Journal of Applied Sciences

ISSN 1812-5654

Journal of Applied Sciences 8 (2): 383-386, 2008
ISSN 1812-5654

Geometry of Ovals in R^2 in Terms of the Support Function

Kamal Al-Banawi
Department of Mathematics and Statistics, Mu'tah University,
Al-Karak, P.O. Box 7, Jordan

Abstract: We study the geometry of ovals in R^2 using the support function. Some results concerning ovals and curves of constant width in R^2 are simply proved using the support function. Also new formulas concerning curvature, focal points and calculation of the support function itself are established.

Key words: Ovals, support function, curvature, focal points, constant width

INTRODUCTION

We deal here with a smooth closed convex curve in R^2, called an oval, which contains the origin in its interior. The support function is the function that measures the perpendicular distance from the tangent line at a point on the oval to the origin (Eggleston, 1958; Fillmore, 1969; Yaglom and Boltyanskii, 1961). The line from the origin that is perpendicular on that tangent is called the line of the support function of the oval. We assume that our oval is parametrized by θ, the angle between the line of the support function of the oval and the positive x-axis. If the oval is defined by $f(\theta) = (f_1(\theta), f_2(\theta))$, then it is natural to say that, $\theta\in[0, 2\pi]$ and f is regular. It has been proved that if α represents the support function, then

$$f(\theta) = (\alpha\cos\theta - \alpha'\sin\theta, \alpha\sin\theta + \alpha'\cos\theta) \tag{1}$$

The previous formula for f in terms of the support function α is derived using the idea of the envelope of the tangent lines of a smooth closed convex curve in R^2 (Hsiung, 1981; Struik, 1950). Moreover, the orientation of f is naturally counterclockwise. We will use such a formula to derive new formulas concerning the differential geometry of ovals in R^2 for a detailed proof of Eq. 1, (Al-Banawi, 2004).

Curvature and focal points: We start with the following theorem concerning curvature and total curvature of ovals.

Theorem 1: Let $f = (f_1, f_2)$ be an oval in R^2 with α as a support function. Let κ be the curvature of f. Let K be the total curvature of f. Then

$$\kappa = \frac{1}{\alpha + \alpha''} \tag{1a}$$

$$\kappa = \frac{1}{f_1''\cos\theta + f_2''\sin\theta - 2f_1'\sin\theta + 2f_2'\cos\theta} \tag{1b}$$

$$K = 2\pi \tag{1c}$$

Proof:
a: Observe that,

$$f' = (\alpha+\alpha'')(-\sin\theta, \cos\theta)$$

and

$$f'' = (\alpha'+\alpha''')(-\sin\theta, \cos\theta)+(\alpha+\alpha'')(-\cos\theta, -\sin\theta)$$

Thus,

$$\kappa = \frac{|f'\times f''|}{|f'|^3} = \frac{(\alpha+\alpha'')^2}{(\alpha+\alpha'')^3} = \frac{1}{\alpha+\alpha''}$$

b: Now

$$f_1 = \alpha\cos\theta - \alpha'\sin\theta$$

and

$$f_2 = \alpha\sin\theta + \alpha'\cos\theta$$

So

$$\alpha = f_1\cos\theta + f_2\sin\theta \tag{2}$$

Now differentiate Eq. 2 twice and substitute in (a) to get the formula for the curvature in (b).
c: Recall that the total curvature of a curve f_1 on [a, b] is

$$K = \int_a^b \kappa|f'|d\theta$$

Corresponding Author: Kamal Al-Banawi, Department of Mathematics and Statistics, Mu'tah University, Al-Karak, P.O. Box 7, Jordan

Thus,

$$K=\int_0^{2\pi}\frac{1}{\alpha+\alpha''}(\alpha+\alpha'')d\theta=\int_0^{2\pi}d\theta=2\pi$$

Corollary 1: Let f be an oval in R^2 with α as a support function. Then the trace of f is a circle iff

$$\alpha(\theta)=c_1\cos\theta+c_2\sin\theta+c$$

Where, c_1, c_2, c are constants.

Proof: Now if the trace of f is a circle, then $\kappa(\theta)$ is constant, $\forall\theta\in[a, b]$. This is equivalent to $\alpha+\alpha''=c$, $\forall\theta\in[a, b]$ where c is constant. Then solving the last second order differential equation. by method of undetermined coefficients, we have

$$\alpha(\theta)=c_1\cos\theta+c_2\sin\theta+c$$

Now if, $\alpha(\theta)=c_1\cos\theta+c_2\sin\theta+c$, then,

$$\kappa(\theta)=\frac{1}{c},$$
$$\forall\theta\in[a, b]$$

Thus, the trace of f is a circle.
Now recall that the curve of focal points of a curve f is

$$g(\theta)=f(\theta)+\frac{1}{\kappa(\theta)}\nu(\theta) \qquad (3)$$

Where, ν is a unit normal of f. Now substitute in Eq. 3 for f as in Eq. 1,

$$\kappa=\frac{1}{\alpha+\alpha''}$$

and choose a unit normal of the oval f to be $\nu(\theta)=(-\cos\theta, -\sin\theta)$. Then the curve of focal points of f is the essence of the next theorem.

Theorem 2: Let f be an oval in R^2 with α as a support function. Then the curve of focal points of f is

$$g(\theta)=(-\alpha'\sin\theta-\alpha''\cos\theta, \alpha'\cos\theta-\alpha''\sin\theta) \qquad (4)$$

Corollary 2: Let f be an oval in R^2. Let g be the curve of focal points of f. Then, $g(\theta)\neq f(\theta)$, $\forall\theta\in[0, 2\pi]$

Proof: First of all, the regularity of f implies that the curvature of f is bounded, hence, $\alpha(\theta)+\alpha''(\theta)\neq 0$, $\forall\theta\in[0, 2\pi]$. Now if $g(\theta_1)=f(\theta_1)$ for some $\theta_1\in[0, 2\pi]$, then by Eq. 1 and 4, we will have:

$$(\alpha(\theta_1)+\alpha''(\theta_1))\cos\theta_1=0$$

and

$$(\alpha(\theta_1)+\alpha''(\theta_1))\sin\theta_1=0$$

which is true only if $\alpha(\theta_1)+\alpha''(\theta_1)=0$, which contradicts the regularity of f.

Calculation of the support function: It looks very easy to calculate the support function using Eq. 2 Also the support function is the solution of the second order differential equation

$$\alpha+\alpha''=\frac{1}{\kappa}$$

Nevertheless, The next theorem gives a good formula in the language of first order differential equations for calculating α.

Theorem 3. Let f be an oval in R^2 with α as a support function. Then α is the solution of the first order differential equation

$$\alpha'=\frac{f\cdot f'}{|f'|} \qquad (5)$$

Proof: Observe that

$$f\cdot f'=|f|^2=\alpha^2+(\alpha')^2$$

Also $|f'|=\alpha+\alpha''$ Thus

$$f\cdot f'=\alpha\alpha'+\alpha'\alpha''=\alpha'(\alpha+\alpha'')$$

So,

$$\alpha'=\frac{f\cdot f'}{|f'|}$$

We give two examples for using Eq. 5 to calculate α.

Example 1: For the unit circle $f=(\cos\theta, \sin\theta)$, $\theta\in[0,2\pi]$, Eq. 5 becomes $\alpha'=0$. So $\alpha=c$ constant. But $\alpha(0)=1$. Thus, $\alpha=1$.

Example 2: Consider the embedding $f: [0, 2\pi]\rightarrow R^2$ defined by:

$$f(\theta)=(9\cos\theta+\cos\theta\cos3\theta+3\sin\theta\sin3\theta, 9\sin\theta$$
$$+\sin\theta\cos3\theta-3\cos\theta\sin3\theta)$$

which was constructed by Fillmore (1969). First of all we show that the trace of f is an oval. For, observe that

$$f'(\theta) = (9-8\cos 3\theta)(-\sin\theta, \cos\theta)$$

and

$$f''(\theta) = 24\sin 3\theta(-\sin\theta, \cos\theta) + (9-8\cos 3\theta)(-\cos\theta, -\sin\theta)$$

Thus,

$$\kappa = \frac{|f' \times f''|}{|f'|^3} = \frac{(9-8\cos 3\theta)^2}{(9-8\cos 3\theta)^3} = \frac{1}{9-8\cos 3\theta}$$

Since $\kappa(\theta)>0$, $\forall\theta\in[0, 2\pi]$, f is convex and its trace is an oval. Using Eq. 5, we have $\alpha' = -3\sin 3\theta$. Since $\alpha(0) = 10$ we have $\alpha = \cos 3\theta+9$.

Ovals of constant width in R^2: Ovals of constant width in R^2 were studied by Mellish (1931). Mellish's work was explained by Robertson (1984). First we introduce the definition.

Definition 1: An oval f in R^2 is of constant width a if the perpendicular distance between support tangent lines at opposite points is always a. That is,

$$\alpha(\theta) + \alpha(\theta+\pi) = a, \quad \forall\theta \in [0, 2\pi].$$

Theorem 4: Let f be an oval in R^2. Then f is of constant width a iff

$$\frac{1}{\kappa(\theta)} + \frac{1}{\kappa(\theta+\pi)} = a, \quad \forall\theta \in [0, 2\pi].$$

Proof: If f is of constant width a, then $\forall\theta\in[0, 2\pi]$,

$$\frac{1}{\kappa(\theta)} + \frac{1}{\kappa(\theta+\pi)} = \alpha(\theta)+\alpha''(\theta)+\alpha(\theta+\pi)+\alpha''(\theta+\pi)$$
$$= [\alpha(\theta)+\alpha(\theta+\pi)]+[\alpha''(\theta)+\alpha''(\theta+\pi)] = \alpha+0 = a$$

Now assume that

$$\frac{1}{\kappa(\theta)} + \frac{1}{\kappa(\theta+\pi)} = a, \quad \forall\theta \in [0, 2\pi]$$

Then

$$\alpha(\theta) + \alpha(\theta+\pi) + \alpha''(\theta) + \alpha''(\theta+\pi) = a$$

Thus,

$$\alpha(\theta) + \alpha(\theta+\pi) = c_1\cos\theta + c_2\sin\theta + a \tag{6}$$

Where, c_1, c_2 are constants. Now replace θ by $\theta+\pi$ in Eq. 6 to get

$$\alpha(\theta+\pi) + \alpha(\theta+2\pi) = -c_1\cos\theta - c_2\sin\theta + a \tag{7}$$

Add Eq. 6 to 7, with the fact that $\alpha(\theta+2\pi) = \alpha(\theta)$ to get

$$\alpha(\theta) + \alpha(\theta+\pi) = a, \quad \forall\theta \in [0, 2\pi].$$

Thus, f is of constant width α.

The next theorem is known historically as Barbier's theorem, 1860 and had been proved in different methods. Even though, we use the idea of the support function to introduce a simple proof.

Theorem 5: All ovals in R^2 of constant width a have the same length $a\pi$.

Proof: Let f be an oval in R^2 with constant width a and length L. Then

$$L = \int_0^{2\pi} |f'| d\theta = \int_0^{2\pi} (\alpha(\theta) + \alpha''(\theta)) d\theta = \int_0^{2\pi} (a - \alpha(\theta+\pi) - \alpha''(\theta+\pi)) d\theta$$
$$= 2a\pi - \int_0^{2\pi} (\alpha(\theta+\pi) + \alpha''(\theta+\pi)) d\theta$$
$$= 2a\pi - \int_\pi^{3\pi} (\alpha(\theta) + \alpha''(\theta)) d\theta = 2a\pi - L$$

Thus, $L = a\pi$.

Now we prove that there is a special curve conjugate to an oval of constant width in R^2.

Theorem 6: Let f be an oval in R^2 with constant width a. Then there exists a periodic curve w (with period π), parallel to f and at a distance $\frac{a}{2}$ from f.

Proof: For $\theta\in[0, 2\pi]$, define w by

$$w(\theta) = f(\theta) - \frac{a}{2}(\cos\theta, \sin\theta) \tag{8}$$

Using Eq. 1, with the fact that $\alpha(\theta+\pi) = \alpha-\alpha(\theta)$, $\alpha'(\theta+\pi) = -\alpha'(\theta)$ we have:

$$f(\theta+\pi) = f(\theta) - a(\cos\theta, \sin\theta)$$

Thus, $w(\theta+\pi) = w(\theta)$ and so w is periodic with period π. Now

$$w'(\theta) = f'(\theta) + \frac{a}{2}(\sin\theta, -\cos\theta)$$

So if $v(\theta) = (-\cos\theta, -\sin\theta)$ is a unit normal of f, then v.w' = 0. Thus, v is also a unit normal of w and so w is parallel to f.
Now

$$|f(\theta) - w(\theta)| = \frac{a}{2}$$

So for $\theta\in[0, 2\pi]$, the distance between $f(\theta)$ and $w(\theta)$ is always $\frac{a}{2}$.

Corollary 3: Let f be an oval in R^2 with constant width a. Let g be the curve of focal points of f and let w be the curve as in Eq. 8. Then the trace of f is a circle iff $g(\theta) = w(\theta), \forall\theta\in[0, 2\pi]$.

Proof: If the trace of f is a circle, then

$$\alpha(\theta) = c_1 \cos\theta + c_2 \sin\theta + c$$

Where, c_1, c_2, c are constants, Corollary 1. Now substitute in Eq. 4 to get $g(\theta) = (c_1, c_2)$. Similarly, substitute in Eq. 8 to get

$$w(\theta) = (c_1 + c\cos\theta, c_2 + c\sin\theta) - \frac{a}{2}(\cos\theta, \sin\theta)$$

Now solving $\alpha(\theta)+\alpha(\theta+\pi) = a$ with, $\alpha(\theta) = c_1 \cos\theta + c_2 \sin\theta + c$, we have $c = \frac{a}{2}$ and so $w(\theta) = (c_1, c_2)$. Thus, $g(\theta) = w(\theta), \forall\theta\in[0, 2\pi]$.

Now assume that $g(\theta) = w(\theta), \forall\theta\in[0, 2\pi]$ Then by Eq. 3 and 8, we have $\kappa(\theta) = \frac{a}{2}$ i.e., constant. Thus, the trace of f is a circle.

We finish this work with the following argument. Since α is continuous with a continuous derivative and period 2π, it has a Fourier expansion

$$\alpha(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\theta + b_k \sin k\theta) \quad (9)$$

Where, a_0, a_1, a_2,...,b_1, b_2,... are constants. The above series converges to $\alpha(\theta)$ for all $\theta\in R$.
Since $a(\theta)+\alpha(\theta+\pi) = a$, then

$$a_0 = a$$
$$a_{2k} = b_{2k} = 0, k = 1, 2, ...$$

Thus,

$$\alpha(\theta) = \frac{a}{2} + \sum_{k=1}^{\infty}(a_{2k-1} \cos(2k-1)\theta + b_{2k-1} \sin(2k-1)\theta) \quad (10)$$

The expansion in Eq. 10 allows a huge number of different curves of constant width in R^2 by just assigning nonzero values, which preserves convexity, to a finite number of the constants and ignoring the rest.

REFERENCES

Al-Banawi, K.A.S., 2004. Generating frames and normal holonomy of transnormal submanifolds in euclidean spaces, Ph.D Thesis, University of Leeds, UK.

Eggleston, H.G., 1958. Convexity. Cambridge Tracts in Mathematics, Cambridge University Press, pp: 47.

Fillmore, J.P., 1969. Symmetries of Surfaces of Constant Width. J. Diff. Geom., 3 (1): 103-110.

Hsiung, C.C., 1981. A First Course in Differential Geometry. John Wiley and Sons.

Mellish, A.P., 1931. Notes on differential geometry. Ann. Math., 32 (2): 181-190.

Robertson, S.A., 1984. Curves of constant width and transnormality. Bull. L.M.S., 16 (3): 264-274.

Struik, D.J., 1950. Lectures on Classical Differential Geometry. Addison-Wesley, New York.

Yaglom, I.M. and M.G. Boltyanskii, 1961. Convex Figures. Kelly, P.J. and L.F. Walton (Eds.). Holt, Rinchart and Winston, New York.