,t) = \frac{1}{2}t^2\sin(x), \tag{49}$$

$$v_{2,3}(x,t) = -\frac{1}{6}t^3\sin(x). \tag{50}$$

According to the HPM, we can conclude that

$$u(x,t) = \lim_{p\to 1} v_1(x,t) = \sum_{k=0}^{3} v_{1,k}(x,t), \tag{51}$$

$$v(x,t) = \lim_{p\to 1} v_2(x,t) = \sum_{k=0}^{3} v_{2,k}(x,t). \tag{52}$$

After putting Eq. 43 through 46 into Eq. 51 and 47 through 50 into Eq. 52 the final results can be obtained as fallows:

$$u(x,t) = \sin(x)(1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + ...) = e^{-t}\sin(x) \tag{53}$$

$$v(x,t) = \sin(x)(1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + ...) = e^{-t}\sin(x) \tag{54}$$

It can easily be seen that using ADM also leads to the same results (Dehghan *et al*., 2007).

CONCLUSION

In this study, the homotopy perturbation method (HPM) was used for finding the approximate solutions of KdV-Burgers equation and coupled Burgers' equations. In can be concluded that the HPM is very powerful and efficient technique in finding exact solutions for wide classes of problems. It is worth pointing out that the HPM presents a rapid convergence for the solutions.

The two solved examples show that the results of the present method are in excellent agreement with those obtained by the exact solution and the ADM. The HPM has got many merits and much more advantages than other ADM and methods. Also the HPM does not require small parameters in the equation, so that the limitations of the traditional perturbation methods can be eliminated and also the calculations in the HPM are simple and straightforward. The reliability of the method and the reduction in the size of computational domain gives this method a wider applicability. The results show that the HPM is a powerful mathematical tool for solving systems of nonlinear partial differential equations having wide applications in engineering.

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