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Assessment of Modified Variational Iteration Method in BVPs High-Order Differential Equations

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Abstract: This study has been devoted to investigate the semi-analytical solution of nonlinear differential equations with boundary value problems (BVPs). Modified variation iteration method has been utilized to solve some BVPs nonlinear differential equations. In this method, general lagrange multipliers have been introduced to construct correction functions for the problems. The multipliers can be identified optimally via the variational theory. The results have been compared with those of exact solutions and Homotopy Analysis Method (HAM). A clear conclusion can be drawn from the numerical results that proposed method provides excellent approximations to the solutions of this kind of problems in the terms of simplicity and accuracy, thus, it can be easily extended to other BVPs nonlinear differential equations and so can be found widely applicable in engineering sciences.

Key words: Modify variational iteration method, fourth-order differential equations, boundary value problems

INTRODUCTION

Boundary value problems (BVPs) in nonlinear differential equations have been one of the major problems in engineering sciences. In the past several decades, both mathematicians and physicists have made significant progress in this direction (He, 2000; Al-Hayani and Casasús, 2005; Liu, 2004). Since these equations are nonlinear, thus do not have precise analytical solutions. On the other hand, solving these nonlinear equations analytically may guide authors to know the described some physical process deeply and sometimes leads them to know some facts which are not simply understood through common observations. As a result, these equations have to be solved using other methods. Many different methods have been presented recently; for instance, the homotopy analysis method (Khatami *et al*., 2008; Abbasbandy, 2006), the Variational Iteration Method (VIM) (Khatami *et al*., 2008; Ganji *et al*., 2006; Sadighi and Ganji, 2007; He, 1999, 2002, 2007), modified variational method (Abassy *et al*., 2007; Tari, 2007), the Adomian's Decomposition Method (ADM) (Adomian, 1994; Wazwaz, 2007), Homotopy perturbation method (He, 2003, 2005, 2006; Tolou *et al*., 2008; Ghasemi, 2007) and Exp-function method (Mahmoudi *et al*., 2008).

This study deals with analytical solution of some BVPs nonlinear equation by the means of modified variational iteration method. Three examples are presented to assess the benefits of method. In the first example, the steady flow of a second grade fluid in a porous channel is considered and solved using MVIM and its results are compared with those of homotopy analysis method (HAM) (Hayat *et al*., 2007). The last two examples are nonlinear equations which are solved by MVIM and the results are compared with the obtained results from the exact solutions from Liu and Wv (2002) and Wazwaz (2001).

MATERIALS AND METHODS

Fundamental of He's variational iteration method: To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(t) \qquad (1)$$

where, L is a linear operator, N is a nonlinear operator and g (t) an inhomogeneous term. According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left[Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\right]d\tau \qquad (2)$$

where, λ is a general Lagrangian multiplier which can be identified optimally via the variational theory. The subscript n indicates the nth approximation and $\tilde{u}_n = 0$ is considered as a restricted variation $\delta\,\tilde{u}_n = 0$.

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Implementation of MVIM

Example one: Consider the following nonlinear fourth-order differential equation:

$$\begin{aligned} & y''' - M^2 y'(x) + \mathrm{Re}[y'(x)^2 - y(x)y''(x)] - \\ & \alpha[2y(x)y'''(x) - y''(x)^2 - y(x)y^{iv}(x)] = 0 \end{aligned} \tag{3}$$

Subject to the boundary conditions:

$$\begin{cases} y(0) = 0 & y'(0) = 0 \\ y(0.5) = 0.5 & y'(0.5) = 0 \end{cases} \tag{4}$$

Its correction variational functional can be expressed as follows:

$$\begin{aligned} & y(x) = y(x) + \int_0^x \lambda \Big\{ y'''(t) - My'(t) + \mathrm{Re}\Big[y'(t)^2 - y(t)y''(t) \Big] \\ & - \alpha\Big[2y(t)y'''(t) - y''(t)^2 - y(t)y^{iv}(t) \Big] \Big\} dt = 0 \end{aligned} \tag{5}$$

After some manipulations, the following stationary conditions have been obtained:

$$\lambda^{(iv)}(t) - \lambda(t) = 0 \tag{6}$$

$$1 + \lambda'''(t)\Big|_{t=x} = 0 \qquad \lambda''(t)\Big|_{t=x} = 0 \tag{7}$$

$$\lambda'(t)\Big|_{t=x} = 0 \qquad \lambda(t)\Big|_{t=x} = 0 \tag{8}$$

The Lagrangian multipliers can therefore be identified as:

$$\lambda(t) = -\frac{(x-t)^3}{6} \tag{9}$$

And the variational iteration formula is obtained in the form:

$$\begin{aligned} & y_{n+1}(x) = y_n(x) + \int_0^x \left(\frac{(t-x)^3}{6} \right) \Big\{ y_n'''(t) - My_n'(t) + \mathrm{Re}\Big[y_n'(t)^2 - y_n(t)y_n''(t) \Big] - \\ & \alpha\Big[2y_n(t)y_n'''(t) - y_n''(t)^2 - y_n(t)y_n^{iv}(t) \Big] \Big\} dt = 0 \end{aligned} \tag{10}$$

We are to start with the initial approximation of $y_0(x)$ Since, no initial approximation of $y_0(x)$ is available, we make one in the form of a polynomial as:

$$y_0(x) = a + bx + cx^2 + dx^3 \tag{11}$$

This depends on the order of differentiation and a, b, c and d are unknown constants to be later determined.

Using the above iteration formula (10), we can directly obtain other components as:

$$\begin{aligned} y_1(x) = & -\left(0.5ax^2\right) - \left(0.016\,\mathrm{Re}\,a^2x^6\right) + cx + \left(x^2a^2d\right) + \left(0.16\,\mathrm{Re}\,bdx^2\right) + ax^3 \\ & + bx^2 + \left(0.1a^3x^5\right) - \left(0.03ab\,\mathrm{Re}\,x^5\right) - \left(0.027b^2\,\mathrm{Re}\,x^4\right) - \left(0.5x^4a^3\right) \\ & + \left(0.041aM^2x^4\right) + \left(0.16a^2bx^4\right) + \left(0.05bM^2x^3\right) - \left(0.33ab^2x^2\right) \\ & - \left(0.083c^2\,\mathrm{Re}\,x^2\right) + \left(0.083cM^2x^2\right)\left(0.33a^2cx^3\right) - \left(0.66a^2bx^3\right) \\ & - \left(0.05bc\,\mathrm{Re}\,x^3\right) + \left(0.16ad\,\mathrm{Re}\,x^3\right) + d \end{aligned} \tag{12}$$

For a special case, M = 2, Re = 0, α = 0.2 will be as follows:

$$\begin{aligned} y_1(x) = & -\left(0.5ax^2\right) + cx + \left(0.2adx^2\right) + ax^3 + bx^2 + \left(0.02a^2x^5\right) \\ & - \left(0.1a^2x^4\right) + \left(0.16ax^4\right) + \left(0.03adx^4\right) + \left(0.22bx^3\right) \\ & - \left(0.06b^2x^2\right) + \left(0.33cx^2\right) + \left(0.06acx^3\right) - \left(0.13abx^3\right) + d \end{aligned} \tag{13}$$

Incorporating the boundary conditions, Eq. 4, into $y_1(x)$, we have:

$$y_1(0) = d = 0 \tag{14}$$

$$y_1{}^2(0) = -a + 0.4da + 2b - 0.13b^2 + 0.66c = 0 \tag{15}$$

$$\begin{aligned} y_1\left(\frac{1}{2}\right) = & (0.01a) + (0.58c) + (0.050\,ad) + (0.27b) - (0.00562a^2) \\ & - (0.014ab) - (0.016\,b^2) + (0.008\,ac) + d = 0.5 \end{aligned} \tag{16}$$

$$\begin{aligned} y_1'(0.5) = & (0.33a) + (1.33c) + (0.2\,ad) + (1.16b) - (0.04a^2) \\ & - (0.08ab) - (0.06b^2) + (0.05ca) = 0 \end{aligned} \tag{17}$$

Solving the above system of equations simultaneously, we obtain:

$$\begin{cases} a = -1.246731807 & b = -1.044604271 \\ c = 1.481954721 & d = 0 \end{cases} \tag{18}$$

Therefore, we obtain the following first–order approximate solution for special case, M = 2, Re = 0, α = 0.2:

$$y_1(x) = \left(0.5\times10^{-9}x^2\right) + \left(1.481954721x\right) - \left(1.77568x^3\right) + \left(0.03108x^5\right) - \left(0.31981x^4\right) \tag{19}$$

In the same manner, the rest of the components of the iteration formula can be obtained.

Example two: Now consider another nonlinear fourth-order BVP (Liu and Wv, 2002):

$$y^{iv}(x) + y(x)y'(x) - 4x^7 - 24 = 0 \tag{20}$$

Subject to the boundary conditions:

$$\begin{cases} y(0) = 0 & y'''(0.25) = 6 \\ y''(0.5) = 3 & y(1) = 1 \end{cases} \tag{21}$$

Its correction variational functional in can be expressed as:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda\left(y_n^{(iv)}(t) + y_n(t)y_n'(t) - 4x^7 - 24\right)dt \tag{22}$$

After some computations, we obtain the following stationary conditions:

$$\lambda^{(iv)}(t) = 0 \tag{23}$$

$$1 + \lambda'''(t)\big|_{t=x} = 0 \qquad \lambda''(t)\big|_{t=x} = 0 \tag{24}$$

$$\lambda'(t)\big|_{t=x} = 0 \qquad \lambda(t)\big|_{t=x} = 0 \tag{25}$$

The Lagrangian multipliers can, therefore, be identified as:

$$\lambda(t) = \frac{(x-t)^3}{6} \tag{26}$$

And the variational iteration formula is obtained in the form of

$$y_{n+1}(x) = y_n(x) + \int_0^x \{[\frac{(x-t)^3}{6}] \times [y_n^{(iv)}(t) + y_n(t)y_n'(t) - 4x^7 - 24]\}dt \tag{27}$$

Assuming initial approximation as following form:

$$y_0(x) = a + bx + cx^2 + dx^3 \tag{28}$$

where, a, b, c and d are unknown constants to be further determined. Using the above iteration formula (27), we can directly obtain the other components as follows:

$$\begin{aligned} y_1(x) = {} & (0.00005x^{11}) - (0.00298\,abx^8) - (0.00476acx^7) - (0.00238bx^7) \\ & - (0.00833adx^6) - (0.0083cbx^6) - (0.016bdx^5db) - (0.0083cx^5) \\ & + x^4 - (0.0416cdx^4) + ax^3 - (0.00009ax^2) + bx^2 + cx + d \end{aligned} \tag{29}$$

Incorporating the boundary conditions, Eq. 21, into $y_1(x)$, we have:

$$y_1(0) = d = 0 \tag{30}$$

$$\begin{aligned} y_1(1) = {} & -(0.00833\,ac) - (0.00297ab) - (0.00833bc) \\ & - (0.00009a^2) - (0.00238\,b^2) \\ & - (0.00833c^2) + 1.005 + a + b + c = 1 \end{aligned} \tag{31}$$

$$\begin{aligned} y_1''(0.5) = {} & (-0.00625ac) - (0.0026041ab) - (0.015625bc) \\ & - (0.00055a^2) - (0.00312b^2) - (0.02083c^2) \\ & + 3a + 2b + 3.00011 = 3 \end{aligned} \tag{32}$$

$$\begin{aligned} y_1'''(0.25) = {} & -(0.00391ac) - (0.00097ab) - (0.015625bc) \\ & - (0.000122a^2) - (0.0019531b^2) - (0.03125c^2) \\ & + 6a + 6.00001 = 6 \end{aligned} \tag{33}$$

Solving the above system of equations simultaneously, we obtain:

$$\begin{cases} a = -\,0.0000012703 & b = -\,0.000052345 \\ c = -\,0.00045143 & d = 0 \end{cases} \tag{34}$$

Therefore, we obtain the following first-order approximate solution, as follows:

$$\begin{aligned} y_1(x) = {} & (5.05050\times10^{-4}x^{11}) - (1.60104\times10^{-15}x^9) - (1.979125\times10^{-13}x^8) \\ & - (9.25486\times10^{-12}x^7) - (1.96921\times10^{-10}x^6) - (1.69826\times10^{-9}x^5) \\ & + x^4 - (0.127\times10^{-4}x^3) - (0.5234\times10^{-4}x^2) - (0.4514\times10^{-3}) \end{aligned} \tag{35}$$

The exact solution is $y(x) = x^4$ (Liu and Wv, 2002).

Example three: Consider the nonlinear boundary value problem as follows:

$$y^{iv}(x) - y(x) - y''(x) - e^x(x-3) = 0 \tag{36}$$

Subject to the following conditions:

$$\begin{cases} y(0) = 1 & y'(0) = 0 \\ y(1) = 0 & y'(1) = -e \end{cases} \tag{37}$$

Its correction variational functional can be expressed as:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda\left(y_n^{(iv)}(t) - y_n(t) - y_n''(t) - e^x(x-3)\right)dt \tag{38}$$

After some computations, we obtain the following stationary conditions:

$$\lambda''(t) = 0 \tag{39}$$

$$1 + \lambda'(t)\Big|_{t=x} = 0 \qquad \lambda(t)\Big|_{t=x} = 0 \tag{40}$$

The Lagrangian multipliers can, therefore, be identified as:

$$\lambda(t) = \frac{(t-x)^3}{6} \tag{41}$$

And the variational iteration formula is obtained in the form of

$$y_{n+1}(x) = y_n(x) + \int_0^x [(t-x)^3/6] \times \{(y_n^{(iv)}(t) - y_n(t)y_n''(t) - e^x(x-3)\}\,dt \tag{42}$$

Now, we assume that the initial approximation has the form of

$$y_0(x) = a + bx + cx^2 + dx^3 \tag{43}$$

where ,a, b, c and d are unknown constants to be further determined. Using the above iteration formula (42), we can directly obtain the other components as follows:

$$\begin{aligned} y_1(x) &= (0.00119ax^7) + (0.0027bx^6) + (0.05ax^5) \\ &+ (0.0083cx^5) + (0.083bx^4) + (0.0416dx^4) \\ &+ ax^3 + (0.66x^3) + bx^2 + (2.5x^2) + [(c+6)x] \\ &+ [e^x(x-7)] + d + 7 \end{aligned} \tag{44}$$

Incorporating the boundary conditions, Eq. 37, into $y_1(x)$ and solving the equations, we obtain:

$$y_1(0) = d = 1 \tag{45}$$

$$y_1(1) = (1.105119a + 1.08611b + 1.00833c - 6e + 17.20833 = 0 \tag{46}$$

$$y_1'(0) = 6 + c - 6e^0 = 0 \tag{47}$$

$$y_1'(1) = (3.25833a) + (2.35b) - (0.42473) = -e \tag{48}$$

Solving the above system of equations simultaneously, we obtain:

$$\begin{cases} a = -0.35486 & b = -0.48394 \\ c = 0 & d = 1 \end{cases} \tag{49}$$

In the same manner, the rest of the components of the iteration formula can be obtained as:

$$\begin{aligned} y_1(x) &= -(0.00042x^7) - (0.00134x^6) - (0.01774x^5) - (0.00134x^4) \\ &+ (0.31180x^3) + (2.01606x^2) + [e^x(x-7)] + 6x + 8 \end{aligned} \tag{50}$$

The exact solution for this problem have already been obtained by Wazwaz (2001) as $(1-x)e^x$.

RESULTS AND DISCUSSION

In this survey, some BVPs type nonlinear equations are solved by using MVIM. Figure 1a and b show the y(x) obtained from MVIM and HAM versus x respectively. In comparison to those of HAM (Fig. 1b) (Hayat *et al*., 2007) this results illustrates high accuracy in different values of M. Figure 2 and 3 show the results of y (x) and $y'(x)$

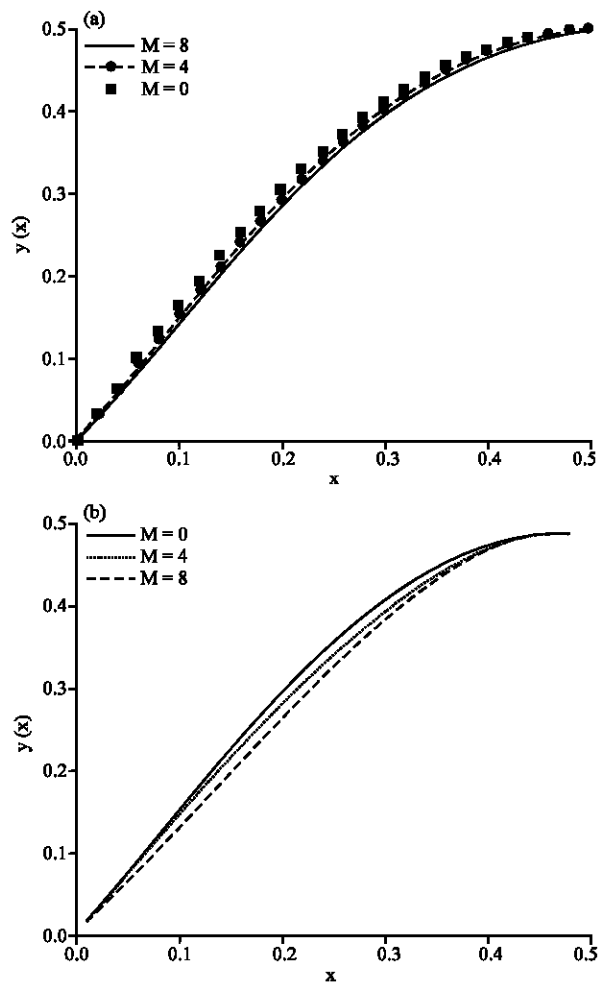

Fig. 1: y (x) values versus x obtained from (a) MVIM and (b) HAM, in different values of M (Hayat *et al*., 2007)

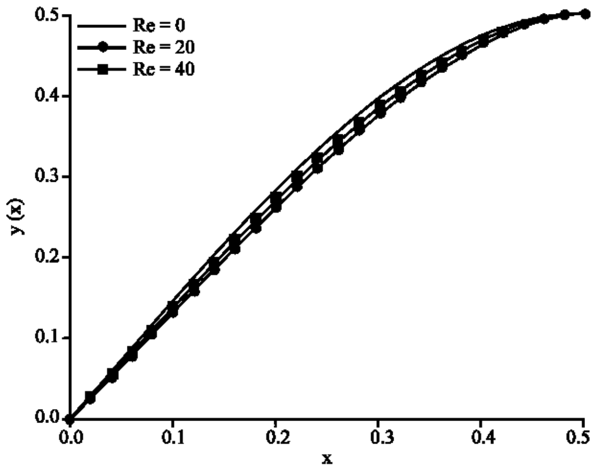

Fig. 2: Plot of the approximated y (x) versus x, this plot has excellent agreement with reference (Hayat *et al*., 2007)

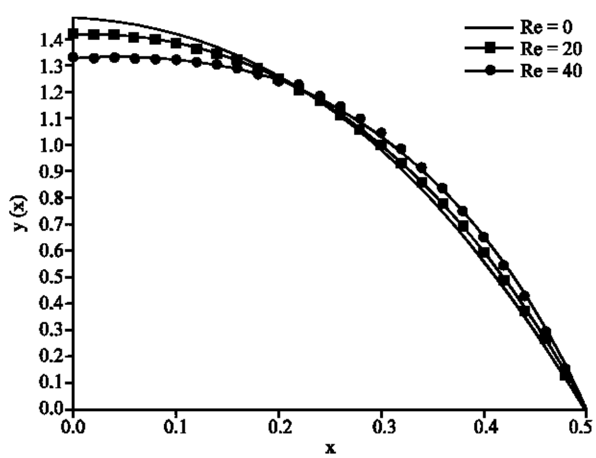

Fig. 3: Plot of the approximated y (x) versus x, this plot has excellent agreement with reference (Hayat *et al*., 2007)

versus x for different values of Re that are in good agreement with those of HAM (Hayat *et al*., 2007). The results of example two from proposed solution and exact solution (Liu and Wv, 2002) are prorated in Fig. 4 which are is in excellent agreement. Finally the results of example three from MVIM and exact solution (Wazwaz, 2001) shown in Fig. 5 which are in good agreement. The result shown in Fig. 1-5 indicates that the MVIM experiences a high accuracy. In addition, in comparison with conventional method, a considerable reduction of the volume of the calculation can be seen in MVIM.

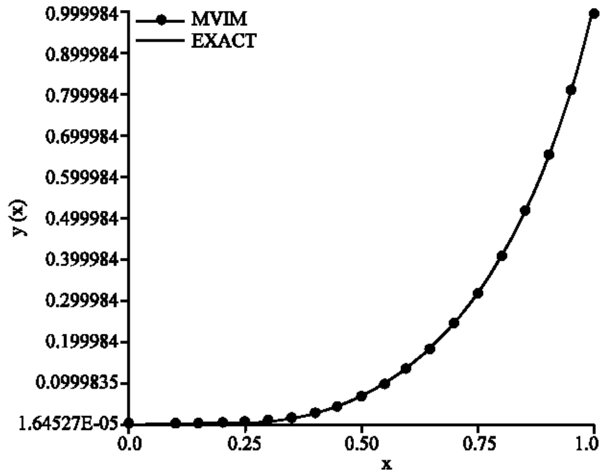

Fig. 4: The comparison between the Exact and MVIM solutions for second example

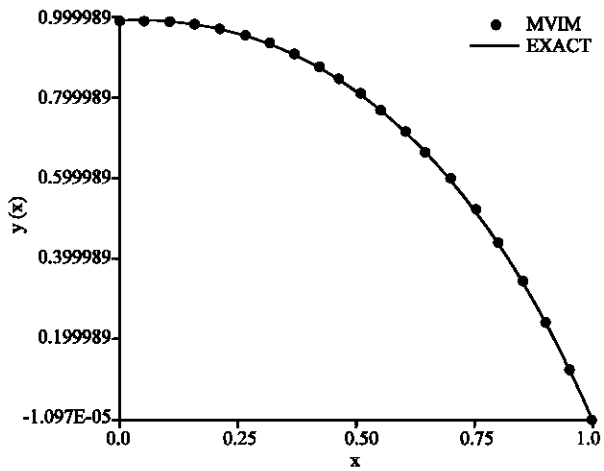

Fig. 5: The comparison between the Exact and MVIM solutions for third example

CONCLUSION

In this research, we studied the application of the modified variation iteration method (MVIM) to nonlinear and linear integral equations and the Blasius problem. It was founded that the approximations obtained by MVIM are valid when compared with the exact solutions.

The Fig. 1-5 clearly show that the results by MVIM are in excellent agreement with the exact solutions. MVIM provides highly accurate numerical solutions in comparison with other methods and it is expected here that this powerful mathematical tool can solve a large class of nonlinear differential systems, especially nonlinear integral systems and equations used in engineering and physics.

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