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Applications of Noether's Theorem to the Equations of Motion of Inclined Sagged Cables

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Abstract: Noether's theorem is the principal systematic procedure for finding conservation laws for the complicated systems of differential equations that derived by the variational method. The non-linear equations of motion of inclined sagged cables are studied by Lie group method. The infinitesimal generators, which are the main tool for this theory, are calculated. After applying the infinitesimal criterion of invariance to linear combination of these generators, the family of all variational symmetries are derived and then used to find conservation laws for the equations of motion of inclined sagged cables by using the Noether's theorem.

Key words: Conservation laws, Lie group theory, infinitesimal generators, variational symmetries, conservation of energy

INTRODUCTION

In the study of differential equations, conservation laws have many significant uses. An important problem is how to find conservation laws for given differential equations. Emmy Noether, 1918 was the first to combine the methods of variational calculus with the theory of Lie groups and to formulate a general approach for constructing conservation laws for Euler-Lagrange equations when their variational symmetries are known. "Noether proved the remarkable result that for systems arising from a variational principle, every conservation law of the system comes from a corresponding symmetry property. For example, invariance of a variational principle under a group of time translations implies the conservation of energy for the solutions of the associated Euler-Lagrange equations and invariance under a group of spatial translations implies conservation of momentum (Olver, 1993). A thorough study of the Noether's theorem with many references can be found in Olver (1993). Numerous examples and applications of the Noether's theorem are presented in Ibragimov (1994). Algorithm of finding an admitted Lie group is demonstrated in Meleshko (2005). Chatanin *et al.* (2008) applied Lie group theory to the non-linear equations of motion of inclined unsagged cables to construct invariant solutions to the problem. In this study, the Lie group theory is applied to the non-linear equations of motion of inclined sagged cables. By using the Noether's theorem, conservation of energy for the solutions of the associated Euler-Lagrange equations is obtained.

MATERIALS AND METHODS

Physical model and equations of motion: The configuration $(\tilde{x},\tilde{y})$ in the global coordinates (X,Y) describes the position of the inclined sagged cable as shown in Fig. 1. The angle θ shows the inclination of the cable with respect to X axis. After the disturbance of an external excitation, the cable configuration is changed into the dynamic configuration which is displaced from the initial configuration. The displacement vectors in X and Y directions are represented by $\tilde{u}(\tilde{x},\tilde{t})$ and $\tilde{v}(\tilde{x},\tilde{t})$, respectively. The horizontal span X_H is fixed and the cable's vertical span Y_H is varied to attain specified values of θ.

In deriving the equation of motion by the virtual work-energy principle the cable is considered to be perfectly flexible, homogeneous, linearly elastic with negligible torsional, bending and shear rigidities. Therefore, the strain energy is due only to stretching of the cable axis. The total strain of the cable at the displaced

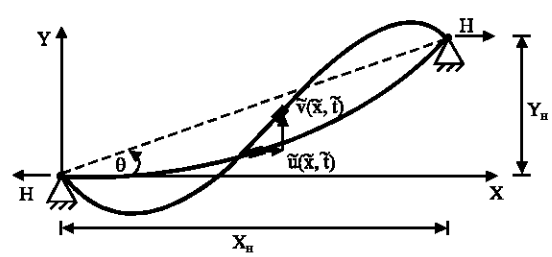

Fig. 1: Configurations of an inclined sagged cable

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state, with assumption of moderately large vibration amplitudes, can be expressed as follows:

$$\bar{e} = e + \left(1+\tilde{y}'^2\right)^{-1}\left[\tilde{u}' + \tilde{y}'\tilde{v}' + \frac{1}{2}\left(\tilde{u}'^2 + \tilde{v}'^2\right)\right] \qquad (1)$$

In this study, the initial strain e is assumed to have a small value and is neglected. The prime (') is used to denote differentiation with respect to $\tilde{x}$. From Srinil *et al.* (2004) and Rega *et al.* (2004), the governing equations of motion are:

$$\rho u_{tt} - \left\{ u_{xx} + \frac{\bar{E}}{\rho^3}\begin{pmatrix} u_{xx} + y_x v_{xx} + y_{xx} v_x + 3u_x u_{xx} + y_x u_x v_{xx} + \\ y_x u_{xx} v_x + y_{xx} u_x v_x + v_x v_{xx} \end{pmatrix} \right\} = 0 \qquad (2)$$

$$\rho v_{tt} - \left\{ v_{xx} + \frac{\bar{E}}{\rho^3}\begin{bmatrix} y_x u_{xx} + y_{xx} u_x + y_x^2 v_{xx} + 2y_x y_{xx} v_x + u_x v_{xx} \\ + u_{xx} v_x + 3y_x v_x v_{xx} + \frac{3}{2} y_{xx} v_x^2 + y_x u_x u_{xx} + \frac{1}{2} y_{xx} u_x^2 \end{bmatrix} \right\} = 0 \qquad (3)$$

Where:

$$x = \frac{\tilde{x}}{X_H}, \quad y = \frac{\tilde{y}}{X_H}, \quad u = \frac{\tilde{u}}{X_H}, \quad v = \frac{\tilde{v}}{X_H},$$

$$t = \frac{\tilde{t}}{X_H}\sqrt{\frac{gH}{w_c}}, \quad \rho = \sqrt{1+\tilde{y}'^2}, \; \bar{E} = \frac{EA}{H},$$

EA is the cable axial stiffness,

$$H = \frac{T_s}{\rho}$$

is the horizontal component of the cable static tension T_s, w_c is the cable weight per unit length, g is the gravity constant. Here, the curve that defines shape of the cable is

$$\tilde{y} = \frac{Y_H}{X_H}\tilde{x}^2$$

This non-linear system describes the coupled longitudinal and vertical displacement's dynamics and is valid also for slightly sagged horizontal cables if one assumes $T_s \approx H$ rendering $\rho = 1$. It contains quadratic and cubic nonlinear terms due to the cable's axial stretching even in the absence of initial sag (taut string case). A method for solving the non-linear equations of motion for horizontal and inclined elastic sagged cables by using a numerical technique is demonstrated in Srinil *et al.* (2005).

Calculation of infinitesimal generators: An infinitesimal generator X for the problem is written in the following form:

$$X = \alpha(x,t,u,v)\frac{\partial}{\partial x} + \beta(x,t,u,v)\frac{\partial}{\partial t} + \gamma(x,t,u,v)\frac{\partial}{\partial u} + \eta(x,t,u,v)\frac{\partial}{\partial v} \qquad (4)$$

The main aim is to determine all functions α, β, γ and η that correspond to the one-parameter symmetry groups of the Eq. 2 and 3. Since the governing equations form a system of second order PDE, the second prolongation of the infinitesimal generator is needed. The second prolongation of X is given by:

$$pr^{(2)}X = X + \gamma^x\frac{\partial}{\partial u_x} + \gamma^t\frac{\partial}{\partial u_t} + \gamma^{xx}\frac{\partial}{\partial u_{xx}} + \gamma^{xt}\frac{\partial}{\partial u_{xt}} + \gamma^{tt}\frac{\partial}{\partial u_{tt}} + \eta^x\frac{\partial}{\partial v_x} + \eta^t\frac{\partial}{\partial v_t} + \eta^{xx}\frac{\partial}{\partial v_{xx}} + \eta^{xt}\frac{\partial}{\partial v_{xt}} + \eta^{tt}\frac{\partial}{\partial v_{tt}} \qquad (5)$$

Where:

$$\begin{aligned}
\gamma^x &= D_x(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xx} + \beta u_{xt}, & \eta^x &= D_x(\eta - \alpha v_x - \beta v_t) + \alpha v_{xx} + \beta v_{xt} \\
\gamma^{xx} &= D_{xx}(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xxx} + \beta u_{xxt}, & \eta^{xx} &= D_{xx}(\eta - \alpha v_x - \beta v_t) + \alpha v_{xxx} + \beta v_{xxt} \\
\gamma^t &= D_t(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xt} + \beta u_{tt}, & \eta^t &= D_t(\eta - \alpha v_x - \beta v_t) + \alpha v_{xt} + \beta v_{tt} \\
\gamma^{tt} &= D_{tt}(\gamma - \alpha u_x - \beta u_t) + \alpha u_{xtt} + \beta u_{ttt}, & \eta^{tt} &= D_{tt}(\eta - \alpha v_x - \beta v_t) + \alpha v_{xtt} + \beta v_{ttt}
\end{aligned} \qquad (5')$$

and

$$\begin{aligned}
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + u_{xxx}\frac{\partial}{\partial u_{xx}} + u_{xxt}\frac{\partial}{\partial u_{xt}} + u_{xtt}\frac{\partial}{\partial u_{tt}} + \ldots \\
&\quad + v_x\frac{\partial}{\partial v} + v_{xx}\frac{\partial}{\partial v_x} + v_{xt}\frac{\partial}{\partial v_t} + v_{xxx}\frac{\partial}{\partial v_{xx}} + v_{xxt}\frac{\partial}{\partial v_{xt}} + v_{xtt}\frac{\partial}{\partial v_{tt}} + \ldots \\
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + u_{txx}\frac{\partial}{\partial u_{xx}} + u_{txt}\frac{\partial}{\partial u_{xt}} + u_{ttt}\frac{\partial}{\partial u_{tt}} + \ldots \\
&\quad + v_t\frac{\partial}{\partial v} + v_{xt}\frac{\partial}{\partial v_x} + v_{tt}\frac{\partial}{\partial v_t} + v_{txx}\frac{\partial}{\partial v_{xx}} + v_{txt}\frac{\partial}{\partial v_{xt}} + v_{ttt}\frac{\partial}{\partial v_{tt}} + \ldots \\
D_x^2 &= D_x(D_x), \quad D_t^2 = D_t(D_t)
\end{aligned}$$

According to theorem 2.31 Olver (1993):

$$pr^{(2)}X\left[\rho u_{tt} - \left\{ u_{xx} + \frac{\bar{E}}{\rho^3}\begin{pmatrix} u_{xx} + y_x v_{xx} + y_{xx} v_x + 3u_x u_{xx} + y_x u_x v_{xx} \\ + y_x u_{xx} v_x + y_{xx} u_x v_x + v_x v_{xx} \end{pmatrix}\right\}\right]\Bigg|_{F=0} = 0 \qquad (6)$$

$$pr^{(2)}X\left[\rho v_{tt} - \left\{ v_{xx} + \frac{\bar{E}}{\rho^3}\begin{pmatrix} y_x u_{xx} + y_{xx} u_x + y_x^2 v_{xx} + 2y_x y_{xx} v_x \\ + u_x v_{xx} + u_{xx} v_x + 3y_x v_x v_{xx} + \\ \frac{3}{2} y_{xx} v_x^2 + y_x u_x u_{xx} + \frac{1}{2} y_{xx} u_x^2 \end{pmatrix}\right\}\right]\Bigg|_{F=0} = 0 \qquad (7)$$

where, the notation $|_{F=0}$ means that the second prolongation of X is applied to the solutions of Eq. 2 and 3. Substituting expressions (5,5') and (2-3) into Eq. 6 and

7 and then equating the coefficients of all independent monomials to zero, we obtain the determining equations. After solving the determining equations, we get the functions $\alpha = 0$, $\beta = k_1$, $\gamma = k_2t+k_3$, $\eta = k_4t+k_5$, where $k_1,\dots,k_5$ are arbitrary constants. Thus, the infinitesimal generators have the form:

$$X = k_1\frac{\partial}{\partial t} + (k_2t + k_3)\frac{\partial}{\partial u} + (k_4t + k_5)\frac{\partial}{\partial v} \quad (8)$$

The infinitesimal generator contains five arbitrary constants. Consequently, the Lie algebra derived from the governing equations is spanned by the following five linearly independent generators:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = t\frac{\partial}{\partial u}, \quad X_3 = \frac{\partial}{\partial u}, \quad X_4 = t\frac{\partial}{\partial v}, \quad X_5 = \frac{\partial}{\partial v}$$

Conservation laws: Consider a system of the second order partial differential equations:

$$F_i(x, u, u_{(1)}, u_{(2)}) = 0 \quad (9)$$

where, $x = (x^1, x^2,\dots,x^n)$ are the independent variables, $u = (u^1, u^2,\dots,u^m)$ are the dependent variables, $u_{(1)} = \{u_i^\alpha\}$ and $u_{(2)} = \{u_{ij}^\alpha\}$ are the first and second order derivatives, respectively.

Definition: Equation

$$D_i[C^i(x,u,u_{(1)})] = 0 \quad (10)$$

is called a conservation law for Eq. 9 if it is satisfied by all solutions u (x) of Eq. 9. The vector $C = (C^1, C^2,\dots,C^n)$ is called a conserved vector and D_i is the total derivative with respect to x^i. Consider a variational integral:

$$\int L(x,u,u_{(1)})dx \quad (11)$$

and its Euler-Lagrange equations:

$$\frac{\delta L}{\delta u^\alpha} \equiv \frac{\partial L}{\partial u^\alpha} - D_i\left(\frac{\partial L}{\partial u_i^\alpha}\right) = 0, \quad \alpha = 1,\dots,m \quad (12)$$

where, the Lagrangian L of the system involves the independent variables $x = (x^1, x^2,\dots,x^n)$, the dependent variables $u = (u^1,u^2,\dots,u^m)$ and the first-order derivatives $u_{(1)} = \{u_i^\alpha\}$ of u with respect to x^i.

Infinitesimal criterion of invariance: The following condition will be necessary and sufficient for a connected group of transformations to be a symmetry group of the variational problem.

Theorem: A connected group of transformations G is a variational symmetry group of the variational integral (11) if and only if:

$$pr^{(1)}X(L)+LDi(\xi^i) = 0 \quad (13)$$

for every infinitesimal generators:

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \quad (14)$$

Noether's theorem: (Ibragimov, 2006) Let the variational integral (11) be invariant under the group with the generator (14). In other words, let the invariance test (13) be satisfied. Then the vector field $C = (C^1,C^2,\dots,C^n)$ defined by:

$$C^i = \xi^i L + \left(\eta^\alpha - \xi^j u_j^\alpha\right)\frac{\partial L}{\partial u_i^\alpha}, \quad i = 1,\dots,n \quad (15)$$

is a conserved vector for Eq. 12, i.e., C satisfies the conservation law $D_i(C^i) = 0$.

RESULTS AND DISCUSSION

Since the equations of motion of the inclined sagged cables are derived by the variational method presented in Srinil *et al.* (2004), we can apply Noether's theorem to this system to obtain conservation laws for the system as follows:

The Lagrangian of the equations of motion of the inclined sagged cables is given by:

$$L = \kappa\left[\frac{1}{2}m\left(u_t^2 + v_t^2\right) - \frac{1}{2}EA\left(\frac{1}{\kappa}\sqrt{(1+u_x)^2 + (y_x + v_x)^2} - 1\right)^2 + w_c v\right], \quad (16)$$

Where:

$$\kappa = \frac{\sqrt{1+y_x^2}}{1+\varepsilon_0} \text{ and } y_x = 2\frac{Y_H}{X_H}x, \text{ m}$$

m is mass per unit length and ε_0 is the initial static strain of the cable centerline. The infinitesimal generators for the problem have the form:

$$X = k_1\frac{\partial}{\partial t} + (k_2t + k_3)\frac{\partial}{\partial u} + (k_4t + k_5)\frac{\partial}{\partial v}$$

The Lie algebra corresponding to the governing equations is spanned by the following five linearly independent generators:

$$X_1 = \frac{\partial}{\partial t},\quad X_2 = t\frac{\partial}{\partial u},\quad X_3 = \frac{\partial}{\partial u},\quad X_4 = t\frac{\partial}{\partial v},\quad X_5 = \frac{\partial}{\partial v}$$

The conservation law for the equations of motion of the inclined sagged cables is:

$$D_t\left(C^t\right) + D_x\left(C^x\right) = 0$$

Let the generator X be a linear combination of five generators $X_1, X_2, ... X_5$.

$$X = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4 + \lambda_5 X_5 = \lambda_1\frac{\partial}{\partial t} + \left(\lambda_2 t + \lambda_3\right)\frac{\partial}{\partial u} + \left(\lambda_4 t + \lambda_5\right)\frac{\partial}{\partial v}$$

The first prolongation of the generator X is given by:

$$pr^{(1)}X == \lambda_1\frac{\partial}{\partial t} + \left(\lambda_2 t + \lambda_3\right)\frac{\partial}{\partial u} + \left(\lambda_4 t + \lambda_5\right)\frac{\partial}{\partial v} + \lambda_2\frac{\partial}{\partial u_t} + \lambda_4\frac{\partial}{\partial v_t}$$

Applying the infinitesimal criterion of invariance (13) to X, we gets:

$$pr^{(1)}X(L) + L\left(D_x\xi^x + D_t\xi^t\right) = pr^{(1)}X(L) = 0$$

This gives:

$$\lambda_1\frac{\partial L}{\partial t} + \left(\lambda_2 t + \lambda_3\right)\frac{\partial L}{\partial u} + \left(\lambda_4 t + \lambda_5\right)\frac{\partial L}{\partial v} + \lambda_2\frac{\partial L}{\partial u_t}$$
$$+\lambda_4\frac{\partial L}{\partial v_t} = \lambda_4 w_c \kappa t + \lambda_5 w_c \kappa + \lambda_2 m\kappa u_t + \lambda_4 m\kappa v_t = 0$$

It follows that $\lambda_2 = \lambda_4 = \lambda_5 = 0$ and we get that the family of all variational symmetries is given by $\lambda_1 X_1 + \lambda_3 X_3$, where, λ_1 and λ_3 are arbitrary constants.

For the variational symmetries

$$\lambda_1 X_1 + \lambda_3 X_3 = \lambda_1\frac{\partial}{\partial t} + \lambda_3\frac{\partial}{\partial u}$$

the associated conserved vectors are given by:

$$C^x = \left(\lambda_3 - \lambda_1 u_t\right)\left[-EA\left(1+u_x\right)\left(\frac{1}{\kappa} - \frac{1}{\sqrt{\left(1+u_x\right)^2 + \left(y_x + v_x\right)^2}}\right)\right]$$
$$+\lambda_1 v_t\left[EA\left(y_x + v_x\right)\left(\frac{1}{\kappa} - \frac{1}{\sqrt{\left(1+u_x\right)^2 + \left(y_x + v_x\right)^2}}\right)\right]$$
$$C^t = \lambda_1\kappa\left[\frac{1}{2}m\left(u_t^2 + v_t^2\right) - \frac{1}{2}EA\left(\frac{1}{\kappa}\sqrt{\left(1+u_x\right)^2 + \left(y_x + v_x\right)^2} - 1\right)^2 + w_c v\right]$$
$$+\left(\lambda_3 - \lambda_1 u_t\right)\left(m\kappa u_t\right) - \left(\lambda_1 v_t\right)\left(m\kappa v_t\right)$$

If we let $\lambda_3 = 0$ and $\lambda_1 = 1$, the variational symmetry of translation in time

$$X = \frac{\partial}{\partial t}$$

are obtained. This gives the law of conservation of energy, $D_t(C^t)+D_x(C^x) = 0$, where, the associated conserved vectors are:

$$C^x = EA\left(\frac{1}{\kappa} - \frac{1}{\sqrt{\left(1+u_x\right)^2 + \left(y_x + v_x\right)^2}}\right)\left[-u_t - u_x u_t + y_x v_t + v_x v_t\right],$$
$$C^t = \kappa\left[\frac{1}{2}m\left(u_t^2 + v_t^2\right) - \frac{1}{2}EA\left(\frac{1}{\kappa}\sqrt{\left(1+u_x\right)^2 + \left(y_x + v_x\right)^2} - 1\right)^2 + w_c v\right] - m\kappa\left(u_t^2 + v_t^2\right)$$

CONCLUSION

Noether's theorem is a powerful method for finding conservation laws for complicated systems of differential equations arising from variational principles. The equations of motion of inclined sagged cables were derived by the variational method in Srinil *et al.*(2004). Thus, Noether's theorem can be applied to obtain the conservation laws, which play an important role in the analysis of basic properties of the solutions of this system. For each variational symmetry, there is a corresponding conservation law. Applying the infinitesimal criterion of invariance, the researchers find all variational symmetries. In particular, the variational symmetry of translation in time is obtained and yields the conservation of energy law.

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