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Coincidence Point Theorem for Non-Linear Hybrid Contractions in Non-Archimedean Menger PM-Space

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Abstract: In this study, we give some point coincidence theorems for non-linear hybrid contractions, that is, contractive conditions involving single-valued and multi-valued mapping in non-Archimedean Menger probabilistic metric space. By using our results, we can also give common fixed theorem for single-valued and multi-valued mapping in metric space. The results presented in this research generalize and improve many results in metric spaces and probabilistic metric spaces.

Key words: Probabilistic metric space, coincidence point, fixed point, compatible mappings, commuting mappings, menger PM-space

INTRODUCTION

Let R denote the set of real numbers and R^+ the non-negative real. A mapping F: $R \rightarrow R^+$ is called distribution function if its non-decreasing and left continuous with inf (F) = 0 and sup(F) = 1. We will denote D by the set of all distribution functions.

Let H denote the specific distribution function defined by:

$$H(x) = \begin{cases} 0, x \leq 0 \\ 1, x \geq 0 \end{cases}$$

We shall also, for convenience, adhere to the convention that for any distribution function F and for any

$$x > 0, F(\frac{x}{0}) = 1, \text{ while } F(\frac{0}{0}) = 0$$

A probabilistic metric space (briefly, a PM-space) is an ordered pair (X, F), where, X is a set and F is a mapping of F×F into D. i.e., F associate a distribution function F(p, q) by $F_{p,q}$, where the symbol $F_{p,q}$ (x) will denote the value of $F_{p,q}$, for the argument x. the functions $F_{p,q}$ are assumed to satisfy the following conditions:

(PM-1) $F_{p,q} = 1$ for all x>0 iff p = q.
(PM-2) $F_{p,q}(0) = 0$.
(PM-3) $F_{p,q} = F_{q,p}$.
(PM-4) If $F_{p,q} = 1$ and $F_{q,r}(y) = 1 \Rightarrow F_{p,r}(x+y) = 1$.

A probabilistically normed space (briefly a PN-space) is an ordered pair (X, F) where X is a real linear space, F is a mapping of X into D. (We shall denote the distribution functions by F (x) by f_x) satisfying the following conditions:

(PN-1) $f_x(t) = 1$ for all t>0 if and only if x = 0.
(PN-2) $f_x(0) = 0$.
(PN−3) $f_{\alpha x}(t) = f_x(\frac{t}{|\alpha|})$ for all $\alpha \in R, \alpha \neq 0$
(PN-4) If $f_x(t_1) = 1, f_y(t_2) = 1 \Rightarrow f_{x+y}(t_1 + t_2) = 1$

If we take $F_{x,y} = f_{x-y}$, then the PN-space must be a PM-space.

A triangle inequality is said to hold in a PM-space if and only if it holds for all triple of points, distinct not, in the space.

Let Δ: [0, 1] be a 2- place function satisfying the following conditions:

(Δ−1) $0 \leq \Delta(a,b) \leq 1$
(Δ−2) $\Delta(c,d) \geq \Delta(a,b)$ for $c \geq a, d \geq b$
(Δ−3) $\Delta(a,b) = \Delta(b,a)$
(Δ−4) $\Delta(1,1) = 1$
(Δ−5) $\Delta(a,1) \geq 1$ for $a > 0$.

Menger (1942) introduced, as the generalized triangle inequality, the following condition:
(PM-5) $F_{p,r}(x+y) \geq \Delta(F_{p,q}(x) F_{q,r}(y))$ for all x, y≥0, where Δ is 2-place function satisfying (Δ-1) to (Δ-5).

A Manger PM-space is a PM-space in which the condition (PM-5) holds universally for some choice of Δ satisfying the conditions:

(Δ−2) $\Delta(c,d) \geq \Delta(a,b)$ for $c \geq a, d \geq b$
(Δ−3) $\Delta(a,b) = \Delta(b,a)$.
(Δ−6) $\Delta(a,1) = a$ and $\Delta(0,0) = 0$
(Δ−7) $\Delta(\Delta(a,b),c) = \Delta(a, \Delta(b,c))$.

A triangular norm (briefly, a t-norm) is a 2-place function Δ: $[0,1]\times[0,1]\rightarrow[0,1]$ satisfying the conditions (Δ-2), (Δ-3), (Δ-6) and (Δ-7).

RESULTS

We introduce the concept for compatibility for single-valued and multi-valued mapping in non-Archimedean Menger probabilistic metric spaces and give some coincidence point theorems for non-linear hybrid contractions that is, contractive conditions involving single-valued and multi-valued mapping in non-Archimedean Menger probabilistic metric space.

By using our results, we can also give some common fixed-point theorem for single-valued and multi-valued mapping in metric space.

The results presented in this study generalize and improve many results of (Kaneko and Sessa, 1989), Nadler and many others in metric spaces and probabilistic metric spaces.

Let G be the family of functions g: $[0,1]\rightarrow[0,\infty]$ such that g is continuous, strictly decreasing, g (1) = 0 and g (0)<∞

Definition 1: Menger (1942); A Menger PM-space (X, F, Δ) is said to be of type $(C)_g$ if there exists a function g ϵ G such that:

$$g(F_{x,z}(t)) \le g(F_{x,y}(t)) + g(F_{y,z}(t)) \quad (1)$$

for all x, y, z ϵ X and t≥0.

Definition 2: Nadler (1969); A non-Archimedean Menger PM-space (X, F, Δ) is said to be of type (D)g if there exists a point g ϵ G such that:

$$g(\Delta(s,t)) \le g(s) + g(t) \quad (2)$$

for all s, tϵ[0,1].

Theorem 1: Chang *et al.* (1994a); If a non-Archimedean Menger PM-space (X, F, Δ) is of type $(D)_g$ and then it is of type $(C)_g$.

$$g(F_{x,z}(t)) \le g(F_{x,y}(t)) + g(F_{y,z}(t)) \quad (3)$$

Theorem 2: Chang (1990, 1985, 1984). If (X, F, Δ) is a Menger PM-space with t-norm $\Delta(a, b) \ge \Delta_m(a, b)$ for all a, b ϵ[0,1] and then it's of type $(D)_g$ for gϵ G type (D)g.

Definition 3: Menger (1942). Let (X, d) be a metric space and CB (X) be the family of all non-empty closed and bounded subsets of X.

Let δ be the Hausdorff metric on CB(X) induced by the metric d, that is:

$$\delta(A, B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\} \quad (4)$$

For all A, B $\in$ CB (X), where $d(x,A) = \inf_{y\in A} d(x,y)$

Theorem 3: Menger (1942); (a) (CB(X), d) is a metric space.

(b) If (X, d) is complete then (CB (X), d) is complete.

Theorem 4: Chang (1985); Let (X, d) be a complete metric space. If we define F: X×X→D as follows:

$$F(x, y)(t) = F_{x,y}(t) = H(t - d(x, y))$$

for all x, y ϵ X and t ϵ R then the space (X, F, Δ) with the t-norm$\Delta(a, b) = \min\{a, b\}$ for all x, y ϵ [0,1] is a τ -complet Menger PM-space.

Theorem 5: Michael (1951); If (X, F, Δ) is τ-complet PM-space with t-norm$\Delta(a, b) = \min\{a, b\}$ for all x, yϵ[0,1], then (X, d) is a d-complete metric space, where the metric d is defined as follows:

$$d(x, y) = \sup\{t \in [0,1]; F_{x,y}(t) \le 1 - t\} \quad (5)$$

for all x, y ϵ X

Definition 4: Chang (1985). Let AϵCB(X) and xϵA. We define the probabilistic distance $F_{x,A}$ between the point x and the set A as follows:

$$F_{x,A}(t) = H(t - d(x,A))$$

For all tϵR.

If we define $\tilde{F}: CB(X)\times CB(X) \rightarrow D$ by:

$$\tilde{F}(A,B)(t) = \tilde{F}_{A,B}(t) = H(t - \delta(A,B))$$

For all A, B ϵ CB (X) and t ϵ R., then $\tilde{F}$ is the Menger -Hausdorf metric induced by F.

Definition 5: Michael (1951); Let (X, F, Δ) be a τ-complete non-Archimedean Menger PM-space of type $(D)_g$ with the continuous τ-norm$\Delta(a, b) = \min\{a, b\}$ for all a, b ϵ [0,1]. Let Φ be the family of mappings ϕ: $(R^+)^5\rightarrow R^+$ such that each ϕ is non-decreasing for each variable, right-continuous and for any $t\geq 0$:

$$\phi(t,t,t,2t,0)\leq\psi(t)$$

where the function ψ: $R^+\to R^+$ is non-decreasing, right continuous and

$\psi^n(t)\to 0$ As $n\to\infty$ for all $t>0$.

Lemma 1: Chang (1990). Let ψ: $R^+\to R^+$ be non-decreasing, right continuous and
$\psi^n(t)\to 0$ As $n\to\infty$ for all $t>0$. Then we have the following:

$$\psi(t)<t \text{ for all } t>0 \tag{6}$$

$$\text{If } t\leq\psi(t) \text{ then } t=0 \tag{7}$$

Definition 6: Chang (1990). Let f be a mapping from X into itself and T be a multi-valued mapping from X into Ω, where Ω is the family of all non-empty τ-closed and probabilistically bounded subsets of X. Then:

- The mappings f and T are said to be commuting if $fTx\in\Omega$ and $fTx = Tfx$ for all $x\in X$.
- The mappings f and T are said to be compatible if $fTx\in\Omega$ and
  $\lim_{n\to\infty} g(\tilde{F}_{fTx_n,Tfx_n}(t))=0$.
  For all $t>0$, whenever $\{x_n\}$ is a sequence in X such that
  $\lim_{n\to\infty} Tx_n = A\in\Omega$ and $\lim_{n\to\infty} fx_n = z\in A$, $g\in G$.

Theorem 6: Chang (1990). Any commuting mappings are compatible, but the converse is not true.

Lemma 2: Chang *et al.* (1994a). Let $(\Omega,\tilde{F},\Delta)$ be a Menger PM-space. Then a mapping $\tilde{F}$ from $\Omega\times\Omega$ into D satisfying the following conditions:

(1) $F_{A,B}(t)=1$ For al $t>0$ if and only if $A=B$.

(2) $F_{A,B}(0)=0$

(3) $\tilde{F}_{A,B}=\tilde{F}_{B,A}$

$\tilde{F}_{A,B}(t_1+t_2)\geq\Delta(\tilde{F}_{A,C}(t_1),\tilde{F}_{C,B}(t_2))$ For all $A,B,C\in\Omega$ and $t_1,t_2\geq 0$.

(4)

Theorem 7: Let f be τ-continuous mapping from X into itself and
$\{T_n\}_{n=1}^{\infty}$
be a sequence of τ-continuous multi-valued mappings from X into Ω satisfying the following conditions:

- $T_n(X)\subset f(X)$ for $n=1,2,3,...$
- f and T_n are complete for $n=1,2,3,...$
- $g(\tilde{F}_{T_ix,T_jy}(t))\leq\psi(\max\{g(F_{fx,fy}(t)),g(F_{fx,T_ix}(t)),$
  $g(F_{fy,T_jy}(t)),g(F_{fx,T_jy}(t)),g(F_{fy,T_ix}(t))\}$

For all $x, y\in X$, $t\geq 0$ and $i\neq j$, $i, j = 12$
where, $g\in G$ and $\psi\in\Psi$.

- Suppose further that for any $x\in X$ and $a\in T_n x$, $n = 1,2, ...$ there exists a point $b\in T_{n+1}$ a Such that $F_{a,b}(t)\geq\tilde{F}_{T_nx,T_{n+1}}(t)$ for all $t>0$. Then there exists a point z $\in$ X such that $fz\in T_n$ for $n = 1,2,3, ...$, that is, z is a coincidence point of f and T_n.

Proof: Since $T_n(X)\subset f(X)$ for $n = 1,2, ...$ by condition (4) and $g\in G$ for an arbitrary $x_0\in X$, we can choose $x_1\in X$ such that $fx_1\in T_1x_0\in\Omega$.

For this point x_1 there exists a point $x_2\in X$ such that $fx_2\in T_2x_1\in\Omega$ and
$g(F_{fx_1,fx_2}(t))\leq g(\tilde{F}_{T_1x_0,T_2x_1}(t))$
for all $t\geq 0$. Similarly, there exists a point $x_3\in X$ such that
$fx_3\in T_3x_2\in\Omega$ and $g(F_{fx_2,fx_3}(t))\leq g(\tilde{F}_{T_2x_1,T_3x_2}(t))$
for all $t\geq 0$. Inductively, we can obtain a sequence $\{x_n\}$ in X such that $fx_n\in T_nx_{n-1}\in\Omega$ and
$g(F_{fx_n,fx_{n+1}}(t))\leq g(\tilde{F}_{T_nx_{n-1},T_{n+1}x_n}(t))$,
for all $t\geq 0$.

Now, we show that the sequence $\{fx_n\}$ is a cauchy sequence in X. In fact by lemma 2 and conditions (3), (4), since $g\in G$ we have:

$$\begin{aligned}g(F_{fx_n,fx_{n+1}}(t)) &\leq g(\tilde{F}_{T_nx_{n-1},T_{n+1}x_n}(t))\\ &\leq\Psi(\max\{g(F_{fx_{n-1},fx_n}(t)),g(F_{fx_{n-1},T_nx_{n-1}}(t)),\\ &g(F_{fx_n,T_{n+1}x_n}(t)),g(F_{fx_{n-1},T_{n+1}x_n}(t)),g(F_{fx_n,T_nx_{n-1}}(t)).\\ &\leq\Psi(\max\{g(F_{fx_{n-1},fx_n}(t)),g(F_{fx_{n-1},fx_n}(t)),g(F_{fx_n,fx_{n+1}}(t))\\ &,g(F_{fx_{n-1},fx_{n+1}}(t)),g(F_{fx_{n-1},fx_n}(t))\}.\end{aligned} \tag{8}$$

If $g(F_{fx_{n-1},fx_n}(t_0))<g(F_{fx_n,fx_{n+1}}(t_0))$ for some $t_0>0$, from Eq. 8 and lemma (1), it follows that:

$$\begin{aligned}g(F_{fx_n,fx_{n+1}}(t)) &\leq\Psi(\max\{g(F_{fx_n,fx_{n+1}}(t_0)),g(F_{fx_{n+1},fx_n}(t_0)),g(F_{fx_n,fx_{n+1}}(t_0))\\ &g(F_{fx_n,fx_{n+1}}(t_0)),g(F_{fx_{n-1},fx_n}(t_0)).\\ &\leq g(F_{fx_n,fx_{n+1}}(t_0)).\end{aligned}$$

Which is a contradiction. Thus, for any $t>0$, we have

$$g(F_{fx_n,fx_{n+1}}(t))\leq g(F_{fx_{n-1},fx_{n+1}}(t_0))$$

For n = 1,2, ... and so, by (8),

$$g(F_{fx_n,fx_{n+1}}(t_0)) \le \Psi(g(F_{fx_{n-1},fx_n}(t)) \\ \vdots \\ \le \Psi^n(F_{fx_0,fx_1}(t)). \quad (9)$$

For all t>0. Hence, for any positive integers m,n with m>n and t>0,

$$g(F_{fx_n,fx_{n+m}}(t)) \le g(F_{fx_n,fx_{n+1}}(t)) + \cdots + g(F_{fx_{n+m-1},fx_{n+m}}(t)) \\ \le \sum_{i=1}^{n+m-1} \Psi^i(g(F_{fx_0,fx_1}(t))) \to 0$$

as n→∞, which implies that $F_{fx_n,fx_{n+m}}(t) \to 1, \text{as } n \to \infty$, for any positive integer m, that is, $\{fx_n\}$ is a cauchy sequence in X.

Since (X,F,Ω) is τ-complete, the sequence $\{fx_n\}$ converges to a point z in X. On the other hand, by condition (7) and (9), since we have:

$$g(F_{fx_n,fx_{n+m}}(t)) \le g(\tilde{F}_{T_nx_{n-1},T_{n+1}x_n}(t)) \le \Psi^n(g(F_{fx_0,fx_1}(t)))$$

Letting $n \to \infty$, $g(\tilde{F}_{T_nx_{n-1},T_{n+1}x_n}(t)) \to 0$, so that $\tilde{F}_{T_nx_{n-1},T_{n+m}x_{n+m-1}}$ (t)→1 as n→∞ for all t>0, that is, $\{T_nx_{n-1}\}$ a cauchy sequence in $(\Omega,\tilde{F},\Delta)$, but $(\Omega,\tilde{F},\Delta)$ is τ-complete the sequence $\{T_nx_{n-1}\}$ converge to a set AϵΩ.

Next, we shall show that zϵA. Indeed, we have:

$$g(F_{z,A}(t)) \le g(F_{z,fx_n}(t)) + g(F_{fx_n,T_nx_{n-1}}(t)) + g(\tilde{F}_{T_nx_{n-1},A}(t)) \\ \le g(F_{z,fx_n}(t)) + g(F_{fx_n,fx_n}(t)) + g(\tilde{F}_{T_nx_{n-1},A}(t))$$

as n→∞ which implies $F_{z,A}(t)$→1, as n→∞ for all t>0. Thus, since AϵΩ, zϵA. Therefore, since f and T_n for n = 1,2, ... , we have

$$g(F_{fz,T_nz}(t)) \le g(F_{fz,ffx_n}(t)) + g(F_{ffx_n,T_nz}(t)) \\ \le g(F_{fz,ffx_n}(t)) + g(\tilde{F}_{ffx_n,T_nz}(t)) \\ \le g(F_{fz,ffx_n}(t)) + g(\tilde{F}_{fT_nx_{n-1},T_nfx_{n-1}}(t)) + g(\tilde{F}_{T_nfx_{n-1},T_nz}(t)) \\ \to 0$$

as n→∞ that is, $F_{fz,T_nz}(t) \to 1$ as n→∞. Since T_nzϵΩ, we have fzϵT_nz for n = 1,2, ... This completes the proof.

Corollary 1: Let f be τ-continuous mapping from X into itself and S, T be τ-continuous multi-valued mappings from X into Ω satisfying the following conditions:

- S(X)∪T(X)⊂f(X),
- The pair f, S and f, T are compatible,
- $g(F_{Sx,Ty}(t)) \le \Psi(\max\{g(F_{fx,fy}(t)), g(F_{fx,Sx}(t)), g(F_{fy,Ty}(t)) \\ g(F_{fx,Ty}(t)), g(F_{fy,Sx}(t))\}.$

  for all x,y ϵ X and t≥0 where gϵG and ϕϵΦ.

- Suppose further that for any xϵX and zϵX there exists a point bϵTa such that:

  $\tilde{F}_{a,b}(t) \ge F_{Sx,Ta}(t)$, for all, then there exists a point zϵX such that fzϵSx∩Tz, that is, z is a coincidence point of the pairs f, S and f,T. t>0.

Proof: Taking T_{2n+1} = S and T_{2n+2} = T, n = 0,1, ..., in theorem 7, the result follows immediately.

Corollary 2: Let f be τ-continuous mapping from X into itself and $\{T_n\}_{n=1}^{\infty}$ be a sequence of τ-continuous multi-valued mappings from X into Ω satisfying the following conditions:

- For any xϵX and aϵT_nx, n = 1,2, ... there exists a point bϵ$T_{n+1}a$ such that $F_{a,b}(t) \ge \tilde{F}_{T_nx,T_{n+1}}(t)$ for all t>0.
- $g(\tilde{F}_{T_ix,T_jy}(t)) \le \Psi(\max\{g(F_{x,y}(t)), g(F_{x,T_ix}(t)), g(F_{y,T_jy}(t)), \\ g(F_{x,T_jy}(t)), g(F_{y,T_ix}(t))\}).$

  for all x,yϵX and t≥0 where gϵG and ϕϵΦ.

Then there exists a point z ϵ X such that zϵT_nz for n = 1,2, ... , that is; the point z is a common fixed point of T_n.

Proof: Taking f = I_X (the identity mapping on X) in Theorem 7, the proof follows immediately.

Definition 7: Menger (1942). A metric space (X, d) is said to be non-Archimedean if the following condition holds:

d(x, y)≤max {d(x,z), d(z,y)}, for all x,y,zϵX

Theorem 8: Chang *et al.* (1994b). Let f be a τ-continuous mapping from X into itself and $\{T_n\}_{n=1}^{\infty}$ be a sequence of τ-continuous multi-valued mappings from X into Ω satisfying the conditions:

- $T_n(X) \subset f(X)$ for n = 1,2,3,......
- f and T_n are compatible for n = 1,2,3, ...
- For any xϵX and aϵT_nx, n = 1,2, ... , there exists a point bϵ$T_{n+1}a$ such that $F_{a,b}(t) \ge \tilde{F}_{T_nx,T_{n+1}}(t)$ for all t>0.
- There exists a constant k> such that :

  $$\tilde{F}_{T_ix,T_jy}(t) \ge \min\{F_{fx,fy}(kt), F_{fx,T_ix}(kt), F_{fy,T_jy}(kt), \\ F_{fx,T_jy}(kt), F_{fy,T_ix}(kt)\}$$

  for all x,yϵX and t≥0.

Then there exists a point zϵX such that fzϵT_nz for n = 1,2,3 ... , That is, z is a coincidence point of f and T_n.

Theorem 9: Let (X,d) be a complete non-Archimedean metric space and C(X) be the family of all non-empty compact subsets of X. Let f be a continuous mapping from X into itself and $\{T_n\}_{n=1}^{\infty}$ be a sequence of continuous multi-valued mappings T_n from X into C(x) satisfying the following conditions:

- $T_n(X)\subset f(X)$ for n = 1,2, ...
- f and T_n are commuting for n = 1,2, ...
- There exists a point $\alpha\in(0,1)$ such that

$$\delta(T_i x, T_j y) \le \alpha \max\{d(fx,fy), d(fx,T_i x), d(fy,T_j y), d(fx,T_j y), d(fy,T_i x)\}$$

For all x,y∈X and I≠j,i,j = 1,2,, ...

Then there exists a point z∈X such that $fz\in T_n Z$ for n = 1,2, ... , that is, z is a coincidence point of f and T_n.

Proof: By definition (4) and condition (3), we have:

$$\begin{aligned}\tilde{F}_{T_i x, T_j y}(t) &= H(t-\delta(T_i,T_j))\\ &\ge H(t-\alpha\max\{d(fx,fy),d(fx,T_i x),d(fy,T_j y), d(fx,T_j y),d(fy,T_i x)\}\\ &= H(\frac{t}{\alpha}-\max\{d(fx,fy),d(fx,T_i x),d(fy,T_j y), d(fx,T_j y),d(fy,T_i x)\}\\ &= \min\{F_{fx,fy}(\frac{t}{\alpha}),F_{fx,T_i x}(\frac{t}{\alpha}),F_{fy,T_j y}(\frac{t}{\alpha}), F_{fx,T_j x}(\frac{t}{\alpha}),F_{fy,T_i x}(\frac{t}{\alpha})\}\end{aligned}$$

For all x, y∈X and i ≠ j,i,j = 1,2, ...

Moreover, for any x∈X and $a\in T_n x$ for n = 1,2, ... , there exists a point $b\in T_{n+1}a$ such that:

$$d(a,b) \le \delta(T_n x, T_{n+1} a)$$

and so we have

$$\begin{aligned}F_{a,b}(t) &= H(t-d(a,b))\\ &\ge H(t-\delta(T_n x, T_{n+1} a))\\ &= \tilde{F}_{Tx_n, T_{n+1}a}(t)\end{aligned}$$

for all t≥0.

Therefore all conditions of Theorem 8 are satisfied and hence this theorem follows immediately. This completes the proof.

Corollary 3: Let (X,d) be a complete non-Archimedean metric space and C (X) be the family of all non-empty compact subsets of X. Let f be a continuous mapping from X into itself and T be a sequence of continuous multi-valued mappings T from X into C (x) satisfying the following conditions:

- T(X)⊂f(X).
- f, T are commuting.
- There exists an h∈(0,1) such that:

δ(Tx,Ty)≤hd (fx, fy), for all x,y∈X. Then there exists a point z∈X such that fz∈Tz.

Proof: By Definition 4 and condition 3, we have:

$$\begin{aligned}\tilde{F}_{Tx,Ty}(t) &= H(t-\delta(Tx,TY))\\ &\ge H(t-hd(fx,fy))\\ &= H(\frac{t}{h}-d(fx,fy)\\ &= F_{fx,fy}(\frac{t}{h})\end{aligned}$$

for all x,y ∈ X.

Moreover, for any x∈X and a∈Tx, there exists a point b∈Ta such that:

$$d(a,b) \le \delta(Tx,Ta)$$

And so we have

$$\begin{aligned}F_{a,b}(t) &= H(t-d(a,b))\\ &\ge H(t-\delta(Tx,Ta))\\ &= \tilde{F}_{Tx,Ta}(t)\end{aligned}$$

for all t≥0. Therefore, by theorem 8 there exists a point z∈X such that fz∈Tz. This complete the proof.

Lemma 3: Kaneko and Sessa (1989). Let (X,d) be metric space. Let f be a mapping from X into itself and T be a multi-valued mapping from X into C(X) such that the mappings f and T are compatible. If fz∈Tz for some z∈Tz for some z∈X, then fTz = Tfz.

Theorem 10: Let (X, d) and C(X) be as in Theorem 9. Let f be a continuous mapping from X into itself and T be a continuous multi-valued mapping from X into C(X) satisfying the following conditions:

- T(X)⊂f(X),
- f and T are compatible,
- There exists an $\alpha\in(0,1)$ such that

$$\delta(Tx,Ty) \le \alpha\max\{d(fx,fy), d(fx,Tx), d(fy,Ty) d(fx,Ty), d(fy,Tx)\}$$

for all $x,y \in X$. Suppose also that, for each $x \in X$ either (a) $fx \neq f^2 x$ implies $fx \notin Tx$ or (b) $fx \in Tx$ implies $\lim_{n\to\infty} f^n x = z$

for some $z \in X$. Then f and T have a common fixed point in X.

Proof: Taking $T_n = T$ in Theorem 9 then there exists a point $z \in X$ such that $fz \in Tz$, i.e., z is a coincidence point of f and T. Then by Lemma 3, we have $fTz = Tfz$.

Now, by definition 4 and condition (3), we have:

$$\begin{aligned}\tilde{F}_{Tx,Ty} &= H(t - \delta(Tx,Ty)) \\ &\geq H(t - \alpha \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \\ &\qquad d(fx,Ty), d(fy,Tx)\} \\ &= H(\frac{t}{\alpha} - \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \\ &\qquad d(fx,Ty), d(fy,Tx)\} \\ &= \min\{F_{fx,fy}(\frac{t}{\alpha}), F_{fx,Tx}(\frac{t}{\alpha}), F_{fy,Ty}(\frac{t}{\alpha}), \\ &\qquad F_{fx,Tx}(\frac{t}{\alpha}), F_{fy,Tx}(\frac{t}{\alpha})\}\end{aligned}$$

For all $x,y \in X$.

Moreover, for any $x \in X$ and $\alpha \in X$ for $n = 1,2, \ldots$, there exists a point $b \in Ta$ such that:

$$d(a,b) \leq \delta(Tx, Ta)$$

and so we have

$$\begin{aligned}F_{a,b}(t) &= H(t - d(a,b)) \\ &\geq H(t - \delta(Tx, Ta)) \\ &= \tilde{F}_{Tx,Ta}(t)\end{aligned}$$

Since $T(X) \subset f(X)$, for arbitrary $x_0 \in X$, we can choose a point $x_1 \in X$ such that $fx_1 \in Tx_0$. For this point x_1, there exists a point $x_2 \in X$ such that $fx_2 \in Tx_1$ and

$$F_{fx_1,fx_2}(t) \geq \tilde{F}_{Tx_0,Tx_1}(t)$$

For all $t \geq 0$ Inductively, we can obtain a sequence in X such that $fx_n \in Tx_{n+1}$ and

$$F_{fx_n,fx_{n+1}}(t) \geq \tilde{F}_{Tx_{n-1}Tx_n}(t)$$

for all $t \geq 0$.

Then $\{x_n\}$ is a cauchy sequence in X. But (X,d) is complete, then $x_n \to x$.

Also the subsequences $\{fx_n\}$, $\{Tx_{n-1}\}$ converges to x.

Now, since f, T are continuous then $fx_n \to fx$, Tx $fx_n \to Tx$. This shows that x is a common fixed point of f, T.

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