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Exact Solutions for Thermal Stresses in a Rotating Thick-Walled Cylinder of Functionally Graded Materials

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Abstract: In this study, thermal stresses in a hollow rotating thick-walled cylinder made of functionally graded material under internal and external pressure are obtained as a function of radial direction to an exact solution by using the theory of elasticity. Material properties are considered as a function of the radius of the cylinder and the Poisson's ratio as constant. The distributions of the thermal stresses are obtained for different values of the powers of the module of elasticity.

Key words: FGM, thermal stress, thick-wall, cylinder, rotating

INTRODUCTION

By using the continuous change in the physical and the mechanical properties of a material, it is possible to prevent from the fracture in composite materials, which causes stress concentration and yield in such materials. These materials which possess gradual change in their material properties are known as Functionally Gradient Materials (FGM). The problems of rotating annular disks or cylinders have been investigated under various assumptions and conditions. This is a topic which can be readily found in most standard elasticity books (Boresi and Chong, 1999). Obata and Noda (1994), through the application of a perturbation approach, investigated the thermal stresses in an FGM hollow sphere and in a hollow circular cylinder. Assuming that the material has a graded modulus of elasticity, while the Poisson's ratio is a constant, Tutuncu and Ozturk (2001) investigated the stress distribution in the axisymmetric structures. They obtained the closed-form solutions for stresses and displacements in functionally graded cylindrical and spherical vessels under internal pressure. Based on approximate solutions of temperatures and thermal stresses, the optimization of the material composition of FGM hollow circular cylinders under thermal loading was discussed (Ootao *et al*., 1999). Applying the Frobenius series method, Zimmerman and Lutz (1999) found a way round the problem of the uniform heating of functionally graded circular cylinder. Another general analysis of one-dimensional steady-state thermal stresses in a hollow thick cylinder made of functionally graded material was obtained (Jabbari *et al*., 2002). An analysis of the thermomechanical behavior of hollow circular cylinders of functionally graded materials was presented (Liew *et al*., 2003). They worked out a solution based on the solutions obtained by a novel limiting process that makes use of the solutions of homogeneous hollow circular cylinders, without resorting to the basic theory or the equations of non-homogeneous thermoelasticity. Tarn and Wang (2004) studied heat conduction in circular cylinders of functionally graded materials and laminated composites. They focused on the end effects and by means of matrix algebra and eigenfunction expansion, the decay length that characterizes the end effects on the thermal filed was assessed. An accurate method for conducting elastic analysis of thick-walled spherical pressure vessels subjected to internal pressure was devised (You *et al*., 2005). It is necessary to point out that two kinds of pressure vessel are considered: one is made up of two homogeneous layers near the inner and outer surfaces of the vessel and the other functionally graded layer in the middle; the latter consists of the functionally graded material only. Jabbari *et al*. (2007), making use of the generalized Bessel function and Fourier series solved the temperature and Navier equations analytically and offered a general theoretical analysis of three-dimensional mechanical and thermal stresses for a short hollow cylinder made of functionally graded material. Given the assumption that the material is isotropic with constant Poisson's ratio and exponentially varying elastic modulus through the thickness, Tutuncu (2007), obtained power series solutions for stresses and displacements in functionally-graded cylindrical vessels subjected to internal pressure alone. Argeso and Eraslon (2008) assuming the different states of material properties including Poisson's ratio υ, modulus of elasticity E, the

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yield strength σ_0, the coefficient of thermal expansion α and the thermal conductivity k, assessed the thermoelastic response of cylinders and tubes.

MATERIALS AND METHODS

The thermal stresses distribution in a hollow rotating thick-walled cylinder in the plane strain condition will be calculated. Consider a thick-walled FGM cylinder with an inner radius a and an outer radius b, subjected to an internal pressure P_i and external pressure P_o that are axisymmetric and rotating at a constant angular velocity ω about its axis. The cylindrical coordinates (r, θ, z) are chosen so, that r and z are the radial and axial coordinates, respectively.

The material properties are assumed to be radically dependent. The module of elasticity, thermal conductivity, linear expansion coefficient and density through the wall thickness are assumed to vary as follows:

$$E = E_0 r^{\beta} \tag{1}$$

$$k = k_0 r^{\xi} \tag{2}$$

$$\alpha = \alpha_0 r^{\eta} \tag{3}$$

$$\rho = \rho_0 r^{\gamma} \tag{4}$$

where, E, k, α and ρ are module of elasticity, thermal conductivity, linear expansion coefficient and density. E_0, k_0, α_0 and ρ_0 are the material constants and β, ξ, η and γ are the power law indices of the material.

It could be assumed that:

$$\xi = m_1\beta \tag{5}$$

$$\eta = m_2\beta \tag{6}$$

$$\gamma = m_3\beta \tag{7}$$

where, m_1, m_2 and m_3 are constant. In the current study, a range of $-2\leq\beta\leq2$ is employed which consists of all the values which has widely been used in the research mentioned above.

To show the effect of inhomogeneity on the stress distributions, different values were considered for β. Since, the variation of Poisson's ratio, υ, for engineering materials is small, it is assumed constant.

It is assumed that plane strain $\varepsilon_z = 0$. The radial strain ε_r and circumferential strain ε_θ are related to the radial displacement u by:

$$\varepsilon_r = \frac{du}{dr} \tag{8}$$

$$\varepsilon_\theta = \frac{u}{r} \tag{9}$$

The equilibrium equations in the absence of body forces reduce to:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = -\rho r\omega^2 \tag{10}$$

where, σ_r and σ_θ are the radial and circumferential stress components, respectively.

In the steady state case, the heat conduction equation for the one-dimensional problem in polar coordinates simplifies to:

$$\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = 0 \tag{11}$$

Boundary condition of temperature is as follows:

$$k\frac{dT}{dr} = h_a(T - T_a) \qquad \text{on} \quad r = a \tag{12}$$

$$-k\frac{dT}{dr} = h_b(T - T_b) \qquad \text{on} \quad r = b \tag{13}$$

where, T_a and T_b are temperatures of the surrounding media, h_a and h_b are the heat transfer coefficients and subscripts a and b correspond to surfaces r = a and r = b, respectively.

The general solution of Eq. 11 with considering relation of thermal transfer coefficient Eq. 2 and boundary conditions into Eq. 12 and 13 is:

$$T = \frac{T_a - T_b}{k_0 m_1\beta\left(\frac{1}{ah_a} + \frac{1}{bh_b}\right) + \left(\frac{1}{a^{m_1\beta}} - \frac{1}{b^{m_1\beta}}\right)} r^{-m_1\beta} + \frac{k_0 m_1\beta\left(\frac{T_b}{ah_a} + \frac{T_a}{bh_b}\right) + a^{-m_1\beta}T_b - b^{-m_1\beta}T_a}{k_0 m_1\beta\left(\frac{1}{ah_a} + \frac{1}{bh_b}\right) + a^{-m_1\beta} - b^{-m_1\beta}} \tag{14}$$

with considering special case in which there is no heat transfer taking place between the inner surface and outer surface with the surrounding medium and that the surface temperature at the inner and outer surfaces prescribed as T_a and T_b, respectively. Thus:

$$T = \frac{1}{a^{-m_1\beta} - b^{-m_1\beta}}\left[(T_a - T_b)r^{-m_1\beta} + a^{-m_1\beta}T_b - b^{-m_1\beta}T_a\right] \tag{15}$$

Applying Hooke's law, the equations for radial and circumferential stresses would be:

$$\sigma_r = A_1 r^{\beta}\varepsilon_r + A_2 r^{\beta}\varepsilon_\theta + A_3 r^{(1-m_1+m_2)\beta} + A_4 r^{(1+m_2)\beta} \tag{16}$$

$$\sigma_\theta = A_2 r^{\beta}\varepsilon_r + A_1 r^{\beta}\varepsilon_\theta + A_3 r^{(1-m_1+m_2)\beta} + A_4 r^{(1+m_2)\beta} \tag{17}$$

where, A_1, A_2, A_3, A_4, A_5 and A_6 are

$$\begin{aligned}
A_1 &= \frac{E_0(1-\upsilon)}{(1+\upsilon)(1-2\upsilon)}\\
A_2 &= \frac{E_0\upsilon}{(1+\upsilon)(1-2\upsilon)}\\
A_3 &= \frac{E_0\alpha_0(T_b-T_a)(ab)^{m_1\beta}}{(1-2\upsilon)(b^{m_1\beta}-a^{m_1\beta})}\\
A_4 &= \frac{E_0\alpha_0(a^{m_1\beta}T_a-b^{m_1\beta}T_b)}{(1-2\upsilon)(b^{m_1\beta}-a^{m_1\beta})}\\
A_5 &= (1-2\upsilon)A_3\\
A_6 &= (1-2\upsilon)A_4
\end{aligned} \tag{18}$$

Using Eq. 1-10 and Eq. 16-18, the Navier equation in term of the radial displacement is:

$$r^2\frac{d^2u}{dr^2} + (\beta+1)r\frac{du}{dr} + (n\beta-1)u = B_1 r^{3+(m_3-1)\beta} + B_2 r^{3+(m_2-m_1)\beta} + B_3 r^{1+m_2\beta} \tag{19}$$

where, $n = \dfrac{\upsilon}{1-\upsilon}$ and B_1, B_2 and B_3 are

$$\begin{aligned}
B_1 &= -\frac{\rho_0\omega^2(1+\upsilon)(1-2\upsilon)}{E_0(1-\upsilon)}\\
B_2 &= \frac{\alpha_0\beta(1-m_1+m_2)(1+\upsilon)(T_a-T_b)(ab)^{m_1\beta}}{(1-\upsilon)(b^{m_1\beta}-a^{m_1\beta})}\\
B_3 &= \frac{\alpha_0\beta(1+m_2)(1+\upsilon)(b^{m_1\beta}T_b-a^{m_1\beta}T_a)}{(1-\upsilon)(b^{m_1\beta}-a^{m_1\beta})}
\end{aligned} \tag{20}$$

Equation 19 is the non-homogeneous Euler-Caushy equation whose complete solution is:

$$u = C_1 r^{5+(m_3-1)\beta} + C_2 r^{5+(m_2-m_1)\beta} + C_3 r^{3+m_2\beta} + C_4 r^{f_1} + C_5 r^{f_2} \tag{21}$$

where, f_1, f_2, C_1, C_2 and C_3 are:

$$\begin{aligned}
f_1 &= \frac{-\beta+\sqrt{\beta^2-4n\beta+4}}{2}\\
f_2 &= \frac{-\beta-\sqrt{\beta^2-4n\beta+4}}{2}
\end{aligned} \tag{22}$$

$$\begin{aligned}
C_1 &= \frac{B_1}{(5+m_3\beta+f_1)(5+m_3\beta+f_2)}\\
C_2 &= \frac{B_2}{(5-m_1\beta+m_2\beta-f_1)(5-m_1\beta+m_2\beta-f_2)}\\
C_3 &= \frac{B_3}{(3+m_2\beta-f_1)(3+m_2\beta-f_2)}
\end{aligned} \tag{23}$$

By substituting Eq. 21 into Eq. 8 and 9 and results in Eq. 16 and 17, the stresses are obtained as:

$$\begin{aligned}
\sigma_r &= C_1[A_1(5+m_3\beta-\beta)+A_2]r^{4+m_3\beta} + C_2[A_1(5+m_2\beta-m_1\beta)\\
&+A_2]r^{4+\beta+(m_2-m_1)\beta} + C_3[A_1(3+m_2\beta)+A_2]r^{2+(1+m_2)\beta} +\\
&C_4[A_1f_1+A_2]r^{\beta+f_1-1} + C_5[A_1f_2+A_2]r^{\beta+f_2-1} + A_3 r^{(1-m_1+m_2)\beta} + A_4 r^{(1+m_2)\beta}
\end{aligned} \tag{24}$$

$$\begin{aligned}
\sigma_\theta &= C_1[A_2(5+m_3\beta-\beta)+A_1]r^{4+m_3\beta} + C_2[A_2(5+m_2\beta-m_1\beta)+\\
&A_1]r^{4+\beta+(m_2-m_1)\beta} + C_3[A_2(3+m_2\beta)+A_1]r^{2+(1+m_2)\beta} +\\
&C_4[A_2f_1+A_1]r^{\beta+f_1-1} + C_5[A_2f_2+A_1]r^{\beta+f_2-1} + A_3 r^{(1-m_1+m_2)\beta} + A_4 r^{(1+m_2)\beta}
\end{aligned} \tag{25}$$

To determine the constants C_4 and C_5 consider the boundary conditions for stresses given by:

$$\sigma_r\big|_{r=a} = -P_i,\ \sigma_r\big|_{r=b} = -P_o \tag{26}$$

By substituting the boundary conditions Eq. 26 into Eq. 24, the constants becomes:

$$C_4 = \frac{g_4g_5-g_2g_6}{g_1g_4-g_2g_3},\quad C_5 = \frac{g_1g_6-g_3g_5}{g_1g_4-g_2g_3} \tag{27}$$

where, the parameters of g_1, g_2, g_3, g_4, g_5 and g_6 are as follows:

$$\begin{aligned}
g_1 &= (A_1f_1+A_2)a^{\beta+f_1-1}\\
g_2 &= (A_1f_2+A_2)a^{\beta+f_2-1}\\
g_3 &= (A_1f_1+A_2)b^{\beta+f_1-1}\\
g_4 &= (A_1f_2+A_2)b^{\beta+f_2-1}\\
g_5 &= -C_1[A_1(5+m_3\beta-\beta)+A_2]a^{4+m_3\beta} - C_2[A_1(5+m_2\beta-m_1\beta)\\
&+A_2]a^{4+\beta+(m_2-m_1)\beta} - C_3[A_1(3+m_2\beta)+A_2]a^{2+(1+m_2)\beta} -\\
&A_3a^{(1-m_1+m_2)\beta} - A_4a^{(1+m_2)\beta} - P_i\\
g_6 &= -C_1[A_1(5+m_3\beta-\beta)+A_2]b^{4+m_3\beta} - C_2[A_1(5+m_2\beta-m_1\beta)+\\
&A_2]b^{4+\beta+(m_2-m_1)\beta} - C_3[A_1(3+m_2\beta)+A_2]b^{2+(1+m_2)\beta} -\\
&A_3b^{(1-m_1+m_2)\beta} - A_4b^{(1+m_2)\beta} - P_o
\end{aligned} \tag{28}$$

RESULTS AND DISCUSSION

Here, the exact solution obtained from the previous part of the study will be followed up by an example. A hollow thick-walled cylinder with the internal radius of a = 100 cm, the outer radius of b = 115 cm is considered, which is rotating around the z-axis at the constant angular velocity of $\omega = 10$ rad sec^{-1}. The modulus of elasticity, the thermal coefficient of expansion and density at internal radius, respectively, have the values of $E_i = 200$ GPa, $\alpha_i = 11.7(10^{-6})$/°C, $\rho_i = 7810$ kg m^{-3}. It is also assumed that

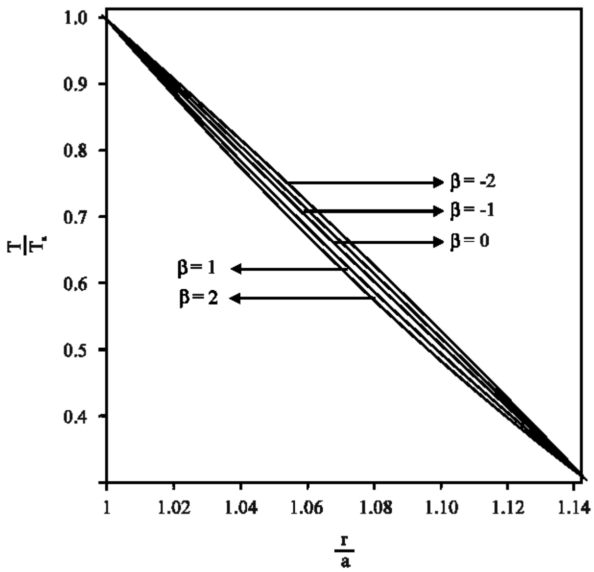

Fig. 1: Distribution of temperature versus radius

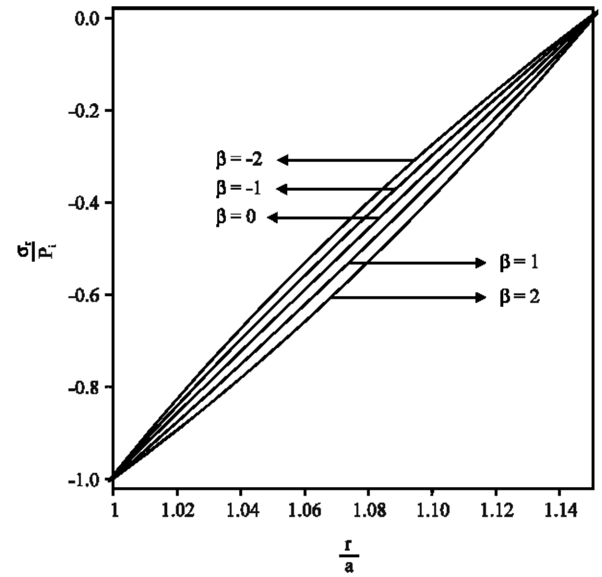

Fig. 3: Distribution of radial stress versus radius

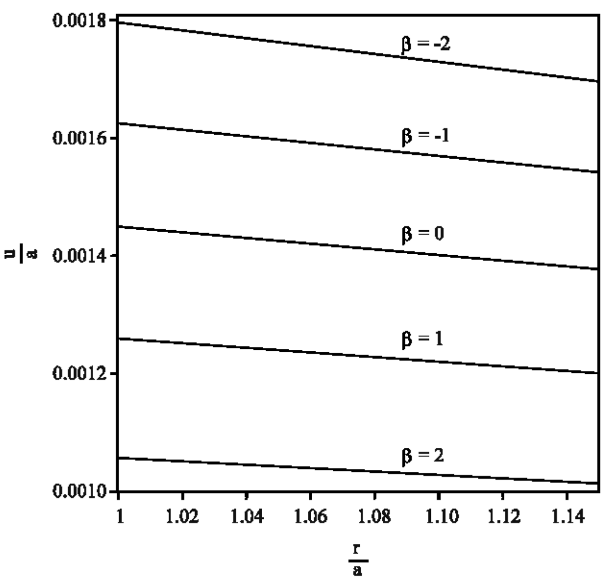

Fig. 2: Distribution of radial displacement versus radius

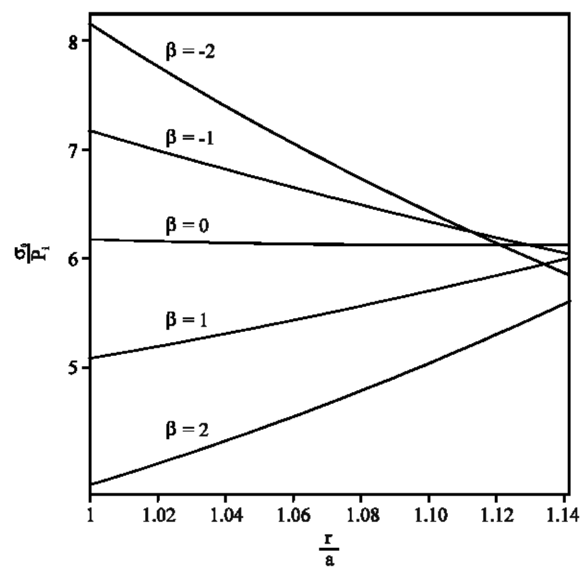

Fig. 4: Distribution of circumferential stress versus radius

the Poisson's ratio, υ, has a constant value of 0.33. For boundary conditions, the internal and external surfaces of the cylinder are taken to be under the pressure of P_i = 40 MPa, P_o = 0 MPa, respectively. In addition, the temperature of the internal and the external surfaces are considered constant as T_a = 15°C, T_b = 5°C. Furthermore, it is assumed that $m_1 = m_2 = m_3 = 1$. In Fig. 1-7, the changes are shown according to the different values for β in the radial direction within the range of $-2 \leq \beta \leq 2$.

The variations of the temperature in the radial direction for different values of the β are shown in Fig. 1. The Fig. 1 shows that with increasing β, the temperature decreases. The radial displacement along the radius is shown in Fig. 2. There is a decrease in the value of the radial displacement as β increases. Figure 3 and 4 show the distribution radial and circumferential stresses in the radial direction. As β is increases, so does the magnitude of the radial stress. For $\beta>0$, the circumferential stress increases as the radius increases whereas for $\beta<0$ the circumferential stress along the radius decreases. Given that $\beta = 0$, the circumferential stress remains nearly constant along the radius. For the purpose of

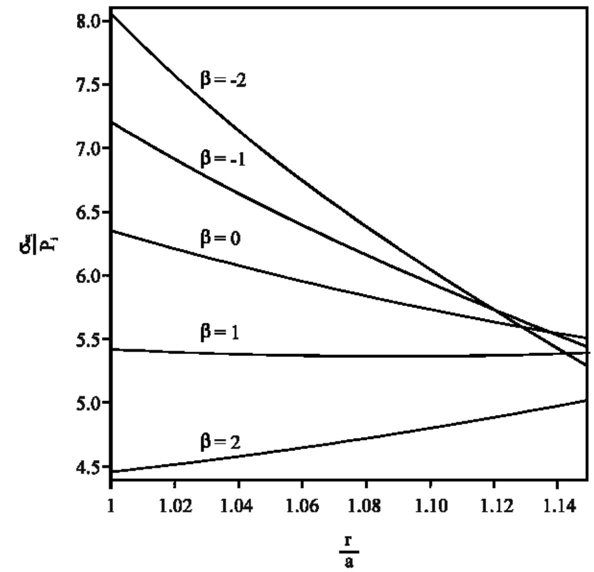

Fig. 5: Distribution of equivalent stress versus radius for b/a = 1.15

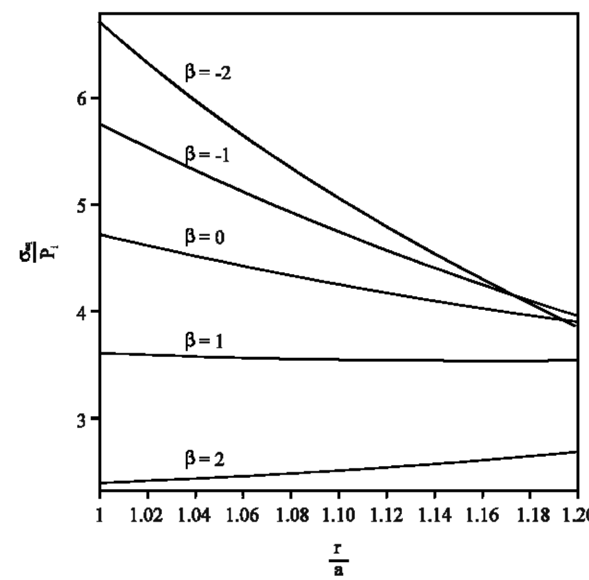

Fig. 6: Distribution of equivalent stress versus radius for b/a = 1.2

studying the stress distribution along the cylinder radius, the Von mises effective stress $\sigma_{eq} = \frac{1}{\sqrt{2}}[(\sigma_r - \sigma_\theta)^2 + (\sigma_\theta - \sigma_z)^2 + (\sigma_r - \sigma_z)^2]^{0.5}$ is plotted in the radial direction for various and b/a and β values. Figure 5-7 are plotted for b/a = 1.15, 1.2 and 1.25, respectively. It must be noted from Fig. 5 that for β = 1 the effective stress remains almost uniform along the radius of the cylinder. As Fig. 6 and 7 suggest, with increasing

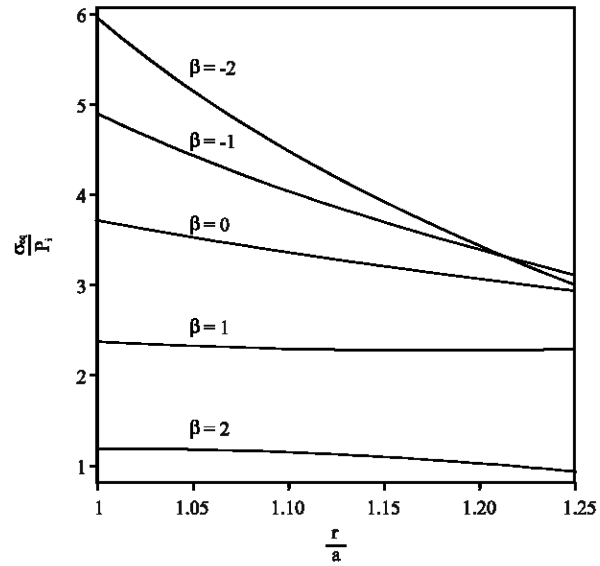

Fig. 7: Distribution of equivalent stress versus radius for b/a = 1.25

b/a ratios, the curves related to β = 1 produce an almost uniform distribution for the effective stress.

CONCLUSION

In the present study, by the application of the elasticity theory, thermal stresses are obtained for an FG rotating cylinder. It is assumed that the material properties change as graded in radial direction to a power law function. The values of ξ, η and γ are taken as coefficients of the value of β. Change in value of β may or may not bring about changes in these coefficients, depending on the value of m_1, m_2 and m_3. Numerical results show that value of β has a great effect on the thermoelastic stresses. This solution could be the most general solution for such problems. The findings of this study can be used to find optimum values of β, ξ, η and γ which can, in turn, be used to minimize and uniform stresses and prevent from yielding.

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