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A New Approach to Compute the Cross-Gramian Matrix and its Application in Input-Output Pairing of Linear Multivariable Plants

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Abstract: In this study, a new approach to solve the Sylvester equation, AX+XA = -BC is derived. The calculated cross-Gramian matrix, which results from the Sylvester equation, proposes a new input-output pairing analysis for stable multivariable plants. This new approach is based on the cross-Gramian matrix of SISO elementary subsystems built from the original MIMO plant and the main advantage of the method is its simplicity to choose the best input-output pair, though, it considers the plant dynamic properties.

Key words: Sylvester equation, cross-Gramian matrix, decentralized control, input-output pairing

INTRODUCTION

Decentralized controllers are used in many complex multivariable plants (Asano and Morari, 1998; Astrom *et al.*, 2002; Takagi and Nishimura, 2003; Tan *et al.*, 2001; Moaveni and Khaki Sedigh, 2007a; Skogestad and Postlethwaite, 2005). An appropriate input-output pairing prior to the commencement of the design is vital for desired closed-loop stability and performance. There are different approaches to input-output selection and Relative Gain Array (RGA) is the first and the most widely used analytical tool for this problem (Skogestad and Postlethwaite, 2005; Van de Wal and De Jagar, 2001). The Relative Gain Array (RGA) was introduced by Bristol as a measure for interactions in decentralized control systems (Bristol, 1966). This seminal work of Bristol, resulted mainly from his engineering background with little theoretical basis and proof. However, in the past two decades there have been extensive theoretical studies about the RGA method (Xiong *et al.*, 2005; Chen and Seborg, 2002; Kariwala *et al.*, 2006; Moaveni and Khaki Sedigh, 2007b).

In state-space approach to input-output pairing for linear multivariable plants, a method to provide the input-output pairing of stable, controllable and observable multivariable plants is introduced (Khaki-Sedigh and Shahmansoorian, 1996). This method is based on the analysis of the elements of a matrix obtained from the cross-Gramian matrix of the system in balanced realization form. Another input-output pairing method in this category is provided in (Conley and Salgado, 2004). This method is proposed for stable multivariable plants and their measure is based on the controllability and observability Gramians. Furthermore, this measure allows the designer to assess the benefits of other controller structures (triangular, block diagonal, sparse etc.). Wittenmark and Salgado introduced another input-output pairing method for stable multivariable plants (Wittenmark and Salgado, 2002). Their proposed method uses the Hankel norm of the SISO elementary subsystems built from the original multivariable plant. The main advantage of the Hankel interaction index is its ability to quantify the frequency dependent interactions.

In this study, a new input output pairing method for stable linear multivariable plants based on the cross-Gramian matrix is introduced. Where, the cross-Gramian matrix is the solution of Sylvester equation, AX+XA = -BC and we propose a new method to solve the Sylvester equation. Finally, simulation results are employed to show the effectiveness of the results.

THE CROSS-GRAMIAN MATRIX

Consider the linear single-input, single-output, asymptotically stable, controllable and observable time invariant system S(A, b, c) described by:

$$\dot{x} = Ax+bu \qquad y = cx \tag{1}$$

using the impulse response of the system, the cross-Gramian matrix W_{co} is defined by Fernando and Nicholson (1983):

$$W_{co} \overset{\Delta}{=} \int_0^\infty (e^{At}b)(e^{A^Tt}c^T)^T dt = \int_0^\infty e^{At}bce^{At}dt \tag{2}$$

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It is easily seen that the matrix W_{co} can be computed by solving the following Sylvester equation (Fernando and Nicholson, 1983):

$$AW_{co} + W_{co}A = -bc \tag{3}$$

Since, matrix A is assumed stable, a unique solution matrix W_{co} exists. It is intuitively clear that matrix W_{co} carries information about both controllability and observability properties (Fernando and Nicholson, 1983).

SOLVING THE SYLVESTER EQUATION

There are several known methods to solve the general Sylvester equation. But these methods are computationally complicated (Hu and Cheng, 2006; Jbilou, 2006; Zhou and Duan, 2005).

In this section, two theorems are introduced to provide a new solution for a class of Sylvester equations, $AW_{co} + W_{co}A = -bc$ where, it is important to compute the cross-Gramian matrix of linear stable systems.

Theorem 1 (distinct eigenvalues): Consider the single-input, single-output, asymptotically stable, linear time invariant system, $S(A_{n\times n}, b_{n\times 1}, C_{1\times n})$ where $\lambda_i(i = 1,\ldots,n)$ are distinct eigenvalues of matrix A and $v_i(i = 1,\ldots,n)$ are the corresponding eigenvectors. The cross-Gramian matrix, W_{co}, of this system can be computed as:

$$W_{co} = -[I \;\; I \;\; \cdots \;\; I]\begin{bmatrix} (A+\lambda_1 I)^{-1}bcv_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \ddots & \vdots \\ \vdots & 0 & (A+\lambda_i I)^{-1}bcv_i & 0 & \vdots \\ \vdots & \ddots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & (A+\lambda_n I)^{-1}bcv_n \end{bmatrix}[v_1 \;\; \cdots \;\; v_i \;\; \cdots \;\; v_n]^{-1} \tag{4}$$

where, I is the n×n identity matrix and 0 is n×1 a zero vector.

Proof: As $S(A_{n\times n}, b_{n\times 1}, C_{1\times n})$ is asymptotically stable, cross-Gramian matrix can be computed using Sylvester Eq. 3. So, multiplying the Sylvester equation by eigenvector v_i gives:

$$AW_{co} + W_{co}A = -bc \;\; \Rightarrow AW_{co}v_i + W_{co}Av_i = -bcv_i \tag{5}$$

hence,

$$(5) \Rightarrow \;\; AW_{co}v_i + \lambda_i W_{co}v_i = -bcv_i \Rightarrow (A+\lambda_i I)W_{co}v_i = -bcv_i \tag{6}$$

and

$$W_{co}v_i = -(A+\lambda_i I)^{-1}bcv_i, \qquad (i = 1,\ldots,n) \tag{7}$$

Rewriting these n equations in a matrix form gives:

$$W_{co}\left[v_1 \;\; \cdots \;\; v_i \;\; \cdots \;\; v_n\right] = -[(A+\lambda_1 I)^{-1}bcv_1 \;\; \cdots \;\; (A+\lambda_i I)^{-1}bcv_i \;\; \cdots \;\; (A+\lambda_n I)^{-1}bcv_n] \tag{8}$$

So, the cross-Gramian matrix can be computed as:

$$W_{co} = -[I \;\; I \;\; \cdots \;\; I]\begin{bmatrix} (A+\lambda_1 I)^{-1}bcv_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \ddots & \vdots \\ \vdots & 0 & (A+\lambda_i I)^{-1}bcv_i & 0 & \vdots \\ \vdots & \ddots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & (A+\lambda_n I)^{-1}bcv_n \end{bmatrix}[v_1 \;\; \cdots \;\; v_i \;\; \cdots \;\; v_n]^{-1} \tag{9}$$

where, I is the n×n identity matrix and 0 is a n×1 zero vector.

Theorem 2 (Repeated Eigenvalues): Consider the single-input, single-output, asymptotically stable, linear time invariant system, $S(A_{n\times n}, b_{n\times 1}, C_{1\times n})$ Assume that λ is a repeated eigenvalue of A with multiplicity n. The Jordan form of A is:

$$J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & 0 & \ddots & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} \tag{10}$$

Also, $v = v^{(1)}$ let be the eigenvector and $v^{(i)}$ $(i = 2,...,n)$ be the corresponding generalized eigenvectors. The cross-Gramian matrix, W_{co}, of this system can be computed as:

$$W_{co} = \begin{bmatrix} I_n & I_n & \cdots & I_n \end{bmatrix} \begin{bmatrix} -(A+\lambda I)^{-1}bcv^{(1)} & (A+\lambda I)^{-2}bcv^{(1)} & \cdots & (-1)^n (A+\lambda I)^{-n}bcv^{(1)} \\ 0 & -(A+\lambda I)^{-1}bcv^{(2)} & \ddots & \vdots \\ \vdots & 0 & \ddots & (A+\lambda I)^{-2}bcv^{(n-1)} \\ 0 & \cdots & 0 & -(A+\lambda I)^{-1}bcv^{(n)} \end{bmatrix} \begin{bmatrix} v^{(1)} & v^{(2)} & \cdots & v^{(n)} \end{bmatrix}^{-1} \tag{11}$$

where, I is the $n \times n$ identity matrix and 0 is a $n \times 1$ zero vector.

Proof: Sylvester Eq. 3, gives:

$$AW_{co}v^{(1)} + W_{co}Av^{(1)} = -bcv^{(1)} \tag{12}$$

using Eq. 6 and 7, Eq. 12 can be rewritten as:

$$W_{co}v^{(1)} = -(A+\lambda I)^{-1}bcv^{(1)} \tag{13}$$

Also, for generalized eigenvectors Eq. 12 gives:

$$AW_{co}v^{(2)} + W_{co}Av^{(2)} = -bcv^{(2)} \tag{14}$$

and it is well known that:

$$(A-\lambda I)v^{(2)} = v^{(1)} \tag{15}$$

Eq. 14 can be rewritten as:

$$AW_{co}v^{(2)} + \lambda W_{co}v^{(2)} + W_{co}v^{(1)} = -bcv^{(2)} \tag{16}$$

using Eq. 13 gives:

$$AW_{co}v^{(2)} + \lambda W_{co}v^{(2)} = (A+\lambda I)^{-1}bcv^{(1)} - bcv^{(2)} \tag{17}$$

and

$$W_{co}v^{(2)} = (A+\lambda I)^{-2}bcv^{(1)} - (A+\lambda I)^{-1}bcv^{(2)} \tag{18}$$

Similarly for $i = 1,...,n$:

$$W_{co}v^{(i)} = (-1)^i (A+\lambda I)^{-i}bcv^{(1)} + (-1)^{i-1}(A+\lambda I)^{-i+1}bcv^{(2)} + \cdots + (A+\lambda I)^{-2}bcv^{(i-1)} - (A+\lambda I)^{-1}bcv^{(i)} \tag{19}$$

Rewriting these equations in matrix form gives:

$$W_{co}\begin{bmatrix} v^{(1)} & v^{(2)} & \cdots & v^{(n)} \end{bmatrix} = \begin{bmatrix} I & I & \cdots & I \end{bmatrix} \begin{bmatrix} -(A+\lambda I)^{-1}bcv^{(1)} & -(A+\lambda I)^{-1}bcv^{(2)} & \cdots & -(A+\lambda I)^{-1}bcv^{(n)} \\ 0 & (A+\lambda I)^{-2}bcv^{(1)} & \ddots & \vdots \\ \vdots & 0 & \ddots & (-1)^{n-1}(A+\lambda I)^{-(n-1)}bcv^{(2)} \\ 0 & \cdots & 0 & (-1)^n (A+\lambda I)^{-n}bcv^{(1)} \end{bmatrix} \tag{20}$$

Hence,

$$W_{co} = [I \quad I \quad \cdots \quad I]\begin{bmatrix} -(A+\lambda I)^{-1}bcv^{(1)} & (A+\lambda I)^{-2}bcv^{(1)} & \cdots & (-1)^n(A+\lambda I)^{-n}bcv^{(1)} \\ 0 & -(A+\lambda I)^{-1}bcv^{(2)} & \ddots & \vdots \\ \vdots & 0 & \ddots & (A+\lambda I)^{-2}bcv^{(n-1)} \\ 0 & \cdots & 0 & -(A+\lambda I)^{-1}bcv^{(n)} \end{bmatrix}\begin{bmatrix} v^{(1)} & v^{(2)} & \cdots & v^{(n)} \end{bmatrix}^{-1} \tag{21}$$

where, I is the n×n identity matrix and 0 is a n×1 zero vector.

Note that extension of the above theorems to the more general cases of different eigenvalues with varied multiplicities is trivial. In above theorems, a method to solve the Sylvester equation and compute the cross-Gramian matrix is given. In the following examples, effectiveness of the proposed method is verified.

Example 1: Consider the linear SISO stable system as:

$$\dot{x} = \begin{bmatrix} -4 & -2 & -0.5 \\ 1 & -1 & 0.5 \\ 4 & 4 & -1 \end{bmatrix}x + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}u \qquad y = \begin{bmatrix} 7 & 3 & 3 \end{bmatrix}x \tag{22}$$

where, u and y are the input and output of system respectively and x is state vector. Diagonal form of matrix A is:

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \tag{23}$$

and corresponding eigenvectors are:

$$[v_1 \quad v_2 \quad v_3] = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 0 & -2 \end{bmatrix} \tag{24}$$

So, the cross-Gramian matrix of this system using Eq. 9 is as follows:

$$W_{co} = -[I \quad I \quad I]\begin{bmatrix} (A-I)^{-1}bcv_1 & 0 & 0 \\ 0 & (A-2I)^{-1}bcv_2 & 0 \\ 0 & 0 & (A-3I)^{-1}bcv_3 \end{bmatrix}[v_1 \quad v_2 \quad v_3]^{-1}$$
$$\Rightarrow W_{co} = \begin{bmatrix} 0.3667 & 0.575 & 0.2708 \\ -0.3667 & -0.575 & -0.2708 \\ 0.9167 & 1.25 & 0.5833 \end{bmatrix} \tag{25}$$

where, it satisfies the corresponding Sylvester equation.

Example 2: Consider the linear SISO stable system as:

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}x + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}u \qquad y = [1 \quad 1 \quad 1]x \tag{26}$$

where, u and y are the input and output of system, respectively and x is state vector. Also corresponding ordinary and generalized eigenvectors are:

$$\begin{bmatrix} v^{(1)} & v^{(2)} & v_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{27}$$

The cross-Gramian matrix of this system using Eq. 21 is as:

$$W_{co} = -\begin{bmatrix} I & I & I \end{bmatrix} \begin{bmatrix} -(A-I)^{-1}bcv^{(1)} & (A-I)^{-2}bcv^{(1)} & 0 \\ 0 & -(A-I)^{-1}bcv^{(2)} & 0 \\ 0 & 0 & -(A-2I)^{-1}bcv_1 \end{bmatrix} \begin{bmatrix} v^{(1)} & v^{(2)} & v_1 \end{bmatrix}^{-1} \tag{28}$$

and so:

$$W_{co} = \begin{bmatrix} 0.25 & 0.5 & 0.1111 \\ 0.5 & 0.75 & 0.3333 \\ -0.3333 & -0.4444 & -0.25 \end{bmatrix} \tag{29}$$

where, it satisfies the corresponding Sylvester equation.

INPUT-OUTPUT PAIRING USING THE CROSS-GRAMIAN MATRIX

Two main characteristics of an effective input-output pairing method are its simplicity and ability to encompass the dynamical behaviour of the plant. Both Gramian-based algorithms (Conley and Salgado, 2004) and Hankel Interaction Index Array method (Wittenmark and Salgado, 2002), analyze dynamical interaction and employ the results in their input-output pairing. Interaction is evaluated using the controllability and observability Gramian matrices for each elementary subsystem. This makes their application rather complicated. Here, a new approach based on the cross-Gramian matrix to simplify the above methods and preserve their dynamical analysis is proposed.

Let (A^b, B^b, C^b) be a balanced realization of the linear stable m×m transfer function matrix G(s) and (A^b, b_i^b, c^b) are the elementary subsystems defined for i, j = 1, 2,....,m. Each subsystem has a corresponding diagonal cross-Gramian matrix W^{ij}_{cob} and its norm defined as the largest singular value, $\bar{\sigma}(W^{ij}_{cob})$ is employed to quantify the ability of input u_j to control output yi.

Definition: The following matrix is defined as the Dynamical Input-Output Pairing Matrix (DIOPM):

$$\Gamma = [\bar{\sigma}(W^{ij}_{cob})]_{m\times m} \tag{30}$$

Where:

$$\bar{\sigma}(W^{ij}_{cob}) = \left\|(A^b, b_i^b, c_j^b)\right\|_H = \sqrt{\lambda_{max}(W_{cb}W_{ob})}$$

is the Hankel norm and can be interpreted as mapping past inputs to future outputs (Wittenmark and Salgado, 2002).

Similar to the approach in (Khaki-Sedigh and Shahmansoorian, 1996), input-output pairing is determined by finding the largest value in each row of matrix Γ and it corresponds to the appropriate input-output pair. In the proposed method if $G_{ij} = 0$ for a given pair (i, j), then W^{ij}_{cob}, leading to $\Gamma_{ij} = 0$ This implies that a block diagonal G gives a block diagonal Γ matrix, with the same structure. Also, it is important to observe that Γ takes the full dynamic effects of the system into account, despite the fact that, in RGA, only the steady-state or the behaviour at a single frequency is considered. Hence, Γ matrix can be used as a dynamic input-output pairing approach for linear multivariable plants. Note that to compute each element of the DIOPM, only one matrix equation should be solved. This considerably reduces the computational task of the methodology in comparison with the two Lyapunov equations to compute the controllability and observability Gramian matrices as in Conley and Salgado (2004) and Wittenmark and Salgado (2002).

Algorithm: Input-output pair selection for stable linear multivariable systems:

- 1st step: Calculate the cross-Gramian matrix W^{ij}_{cob}, for each SISO elementary subsystem.
- 2nd step: Compute the largest singular value of each cross-Gramian matrices Γ_{ij} and compute the Dynamical Input-Output Pairing Matrix.

- 3rd step: Find the largest value in each row of matrix Γ, which corresponds to the appropriate input-output pair.

Any minimal and stable state space realisation of the plant can be used in the proposed algorithm. This makes the algorithm invariant under state space realisations. To show this, it is obvious that,

$$\Gamma = [\Gamma_{ij}] = [\bar{\sigma}(W_{cob}^{ij})] = [\sqrt{\lambda_{max}((W_{co}^{ij})^2)}] \quad (31)$$

and

$$\bar{\sigma}(W_{cob}) = \sqrt{\lambda_{max}(W_{cob}W_{cob}^T)} = \sqrt{\lambda_{max}(W_{cob}^2)} \quad (32)$$

Where:

$$W_{cob}A^b + A^bW_{cob} = -B^bC^b \Rightarrow W_{cob}T^{-1}AT + T^{-1}ATW_{cob} = -T^{-1}BCT \Rightarrow$$
$$\left.\begin{array}{l} TW_{cob}T^{-1}A + ATW_{cob}T^{-1} = -BC \\ W_{co}A + AW_{co} = -BC \end{array}\right\} \Rightarrow W_{co} = TW_{cob}T^{-1} \quad (33)$$

So:

$$\bar{\sigma}(W_{cob}) = \sqrt{\lambda_{max}(W_{cob}^2)} = \sqrt{\lambda_{max}(T^{-1}W_{co}TT^{-1}W_{co}T)} = \sqrt{\lambda_{max}(T^{-1}W_{co}^2T)} = \sqrt{\lambda_{max}(W_{co}^2)} \quad (34)$$

Also, it is straightforward to show that the largest singular value of the cross-Gramian matrix of the balanced realization is equivalent to the maximum of the absolute values of the eigenvalues of the cross-Gramian for any realization as:

$$\bar{\sigma}(W_{cob}) = \sqrt{\lambda_{max}(W_{co}^2)} = \max\{|\lambda(W_{co})|\} \quad (35)$$

because, the eigenvalues of, are given by:

$$\det(\lambda I - W_{co}^2) = 0 \quad (36)$$

Let, $\lambda = \delta^2$ then:

$$\det(\delta^2 I - W_{co}^2) = \det(\delta I - W_{co})\det(\delta I + W_{co}) = 0 \Rightarrow \begin{cases} \det(\delta I - W_{co}) = 0 \rightarrow \delta = \lambda(W_{co}) \\ \vee \\ \det(\delta I + W_{co}) = 0 \rightarrow \delta = -\lambda(W_{co}) \end{cases} \quad (37)$$

hence:

$$\lambda(W_{co}^2) = \delta^2 \Rightarrow \sqrt{\lambda_{max}(W_{co}^2)} = \max\{|\lambda(W_{co})|\} \quad (38)$$

So, the Dynamical Input-Output Pairing Matrix (DIOPM) can be computed as:

$$\Gamma = [\Gamma_{ij}] = [\bar{\sigma}(W_{cob}^{ij})] = [\sqrt{\lambda_{max}((W_{co}^{ij})^2)}] = [\max\{|\lambda(W_{co}^{ij})|\}] \quad (39)$$

The following two examples show the effectiveness of the proposed input-output pairing methodology.

Example 3: Consider the system with transfer function matrix:

$$G(s) = \begin{bmatrix} \frac{-0.9019s + 15.47}{s^2 + 9.163s + 15.47} & \frac{-3.327}{s + 6.931} \\ \frac{0.8926}{s + 2.231} & \frac{0.7549s + 13.92}{s^2 + 9.163s + 15.47} \end{bmatrix} \quad (40)$$

RGA for this system is:

$$RGA(G(0)) = \begin{bmatrix} 0.8242 & 0.1758 \\ 0.1758 & 0.8242 \end{bmatrix} \quad (41)$$

and DIOPM is:

$$\Gamma = \begin{bmatrix} 0.3728 & 0.0576 \\ 0.0400 & 0.2366 \end{bmatrix} \quad (42)$$

It is easily seen that both Eq. 41 and 42 propose, (u_1-y_1, u_2-y_2) ppropriate input-output pair.

Example 4: Consider a process given in Grosdidier and Morari (1986) as:

$$G(s) = \begin{bmatrix} \frac{5}{4s+1} & \frac{2.5e^{-5s}}{(2s+1)(15s+1)} \\ \frac{-4e^{-6s}}{20s+1} & \frac{1}{3s+1} \end{bmatrix} \quad (43)$$

The conventional RGA implies the as the appropriate input-output pair. But, DIOPM, using (39), is:

$$\Gamma = \begin{bmatrix} 2.5 & 1.6027 \\ 2.4295 & 0.5 \end{bmatrix} \quad (44)$$

where, it shows that, (u_1-y_1, u_2-y_2) is an appropriate input-output pair. This loop pairing decision was obtained by Grosdidier and Morari (1986) through analyzing both magnitude and phase characteristics of the interaction between the two loops and by Xiong *et al*. (2005) using the effective relative gain array (ERGA).

CONCLUSION

In this study, we propose a new approach to compute the cross-Gramian matrix, which is simple to implement. Also, in this study a new approach based on the cross-Gramian matrix is introduced to compute the Dynamic Input-Output Pairing Matrix of the plant. The proposed approach does not require the controllability and observability Gramian matrices and only computes a cross-Gramian matrix for each elementary subsystem. Also, it does not require a balanced realization of the process.

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