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Impact of Lead Time and Safety Factor in Mixed Inventory Models with Backorder Discounts

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Abstract: This study investigates the impact of safety factor on the continuous review inventory model involving controllable lead time with mixture of backorder discount and partial lost sales. The objective is to minimize the expected total annual cost with respect to order quantity, backorder price discount, safety factor and lead time. A model with normal demand is also discussed. Numerical examples are presented to illustrate the procedures of the algorithms and the effects of parameters on the result of the proposed models are analyzed.

Key words: Order quantity, lead time, backorder price discounts, safety factor

INTRODUCTION

Competition features of 90's have evolved into Time-Based Manufacturing (TBM). The two time elements considered in TBM are the replenishment lead time to supply a product to the customer for a specific order and the time to develop a product from concept to final market delivery. Therefore, reducing lead time on product supply is the strategic objective of the TBM company (Bockerstette and Shell, 1993). Although the time compression will inevitably raise the cost, a customer will pay a premium to the supplier who can furnish its product faster and more reliably than the competition and the premium may be respectable.

Silver *et al.* (1998) defined the replenishment lead time as the time elapsed from the moment at which it is decided to place an order, until it is physically on the shelf to satisfy customer demands. Although lead time can be a constant or a random variable, it is often treated as a prescribed parameter (Silver *et al.*, 1998) thus not controllable. However, lead time can be reduced at extra cost and shorter lead time is the primary driver to achieving customer satisfaction in successful TBM operations (Bockerstette and Shell, 1993). The benefits resulting from reduced lead time include lower cost, less waste and less obsolescence, greater flexibility to response to change, closely linked organization priorities to customers' needs, improved service, quality and reliability and substantially accelerated supply system improvements (Blackburn, 1991). Tersine (1994) and Vollmann *et al.* (1992) attributed the replenishment lead time mostly to manufacturing considerations and addressed some guidelines for its reduction. Liao and Shyu (1991) suggested that lead time could be decomposed into n components each having a different crashing cost for reduction. Ben-Daya and Raouf (1994) generalized the Liao and Shyu model (1991) by considering both lead time and the order quantity as decision variables.

Ouyang *et al.* (1996) extended the Ben-Daya and Raouf's model (1994) to include a mixture of backorder and lost sales in the model by assuming a predetermined service level with both reorder point and backorder rate being fixed. Moon and Choi (1998) suggested that it was not appropriate to include the service constraint if the shortage cost was explicitly specified and claimed that a better solution could be obtained by allowing the reorder point to be variable. Hariga and Ben-Daya (1999) also revised Ouyang *et al.* (1996) model by including the reorder point as a decision variable. Pan and Hsiao (2001) presented two inventory models where shortage was allowed with controllable back ordering.

This study considers a continuous review inventory system in which the lead time is controllable and can be decomposed into several components each having a crashing cost function. In addition, shortage is permitted and the total amount of stockout is a combination of backorder and lost sale. It is further assumed that the patient customers with outstanding orders during the shortage period are offered a backorder price discount

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and consequently the backorder ratio is proportional to the magnitude of this discount (Pan and Hsiao, 2001). Since the shortage cost is explicitly included, the reorder point is also treated as a decision variable in this study (Moon and Choi, 1998).

There is form of lead time demand considered following a normal distribution in the study. In this models, the objectives are to simultaneously optimize the order quantities, back order discounts, reorder points and lead times such that the total expected annual costs are minimized. Furthermore, an iterative algorithm is applied to find the optimal solution for the case where the lead time demand follows a normal distribution. Our model serves as a pioneering work investigating the effects backorder discounts and safety factor have on the integrated inventory system.

NOTATIONS AND ASSUMPTIONS

The following notations are used throughout the study.

L	=	The length of lead time (decision variable)
Q	=	The order quantity (decision variable)
π_x	=	The backorder price discount offered by the supplier per unit (decision variable)
k	=	The safety factor (decision variable)
r	=	The reorder point
π_0	=	The gross marginal profit per unit
D	=	The average demand per year
A	=	The fixed ordering cost per order
h	=	The inventory holding cost per unit per year
β	=	The backorder ratio
β_0	=	The upper bound of the backorder ratio
ϕ	=	The standard normal distribution;
Φ	=	The standard normal cumulative distribution function
SS	=	The safety stock
B(r)	=	The expected shortage of a cycle
R(L)	=	The total crashing cost of a cycle

The assumptions made in the research are defined as follow:

- The reorder point r = expected demand during lead time + safety stock, that is, $r = DL + k\sigma\sqrt{L}$, where k is a safety factor
- The lead time L has n mutually independent components, where the ith component has a normal duration T_i and a minimum duration t_i, i = 1, 2, ..., n and a crashing cost per unit time a_i. These a_i's are arranged such that $a_1 \le a_2 \le \ldots \le a_n$. The lead times are crashed that should be first on component 1 and then component 2 and so on.
- Let L_i denote the length of lead time with component 1, 2, ...n, i crashed to their minimum values, then L_i can be expressed as:

$$L_i = \sum_{j=1}^{n} T_j - \sum_{j=1}^{i} (T_j - t_j)$$

and L_0 is not cashed the length of lead time. Thus, the lead time crashing cost R(L) per replenishment cycle is given by:

$$R(L) = \alpha_i (L_{i-1} - L) + \sum_{j=1}^{i-1} a_j (T_j - t_j), \text{ for } L \in (L_i, L_{i-1})$$

- The backorder ratio β is variable and is in proportion to the backorder price discount offered by the supplier per unit π_x; thus, $\beta = \beta_0 \pi_x / \pi_0$, for $0 < \beta_0 \le 1$, $0 \le \pi_x \le \pi_0$ (Pan and Hsiao, 2001)

NORMAL DEMAND MODEL

The problem under study is to minimize the following total expected annual cost:

$$EAC(Q, \pi_x, k, L) = OC + HC + SC + CC$$

Where:

OC	=	Stands for ordering cost
HC	=	Holding cost
SC	=	Shortage cost
CC	=	Crashing cost

The annual expected ordering cost can be expressed as:

$$OC = A\frac{D}{Q}$$

The lead time demand X is assumed to follow the normal distribution with mean μL and standard deviation $\sigma\sqrt{L}$. Since shortage is allowed, the expected inventory shortage at the end of a cycle is given by:

$$E(X-r)^+ = B(r) = \sigma\sqrt{L}\Psi(k),$$

Where:

$\Psi(k)$	=	$\phi(k)-k[1 - \Phi(k)]$
ϕ, Φ	=	Standard normal distribution and cumulative distribution function, respectively (Ravindran *et al*., 1987)

For backorder ratio β, the expected inventory at the end of a cycle is $\beta B(r)$ and the expected lost sale is

(1-β)B(r). Therefore, the average inventory of a cycle is:

$$\frac{Q}{2}+E(r-X)+(1-\beta)B(r)=\frac{Q}{2}+(r-DL)+(1-\beta)\sigma\sqrt{L}\Psi(k)$$
$$=\frac{Q}{2}+k\sigma\sqrt{L}+(1-\beta)\sigma\sqrt{L}\Psi(k)$$

Hence, the expected annual holding cost is:

$$HC = h[\frac{Q}{2}+k\sigma\sqrt{L}+(1-\beta)\sigma\sqrt{L}\Psi(k)]$$

The backorder price discount of a cycle is $\beta\pi_x B(r)$ and the expected lost sales is $(1-\beta)B(r)$, so the profit lost due to shortage is $\pi_0(1-\beta)B(r)$ and the expected annual shortage cost can be expressed as:

$$SC=\frac{D}{Q}[\beta\pi_x+(1-\beta)\pi_0]\sigma\sqrt{L}\Psi(k)$$

Following the definition of L_i, the lead time crashing cost R(L) per replenishment cycle is given by:

$$R(L)=\alpha_i(L_i-1-L)+\sum_{j=1}^{i-1}a_j(T_j-t_j),\ \text{for}\ L_i<L\le L_{i-1}$$

Hence, the expected annual crashing cost is:

$$CC=\frac{D}{Q}R(L)$$

$$=\frac{D}{Q}\left[a_i(L_{i-1}-L)+\sum_{j=1}^{i-1}a_j(T_j-t_j)\right]\ \text{for}\ L_i<L\le L_{i-1}$$

Consequently, the total expected annual cost EAC(Q, π_x, k, L) is:

$$EAC(Q,\pi_x,k,L)=A\frac{D}{Q}+h[\frac{Q}{2}+k\sigma\sqrt{L}+(1-\beta)\sigma\sqrt{L}\Psi(k)]$$
$$+\frac{D}{Q}[\beta\pi_x+(1-\beta)\pi_0]\sigma\sqrt{L}\Psi(k) \tag{1}$$
$$+\frac{D}{Q}\left[a_i(L_{i-1}-L)+\sum_{j=1}^{i-1}a_j(T_j-t_j)\right]\ \text{for}\ L_i<L\le L_{i-1}$$

Taking the first partial derivatives of EAC(Q, π_x, k, L) with respect to Q, π_x, k and L, respectively, it follows that:

$$\frac{\partial EAC(Q,\pi_x,k,L)}{\partial Q}=-\frac{AD}{Q^2}+\frac{h}{2}-\frac{D}{Q^2}\left[\frac{\beta_0}{\pi_0}\pi_x^2+\pi_0-\beta_0\pi_x\right]\sigma\sqrt{L}\Psi(k)$$
$$-\frac{D}{Q^2}\left[a_i(L_{i-1}-L)+\sum_{j=1}^{i-1}a_i(T_j-t_j)\right] \tag{2}$$

$$\frac{\partial EAC(Q,\pi_x,k,L)}{\partial \pi_x}=-\frac{\beta_0}{\pi_0}h\sigma\sqrt{L}\Psi(k)+\frac{D}{Q}\left[\frac{2\beta_0}{\pi_0}\pi_x-\beta_0\right]\sigma\sqrt{L}\Psi(k) \tag{3}$$

$$\frac{\partial EAC(Q,\pi_x,k,L)}{\partial k}=h\sigma\sqrt{L}-\left[h\left(1-\frac{\beta_0}{\pi_0}\pi_x\right)+\frac{D}{Q}\left(\frac{\beta_0}{\pi_0}\pi_x^2+\pi_0-\beta_0\pi_x\right)\right]$$
$$\times\sigma\sqrt{L}(1-\Phi(k)) \tag{4}$$

and

$$\frac{\partial EAC(Q,\pi_x,k,L)}{\partial L}=\frac{1}{2}\left[h\left(1-\frac{\beta_0}{\pi_0}\pi_x\right)+\frac{D}{Q}\left(\frac{\beta_0}{\pi_0}\pi_x^2+\pi_0-\beta_0\pi_x\right)\right]$$
$$\sigma L^{-1/2}\Psi(k)+\frac{1}{2}hk\sigma L^{-1/2}-\frac{D}{Q}a_i \tag{5}$$

Setting Eq. 3 to zero and solving for π_x, it follows that:

$$\pi_x=\frac{hQ}{2D}+\frac{\pi_0}{2} \tag{6}$$

Setting Eq. 2 to zero and substituting Eq. 6 into 2 to solve for Q, thus:

$$Q=\left[\frac{2D\left[A+\pi_0\left(1-\frac{\beta_0}{4}\right)\sigma L^{1/2}\Psi(k)+a_i(L_{i-1}-L)+\sum_{j=1}^{i-1}a_j(T_j-t_j)\right]}{h\left[1-\frac{h\beta_0}{2D\pi_0}\sigma L^{1/2}\Psi(k)\right]}\right]^{1/2} \tag{7}$$

Setting Eq. 4 to zero and substituting Eq. 6 into 4 to solve for k, then:

$$\Phi(k)=1-\frac{h}{\left[h\left(1-\frac{\beta_0 hQ}{2\pi_0 D}-\frac{\beta_0}{2}\right)+\frac{\pi_0 D}{Q}\left(\frac{\beta_0}{4}(\frac{hQ}{\pi_0 D})^2+(1-\frac{\beta_0}{4})\right)\right]} \tag{8}$$

It can be shown from Eq. 8 that $hQ/D\le\pi_0$, so $\Phi(k)\ge0.5$ holds for nonnegative safety factor k. Consequently, the value of π_x derived in Eq. 6 will automatically lie between 0 and π_0 as specified in assumption 4th.

For fixed values of Q, π_x and k, EAC(Q, π_x, k, L) is concave in the range $L\in(L_i, L_{i-1}]$, since:

$$\frac{\partial EAC(Q,\pi_x,k,L)}{\partial L^2}=-\frac{1}{4}hk\sigma L^{-\frac{3}{2}}-\frac{1}{4}$$
$$\left[h\left(1-\frac{\beta_0}{\pi_0}\pi_x\right)+\frac{D}{Q}\left(\frac{\beta_0}{\pi_0}\pi_x^2+\pi_0-\beta_0\pi_x\right)\right]\sigma L^{-\frac{3}{2}}\Psi(k)<0 \tag{9}$$

For fixed $L\in(L_i, L_{i-1}]$, denote the values of Q, π_x and k found from Eq. 6-8 by Q*, πx* and k*, respectively. In addition, for fixed $L\in(L_i, L_{i-1}]$, the determinant of

Hessian matrix of EAC(Q, π_x, k, L) is positive definite at (Q*, π_x*, k*) as shown in Appendix.

The following algorithm can be used to find the optimal values of the order quantity, backorder discount, reorder point and lead time.

Step 1: For i = 0, 1, 2, ..., n

(i) Set k_{io} = 0 (implies $\Psi(k_{io})$ = 0.39894)
(ii) Substitute $\Psi(k_{io})$ into Eq. 8 to evaluate Q_{io}
(iii) Use Q_{io} to determine $\Phi(k_{in})$ from Eq. 8, then find kin from $\Phi(k_{in})$ by checking the normal table. Let k_{io} = k_{in}
(iv) Repeat (ii) and (iii) until no change occurs in the values of Q_i and k_i. Denote these resulting solutions by Q_i and k_i.
(v) Use Q_i and Eq. 7 to compute the backorder price discount π_{xi}.
(vi) Use Eq. 1 to compute the expected total annual cost EAC(Q_i, π_{xi}, k_i, L_i).

Step 2: Set EAC(Q*, π_x*, k*, L^*) = Min{EAC(Q_i, π_{xi}, k_i, L_i), i = 0, 1, 2, ..., n}.

Step 3: (Q*, π_x*, k*, L*) is a set of optimal solutions.

NUMERICAL EXAMPLE AND ITS SENSITIVITY ANALYSIS

Consider an inventory system with the following data: D = 600 units/year, A = \$200 per order, h = \$20 per unit per year, π0 = \$150 per unit, σ = 7 unit/week and the lead time has three components with data shown in Table 1 (Pan and Hsiao, 2001). Apply the proposed algorithm to solve the problem for the upper bound of the backorder ratio β_0 = 0.95, 0.80, 0.65, 0.50, 0.35 and 0.20, respectively. The resulting optimal solutions are summarized in Table 2. Also included in Table 2 are the results obtained from the associated Pan and Hsiao model (2001) by setting k fixed at 0.85, along with the corresponding saving on the total expected annual cost of the proposed model over that of Pan and Hsiao (2001). It is interesting to observe that the saving increases as β_0 decreases.

Next, we study the effect of change in the model parameters such as D, A, h, σ and π_0 under β_0 = 0.5, the optimal order quantity (Q* = 121), the optimal backorder price discount (π_x* = 77.0157), the optimal safety factor (k* = 1.88), the optimal lead time (L* = 4) and the corresponding total cost (EAC* = EAC(Q*, π_x*, k*, L^*) = 2947.72) keeping the same parameter values in Example. The sensitivity analysis is performed by changing each of the parameter by -50, -25, +25 and +50%, taking one at a time while keeping remaining unchanged. The results are shown in Table 3. Let EAC(δ) denote the percentage difference between the new total expected annual cost obtained from changing the values of these factors and the original total expected annual cost, that is, EAC(δ) = (the new total expected annual cost-the original j total expected annual cost)/(the original total expected annual cost).

Makes the sensitivity analysis in view of various parameters, the obtained data result has the following discovery.

- k*, Q* and EAC* all increases while π_x* decreases with an increase in the value of the model parameter demand rate D. The obtained results show that π_x* and k* are lowly sensitive and Q* and EAC* are middling sensitive to changes in the value of D. Moreover, L_i* is unchanged in D.
- π_x*, Q* and EAC* increases while k* decreases with an increase in the value of the model parameter A. The obtained results show that π_x* and k* are lowly sensitive and Q* and EAC* are middling sensitive to changes in the value of A. Moreover, L_i* is unchanged in A.
- π_x* and EAC* increases while Q* and k* decreases with an increase in the value of the model parameter h. The obtained results show that π_x* and k* are lowly sensitive and Q* and EAC* are middling sensitive to changes in the value of h. Moreover, L_i* is unchanged in h.

Table 1: Lead time data of the examples

Lead time component i	1	2	3
Normal duration Ti (days)	20.0	20.0	16.0
Minimum duration ti (days)	6.0	6.0	9.0
Unit fixed crashing cost ai (\$/day)	0.4	1.2	5.0

Table 2: Summary of the results for example (L_i in weeks)

The proposed model						Pan and Hsiao (k = 0.85)				
β_0	Q*	π_x*	k*	L_i*	EAC(•)(1)	Q*	π_x*	L_i*	EAC(•)(2)	Savings (%) ((2)-(1))/(2)
0.95	121	77.0180	1.82	4	2932.15	155	77.58	4	3346.64	12.39
0.80	121	77.0171	1.84	4	2937.62	156	77.61	4	3382.41	13.15
0.65	121	77.0164	1.86	4	2942.81	160	77.66	3	3417.50	13.89
0.50	121	77.0157	1.88	4	2947.72	161	77.69	3	3447.60	14.50
0.35	121	77.0150	1.90	4	2952.40	162	77.71	3	3477.46	15.10
0.20	121	77.0144	1.92	4	2956.85	164	77.73	3	3507.08	15.69

Table 3: Effect of changes in the parameters of the continuous review inventory models

Parameters	Change in Parameter values (%)	Change in (%) Q* (%)	πk_x* (%)	k* (%)	L_i* (%)	EAC ($Q^*, \pi k_x^*, k^*, L^*$)
D	-50.00	-28	1.17	-8.68	0	-24.26
	-25.00	-13	0.43	-3.52	0	-11.04
	+25.00	11	-0.29	2.66	0	9.66
	+50.00	21	-0.50	4.79	0	18.35
A	-50.00	-25	-0.65	6.39	0	-19.20
	-25.00	-11	-0.30	2.77	0	-8.93
	+25.00	10	0.27	-2.25	0	8.01
	+50.00	20	0.51	-4.15	0	15.33
h	-50.00	39	-0.79	8.07	0	-33.15
	-25.00	15	-0.37	3.42	0	-15.51
	+25.00	-10	0.33	-2.72	0	14.14
	+50.00	-17	0.63	-4.99	0	27.29
σm	-50.00	-5	-0.06	1.28	50	-11.30
	-25.00	-1	-0.03	0.26	0	-5.39
	+25.00	1	0.03	-0.26	0	5.38
	+50.00	2	0.06	-0.53	0	10.75
πk_0	-50.00	0	-48.68	-16.85	0	-2.60
	-25.00	0	-24.34	-6.77	0	-1.05
	+25.00	0	24.34	5.06	0	0.79
	+50.00	0	48.69	9.08	0	1.43

- π_x*, Q* and EAC* increases while k*and L_i* decreases with an increase in the value of the model parameter σ. The obtained results show that π_x* and k* are lowly sensitive and Q* and EAC* are middling sensitive to changes in the value of σ. Moreover, L_i* is highly sensitive to changed in σ.
- π_x*, k* and EAC* all increase with an increase in the value of the model parameter π_0. The obtained results show that EAC* are lowly sensitive, k* are middling sensitive and π_x* highly sensitive to changes in the value of π_0. Moreover, L_i*and Q* is unchanged in π_0.

CONCLUSIONS

This study extends the Pan and Hsiao model (2001) to study the impact of safety factor on the continuous review inventory model involving controllable lead time and backorder price discount with mixture of backorder and partial lost sales. The objective is to minimize the expected total annual cost by simultaneously optimizing order quantity, backorder price discount, safety factor and lead time. It is found that the expected total inventory cost tends to decrease as the upper bound of the backorder ratio increases while all the other parameters stay fixed. The results of the illustrative example also show that the expected total inventory cost and the safety factor increase for a given the upper bound of the backorder ratio decreases. The savings of the total cost on considering safety factor that is decision variable are demonstrated in the examples as well.

APPENDIX

The Hessian matrix H of EAC(Q, π_x, k, L) for a given value of L is:

$$H = \begin{bmatrix} \frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial Q^2} & \frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial Q \partial \pi_x} & \frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial Q \partial k} \\ \frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial \pi_x \partial Q} & \frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial \pi_x^2} & \frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial \pi_x \partial k} \\ \frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial k \partial Q} & \frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial k \partial \pi_x} & \frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial k_x^2} \end{bmatrix} \quad (10)$$

Where:

$$\frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial Q^2} = \frac{2AD}{Q^3} + \frac{2D}{Q^3}\left[\frac{\beta_0}{\pi_0}\pi_x^2 + \pi_0 - \beta_0\pi_x\right]\sigma\sqrt{L}\Psi(k) + \frac{2D}{Q^3}\left[a_i(L_{i-1} - L) + \sum_{j=1}^{i-1} a_j(T_j - t_j)\right]$$

$$\frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial \pi_x^2} = \frac{2D\beta_0}{Q\pi_0}\sigma\sqrt{L}\Psi(k)$$

$$\frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial k^2} = \left[h\left(1 - \frac{\beta_0}{\pi_0}\pi_x\right) + \frac{D}{Q}\left(\frac{\beta_0}{\pi_0}\pi_x^2 + \pi_0 - \beta_0\pi_x\right)\right]\sigma\sqrt{L}\phi(k)$$

$$\frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial Q \partial \pi_x} = \frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial \pi_x \partial Q} = -\frac{D}{Q^2}\left[\frac{2\beta_0}{\pi_0}\pi_x - \beta_0\right]\sigma\sqrt{L}\Psi(k)$$

$$\frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial k \partial \pi_x} = \frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial \pi_x \partial k} = \left[h\frac{\beta_0}{\pi_0} - \frac{D}{Q}\left(\frac{2\beta_0}{\pi_0}\pi_x - \beta_0\right)\right]\sigma\sqrt{L}(1 - \Phi(k))$$

and

$$\frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial Q \partial k} = \frac{\partial^2 EAC(Q,\pi_x,k,L)}{\partial k \partial Q} = \frac{D}{Q^2}\left[\frac{\beta_0}{\pi_0}\pi_x^2 + \pi_0 - \beta_0\pi_x\right]\sigma\sqrt{L}(1 - \Phi(k))$$

Next, evaluate the principal minor of H at point (Q*, πx*, k*). The first principal minor of H is

$$|H_{11}| = \frac{2AD}{Q^{*3}} + \frac{2D}{Q^{*3}}\left[\frac{\beta_0}{\pi_0}\pi^{*2}_x + \pi_0 - \beta_0\pi^*_x\right]\sigma\sqrt{L}\Psi(k^*) + \frac{2D}{Q^{*3}}\left[a_i(L_{i-1} - L) + \sum_{j=1}^{i-1} a_j(T_j - t_j)\right] > 0 \quad (11)$$

Since from Eq. 7 that $\pi_x^* = hQ^*/2D + \pi_0/2$, the second principal minor of H is:

$$|H_{22}| = \frac{4\beta_0 D^2 \sigma L^{1/2}\Psi(k^*)}{Q^{*4}\pi_0}\left\{A + \left[a_i(L_{i-1} - L) + \sum_{j=1}^{i-1} a_j(T_j - t_j)\right]\right\} + \frac{\beta_0 D^2 \sigma^2 L \Psi^2(k^*)}{Q^{*4}}(4 - \beta_0) > 0 \quad (12)$$

Therefore, after substituting π_x^* from Eq. 6 and $\Phi(k^*)$ from Eq. 8, we obtain

$$\frac{hQ}{\pi_0 D}=\frac{\left[\left(2-\beta_0-\frac{2}{1-\Phi(k)}\right)+\sqrt{\left((2-\beta_0-\frac{2}{1-\Phi(k)})^2+\beta_0(4-\beta_0)\right)}\right]}{\beta_0}$$

The third principal minor of H is:

$$|H_{33}|=\left\{\frac{4\beta_0 D^2\sigma L^{1/2}\Psi(k^*)}{Q^{*4}\pi_0}\left\{A+\left[a_i(L_{i-1}-L)+\sum_{j=1}^{i-1}a_j(T_j-t_j)\right]\right\}+\frac{\beta_0 D^2\sigma^2 L\Psi^2(k^*)}{Q^{*4}}(4-\beta_0)\right\}$$

$$\times\left[h\left(1-\frac{\beta_0}{\pi_0}\pi^*_x\right)+\frac{D}{Q^*}\left(\frac{\beta_0}{\pi_0}\pi^{*2}_x+\pi_0-\beta_0\pi^*_x\right)\right]\sigma\sqrt{L}\phi(k^*)$$

$$-\left[\frac{D}{Q^{*2}}\left[\frac{\beta_0}{\pi_0}\pi^{*2}_x+\pi_0-\beta_0\pi^*_x\right]\sigma\sqrt{L}P_Z(k)\right]^2\times\frac{2D\beta_0}{Q^*\ \pi_0}\sigma\sqrt{L}\Psi(k^*)$$

$$=\frac{4\beta_0 D^2\sigma L^{1/2}\Psi(k^*)}{Q^{*4}\ \pi_0}\left\{A+\left[a_i(L_{i-1}-L)+\sum_{j=1}^{i-1}a_j(T_j-t_j)\right]\right\}$$

$$\times\left[h\left(1-\frac{\beta_0}{\pi_0}\pi^*_x\right)+\frac{D}{Q^*}\left(\frac{\beta_0}{\pi_0}\pi^{*2}_x+\pi_0-\beta_0\pi^*_x\right)\right]\sigma\sqrt{L}\phi(k^*)$$

$$+\frac{\beta_0 D^2\sigma^2 L\Psi^2(k^*)}{Q^{*4}}(4-\beta_0)\times h\left(1-\frac{\beta_0}{\pi_0}\pi^*_x\right)\times\sigma\sqrt{L}\phi(k^*)$$

$$+\frac{D^3\sigma^2 L}{Q^{*5}}\sigma\sqrt{L}\Psi(k^*)\left[\frac{\beta_0\pi_0}{4}\left(\frac{hQ^*}{\pi_0 D}\right)^2+\pi_0\left(1-\frac{\beta_0}{4}\right)\right]\times F(k^*) \quad (13)$$

Where, $F(k^*) = \beta_0(4-\beta_0)\phi(k^*)\Psi(k^*) - 2\beta_0(1-\Phi(k^*))^2$

$$\times\left[\frac{1}{4}\times\left[2-\frac{2}{1-\Phi(k^*)}-\beta_0+\sqrt{\left(2-\frac{2}{1-\Phi(k^*)}-\beta_0\right)^2+\beta_0(4-\beta_0)}\right]^2+\beta_0\left(1-\frac{\beta_0}{4}\right)\right]$$

For $\forall k^* \in [0, \infty)$ and $0 < \beta_0 \le 1$, $F(k^*)$ is positive. Hence, we have $|H_{33}|>0$. Consequently, the results from 11-13 show that the Hessian matrix H is positive define at (Q^*, π_x^*, k^*).

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