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Variational Iteration Method and Homotopy-Perturbation Method for Solving Burgers Equation in Fluid Dynamics

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Abstract: Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM) are employed to approximate the solution of the Burgers equation which is a one-dimensional non-linear partial differential equation in fluid dynamics. The explicit solutions obtained were compared with the exact solutions. While the exact solution was not available for viscosity smaller than 0.01, it was shown that mathematical structure of the equation for the obtained explicit solutions did not decay. The results reveal that the HPM and VIM are very effective, convenient and quite accurate to systems of partial differential equations. It is predicted that the HPM and VIM can be found widely applicable in engineering.

Key words: Burgers equation, Homotopy-Perturbation Method (HPM), Variational Iteration Method (VIM), fluid dynamics

INTRODUCTION

The one-dimensional non-linear partial equation

$$\frac{\partial}{\partial t}u(x,t)+u(x,t)\frac{\partial}{\partial x}u(x,t)=\varepsilon\frac{\partial^2}{\partial x^2}u(x,t), \quad (1)$$

is known as Burgers equation. Burgers model of turbulence is a very important fluid dynamics model and the study of this model and the theory of shock waves have been considered by many authors both for conceptual understanding of a class of physical flows and for testing various numerical methods. The distinctive feature of Eq. 1 is that it is the simplest mathematical formulation of the competition between non-linear advection and the viscous diffusion. It contains the simplest form of non-linear advection term $u\ u_x$ and dissipation term εu_{xx} where $\varepsilon = 1/Re$ (ε: kinematics viscosity and Re: Reynolds number) for simulating the physical phenomena of wave motion and thus deteermines the behavior of the solution. The mathematical properties of Eq. 1 have been studied by Cole (1951). Particularly, the detailed relationship between Eq. 1 and both turbulence theory and shock wave theory were described by Cole. He also gave an exact solution of Burgers equation. Benton and Platzman (1972) have demonstrated about 35 distinct exact solutions of Burgers-like equations and their classifications. It is well known that the exact solution of Burgers equation can only be computed for restricted values of ε which represent the kinematics viscosity of the fluid motion. Because of this fact, various numerical methods were employed to obtain the solution of Burgers' equation with small ε values.

Many numerical solutions for Eq. 1 have been adopted over the years. Finite element techniques have been employed frequently. For example, Varoglu and Finn (1980) presented an isoparametric spaceBtime finite-element approach for solving Burgers equation, utilizing the hyperbolic differential equation associated with Burgers equation. Another approach which has been used by Caldwell *et al.* (1981) is the finite-element method such that by altering the size of the element at each stage using information from the previous steps. Caldwell *et al.* (1980) give an indication of how complementary variational principles can be applied to Burgers equation. Later, Saunders *et al.* (1984) have demonstrated how a variational-iterative scheme based on complementary variational principles can be applied to non-linear partial differential equations and the test problem chosen is the steady-state version of Burgers equation Özis and Özdes (1996) applied a direct variational method to generate limited form of the solution of Burgers equation. Özis *et al.* (2003) applied a simple finite-element approach with linear elements to Burgers equation reduced by Hopf-Cole transformation. Aksan and Özdes (2004) have reduced Burgers equation to the system of non-linear ordinary differential equations by discretization in time and solved each non-linear ordinary differential equation by Galerkin method in each time step. As they claimed, for

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moderately small kinematics viscosity, their approach can provide high accuracy while using a small number of grid points (i.e., N = 5) and this makes the approach very economical computational wise. In the case where the kinematics viscosity is small enough i.e., $\varepsilon = 0.0001$, the exact solution is not available and a discrepancy exists in the literature, their results clarify the behavior of the solution for small times, i.e., $T = t_{max} \leq 0.15$. Also it is demonstrated that the parabolic structure of the equation decayed for $t_{max} = 0.5$. And finally, Aksan *et al*. (2006) applied least squares method to solution this equation.

In this study, again, the reduced Burgers equation is solved by He's homotopy perturbation method and variational iteration method. The Variational Iteration Method (VIM) has been previously employed by Abu and Soliman (2005) to obtain a solution to Burgers equation in the form of an infinite power series. It is well-known that the HPM and VIM converge very fast to the results. Moreover, contrary to the conventional methods which require the initial and boundary conditions, the HPM and VIM provide an analytical solution by using only the initial conditions. The boundary conditions can be used only to justify the obtained result. In the present study, it is aimed to establish the existence of the solution first using the Homotopy Perturbation Method (HPM) (He, 1999b, 2006; Zhang and He, 2006) and then by the variational iteration method (VIM) (He, 1999a, 2000; Momani and Abuasad, 2006; Ganji and Sadighi, 2007; Ganji *et al*., 2007; Sweilam and Khader, 2007; Bildik and Konuralp, 2006). A comparison will be made between the two methods to show that both methods are equally able to arrive at exact solutions of Burgers' equation. Numerical examples are also presented for moderate values of ε since the exact solution is not available for lower values.

BASIC IDEA OF HE'S HOMOTOPY-PERTURBATION METHOD

To illustrate the basic ideas of HPM, we consider the following nonlinear differential equation :

$$A(u)-f(r) = 0, \quad r \in \Omega \tag{2}$$

with the boundary conditions of

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{3}$$

Where, A, B, f(r) and Γ are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively.

Generally speaking the operator A can be divided into a linear part L and a nonlinear part N(u). Eq. 2 can therefore, be rewritten as:

$$L(u) + N(u) - f(r) = 0, \tag{4}$$

By the Homotopy technique, we construct a homotopy $v(r,p)$: $\Omega \times [0,1] \rightarrow R$, which satisfies:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], \; r \in \Omega, \tag{5}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{6}$$

Where, $p \in [0, 1]$ is an embedding parameter, while u_0 is an initial approximation of Eq. 2, which satisfies the boundary conditions. Obviously, from Eq. 5 and 6 we will have:

$$H(v,0) = L(v) - L(u_0) = 0, \tag{7}$$

$$H(v,1) = A(v) - f(r) = 0, \tag{8}$$

The changing process of p from zero to unity is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, while $L(v)$-$L(u_0)$ and $A(v)$-$f(r)$ are called homotopy.

According to the HPM, we can first use the embedding parameter p as a small parameter and assume that the solution of Eq. 5 and 6 can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{9}$$

Setting p = 1 yields in the approximate solution of Eq. 2 to:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{10}$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantage.

The series (10) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $A(v)$. Moreover, He (1999a) made the following suggestions:

- The second derivative of $N(v)$ with respect to v must be small because the parameter may be relatively large, i.e., $p \rightarrow 1$.

- The norm of $L^{-1}\frac{\partial N}{\partial v}$ must be smaller than one so that the series converges.

BASIC IDEA OF VARIATIONAL ITERATION METHOD

To clarify the basic ideas of VIM (He, 1999a, 2000; Momani and Abuasad, 2006; Ganji and Sadighi, 2006; Sweilam and Khader, 2007; Bildik and Konuralp, 2006) we consider the following differential equation:

$$Lu+Nu = g(t), \tag{11}$$

Where, L is a linear operator, N is a nonlinear operator and g(t) is an inhomogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t)=u_n(t)+\int_0^t \lambda\left(Lu_n(\tau)+N\tilde{u}_n(\tau)-g(\tau)\right)d\tau \tag{12}$$

Where, λ is a general lagrangian multiplier which can be identified optimally via the variational theory. The subscript n indicates the nth approximation and u_n is considered as a restricted variation, i.e., $\delta\tilde{u}_n = 0$.

HPM APPLICATIONS IN SOLVING OF BURGERS EQUATION

Let us consider Burgers Eq. 1 with the following initial and boundary conditions:

$$u(x,0) = \sin \pi x \quad \text{in } \Omega, \tag{13}$$

$$u(0,t) = u(1,t) = 0, \quad t > 0, \tag{14}$$

Where, Ω, is the interval (0, 1).

The exact solution of Eq. 1 with conditions 13 and 14 was given by Cole (1951) as:

$$u(x,t) = 2\pi\varepsilon\frac{\sum_{n=1}^{\infty} a_n e^{-n^2\pi^2\varepsilon t} n \sin n\pi x}{a_0 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2\varepsilon t} n \cos n\pi x},$$

Where:

$$a_0 = \int_0^1 \exp\left[-(2\pi\varepsilon)^{-1}(1-\cos\pi x)\right]dx$$

and

$$a_n = 2\int_0^1 \exp\left[-(2pe)^{-1}(1-\cos px)\right]\cos npx\, dx, \ n \ge 1.$$

Upon separating the linear and nonlinear parts of the Eq. 1, we apply homotopy-perturbation to Eq. 5. A homotopy can be constructed as follows:

$$(1-p)\left(\frac{\partial}{\partial t}v(x,t)-\frac{\partial}{\partial t}v_0(x,t)\right)+p\left(\frac{\partial}{\partial t}v(x,t)+v(x,t)\frac{\partial}{\partial x}v(x,t)-\varepsilon\frac{\partial^2}{\partial x^2}v(x,t)\right)=0, \quad p\in[0,1]. \tag{15}$$

Substituting the value of v from Eq. 9 into 15 and rearranging based on powers of p-terms yields:

$$p^0 : \frac{\partial}{\partial t}v_0(x,t) = 0, \tag{16}$$

$$p^1 : (\frac{\partial}{\partial t}v_1(x,t)) - \varepsilon(\frac{\partial^2}{\partial x^2}v_0(x,t)) + v_0(x,t)(\frac{\partial}{\partial x}v_0(x,t)) = 0, \tag{17}$$

$$p^2 : (\frac{\partial}{\partial t}v_2(x,t)) - \varepsilon(\frac{\partial^2}{\partial x^2}v_1(x,t)) + v_1(x,t)(\frac{\partial}{\partial x}v_0(x,t)) + v_0(x,t)(\frac{\partial}{\partial x}v_1(x,t)) = 0, \tag{18}$$

$$p^3 : v_1(x,t)(\frac{\partial}{\partial x}v_1(x,t)) - \varepsilon(\frac{\partial^2}{\partial x^2}v_2(x,t)) + v_2(x,t)(\frac{\partial}{\partial x}v_0(x,t)) + v_0(x,t)(\frac{\partial}{\partial x}v_2(x,t)) + \frac{\partial}{\partial t}v_3(x,t) = 0, \tag{19}$$

With the following conditions

$$\begin{aligned} &v_0(x,0) = \sin(\pi x), \ \ v_0(0,t) = 0, \ \ v_0(1,t) = 0, \\ &v_i(x,0) = 0, \ \ v_i(0,t) = 0, \ \ v_i(1,t) = 0, \ \ i = 1,2,... \end{aligned} \tag{20}$$

The solutions of Eq. 16-19 by using the conditions (20), may be re-written as follows:

$$v_0(x,t) = \sin(\pi x), \tag{21}$$

$$v_1(x,t) = -t\varepsilon\sin(\pi x)\pi^2 - \frac{1}{2}t\pi\sin(2\pi x), \tag{22}$$

$$v_2(x,t) = \frac{1}{8}\pi^2t^2(4\varepsilon^2\sin(\pi x)\pi^2 + 12\varepsilon\pi\sin(2\pi x) + 3\sin(3\pi x) - \sin(3\pi x)), \tag{23}$$

$$v_3(x,t) = -\frac{1}{24}\pi^3t^3(56\varepsilon^2\pi^2\sin(2\pi x) + 51\varepsilon\pi\sin(3\pi x) - 9\varepsilon\pi\sin(\pi x) + 8\sin(4\pi x) - 4\sin(2\pi x) + 4\varepsilon^3\pi^3\sin(\pi x)), \tag{24}$$

Similarly, the other components were obtained using the maple software package.

Substituting Eq. 21-24 into 10, then re-written as follows:

$$u(x,t) = \sin(\pi x) - t\varepsilon\sin(\pi x)\pi^2 - \frac{1}{2}t\pi\sin(2\pi x) + \frac{1}{8}\pi^2t^2(4\varepsilon^2\sin(\pi x)\pi^2 + 12\varepsilon\pi\sin(2\pi x) + 3\sin(3\pi x) - \sin(3\pi x)) - \frac{1}{24}\pi^3t^3(56\varepsilon^2\pi^2\sin(2\pi x) + 51\varepsilon\pi\sin(3\pi x) - 9\varepsilon\pi\sin(\pi x) + 8\sin(4\pi x) - 4\sin(2\pi x) + 4\varepsilon^3\pi^3\sin(\pi x)) \tag{25}$$

It is noteworthly that this exact solution (Cole, 1951) is obtained by using only the initial conditions. Moreover, the solution can be used to justify the given boundry conditions.

APPLICATION OF VARIATIONAL ITERATION METHOD

To solve the Eq. 1 by means of VIM, one can construct the following correction functional:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\left((\frac{\partial u_n(x,\tau)}{\partial \tau}) + u_n(x,\tau)(\frac{\partial u_n(x,\tau)}{\partial x}) - \varepsilon \frac{\partial^2 u_n(x,\tau)}{\partial x^2} \right) d\tau \tag{26}$$

Its stationary conditions can be obtained as follows:

$$\begin{aligned} 1 - \lambda'\big|_{\tau=t} &= 0 \\ \lambda\big|_{\tau=t} &= 0 \\ \lambda'' &= 0 \end{aligned} \tag{27}$$

We obtain the lagrangian multiplier:

$$\lambda = -1 \tag{28}$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left((\frac{\partial u_n(x,\tau)}{\partial \tau}) + u_n(x,\tau)(\frac{\partial u_n(x,\tau)}{\partial x}) - \varepsilon \frac{\partial^2 u_n(x,\tau)}{\partial x^2} \right) d\tau \tag{29}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$u_0(x,t) = \sin(\pi x) \tag{30}$$

Using the above variational formula (29), we have:

$$u_1(x,t) = u_0(x,t) - \int_0^t \left((\frac{\partial u_0(x,\tau)}{\partial \tau}) + u_0(x,\tau)(\frac{\partial u_0(x,\tau)}{\partial x}) - \varepsilon \frac{\partial^2 u_0(x,\tau)}{\partial x^2} \right) d\tau \tag{31}$$

Substituting Eq. 30 into 31 and after simplifications, we have:

$$u_1(x,t) = \sin(\pi x) - \sin(\pi x)\cos(\pi x)\pi t - \varepsilon \sin(\pi x)\pi^2 t \tag{32}$$

In the same way, we obtain u_2 (x,t) as follows:

$$\begin{aligned} u_2(x,t) = {} & \sin(\pi x) - \frac{2}{3}t^3 \sin(\pi x)\cos^3(\pi x)\times\pi^3 + \frac{1}{3}t^3\sin(\pi x)\cos(\pi x)\times\pi^3 - \\ & \varepsilon t^3 \sin(\pi x)\cos^2(\pi x)\times\pi^4 + \frac{1}{3}t^3\varepsilon\sin(\pi x)\pi^4 - \frac{1}{3}t^3\varepsilon^2\sin(\pi x)\cos(\pi x)\times\pi^5 + \\ & t^2\sin(\pi x)\cos^2(\pi x)\pi^2 - \frac{1}{2}t^2\sin(\pi x)\pi^2 + \frac{1}{2}t^2\sin(\pi x)\varepsilon\cos(\pi x)\times\pi^3 + \\ & \frac{1}{2}t^2\cos^2(\pi x)\sin(\pi x)\pi^2 + \frac{1}{2}\pi^3\varepsilon t^2\cos(\pi x)\sin(\pi x) + 2\varepsilon t^2\pi^3\cos(\pi x)\sin(\pi x) + \\ & \frac{1}{2}t^2\varepsilon^2\pi^4\sin(\pi x) - \sin(\pi x)\cos(\pi x)\pi t - \varepsilon\pi^2 t\sin(\pi x) \end{aligned} \tag{33}$$

and so on. In the same way the rest of the components of the iteration formula can be obtained.

COMPARISON OF HPM, VIM AND EXACT SOLUTIONS

In order to demonstrate the adoptability and accuracy of the present approaches, we have applied it to the problem given by Eq. 1 which exact solution exists and is given by Cole (1951) in terms of infinite series. To emphasize the accuracy of the method for moderate size viscosity values, we have given the comparisons with analytical solutions obtained from the infinite series of Cole (1951) for ε = 1 and 0.05. Both Table 1-2 show that solutions are in good agreement with analytical solutions. In the case ε is smaller than 0.01, the exact solution is not available and a discrepancy exists in the literature Also, it is not practical to evaluate the analytical solution at these values due to slow convergence of the infinite series and thus the exact solution in this regime is unknown.

Table 1: Comparison of the HPM solutions obtained for ε = 1 at different times with the exact solutions

	t = 0.001			t = 0.01		
x	Exact	HPM	VIM	Exact	HPM	VIM
0.1	0.30509	0.30509	0.30508	0.27324	0.27325	0.27373
0.2	0.58057	0.58057	0.58056	0.52156	0.52158	0.52233
0.3	0.79962	0.79962	0.79962	0.72185	0.72186	0.72256
0.4	0.94082	0.94082	0.94081	0.85459	0.85458	0.85498
0.5	0.99018	0.99018	0.99017	0.90571	0.90570	0.90571
0.6	0.94261	0.94261	0.94260	0.86833	0.86833	0.86803
0.7	0.80252	0.80252	0.80252	0.74410	0.74410	0.74368
0.8	0.58347	0.58346	0.58346	0.54382	0.54382	0.54346
0.9	0.30688	0.30688	0.30688	0.28700	0.28700	0.28679

Table 2: Comparison of the HPM solutions obtained for ε = 0.05 at different times with the exact solutions

	t = 0.001			t = 0.01		
x	Exact	HPM	VIM	Exact	HPM	VIM
0.1	0.30795	0.30795	0.30794	0.29865	0.29865	0.29865
0.2	0.58601	0.58601	0.58600	0.57044	0.57044	0.57045
0.3	0.80713	0.80713	0.80712	0.79034	0.79034	0.79033
0.4	0.94966	0.94966	0.94966	0.93696	0.93696	0.93694
0.5	0.99950	0.99950	0.99950	0.99460	0.99460	0.99458
0.6	0.95151	0.95151	0.95150	0.95513	0.95513	0.95513
0.7	0.81011	0.81011	0.81010	0.81976	0.81976	0.81976
0.8	0.58899	0.58899	0.58898	0.59988	0.59988	0.59989
0.9	0.30979	0.30979	0.30978	0.31686	0.31686	0.31685

CONCLUSION

In this study, the homotopy perturbation method and variational iteration method have been successfully applied to the Burgers equation with specified initial conditions. The results showed that these method are powerful mathematical tools for solving Burgers equation and very effective, convenient and quite accurate to systems of partial differential equations. They provide more realistic series solutions that converge very rapidly in real physical problems. The obtained results reinforce the conclusions made by many researchers about the efficiency of the HPM and VIM. Therefore these methods can be widely applied to engineering problems.

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