# Journal of Applied Sciences

ISSN 1812-5654

science alert

ANSInet
an open access publisher
http://ansinet.com

Journal of Applied Sciences 9 (6): 1185-1189, 2009
ISSN 1812-5654


# Competitive Analysis of Two Special Bahncard Replacement Problem

[1,2]Lili Ding, [1]Xinmin Liu and [1]Wanglin Kang
[1]College of Economics and Management,
[2]College of Information and Electrical Engineering,
Shandong University of Science and Technology, Shandong 266510, China

**Abstract:** This study provides a new competitive analysis framework for the Bahncard problem through introducing the two-stage discount rate and the risk management. For the online decision-makers, who have not any information about future demand, a new online algorithm is present to help them choose an optimal replacement strategy. Furthermore, when the online decision-makers are willing to increase their risk for some reward, an optimal online risk algorithm is developed, which help them manage risk based on their risk tolerance and forecast.

**Key words:** Online algorithm, risk management, online risk algorithm, forecast, competitive ratio

## INTRODUCTION

Suppose that there is a discount card, which costs C and entitles its holder to β price reduction of goods for the time of T. For the common decision-maker, the decision at which time to buy a discount card is a typical online problem, because he usually does not know when to go shopping and how many he will buy. This problem, called the Bahncard problem, was proposed by Fleischer (2001) in the computer science field. In his study two optimal deterministic online algorithms which achieve an optimal competitive ratio of 2-β were present. Fujiwara and Iwama (2005) integrated a possibility distribution assumption into the traditional competitive analysis, where they assume that the input sequences were subject to an exponential distribution. Ding *et al.* (2005) pointed out that the decision-maker was often confronted with the interest rate, which may be an essential feature of any reasonable economic models and gave the optimal deterministic online algorithm. The above Bahncard problem has various interesting applications. In all of these applications the basic question is when to switch from one activity to another more rewarding one. For example, when β = 0 and the rental is equal, the Bahncard problem is of course precisely Ski-rental problem. There are many extensible researches for this problem. Karlin *et al.* (2003) gave a randomized algorithm with a competitive ratio of e/e-1. Xu *et al.* (2007) considered the discrete and continuous model, respectively and present the optimal strategies. If the discount rate becomes the weight of packets, then this problem also can be considered as the TCP problem. Albers and Bals (2003) investigated a new objective function for TCP problem and achieve a deterministic 1.644-competitive online algorithm. Edmonds *et al.* (2003) present the competitive analysis against limited adversary and gave the optimal competitive ratios of some special cases.

A systematic study of online algorithms was given by Sleator and Tarjan (1985), who compared the performance of an online algorithm with that of an optimal offline algorithm. Karlin *et al.* (1988) introduced the notion of a competitive ratio. Note that the use of the competitive ratio for the evaluation of online algorithm is called competitive analysis. An online algorithm is said to be r-competitive (r≥1), if, given any instance of the problem denoted by σ, the cost of the solution given by the online algorithm is no more than r multiplied by that of an optimal offline algorithm: $Cost_{online}(\sigma) \leq r\, Cost_{optimal}(\sigma)$. The infimum over all r such that an online algorithm is r-competitive is called the competitive ratio of the online algorithm. An online algorithm is said to be best-possible if there does not exist another online algorithm with a strictly smaller competitive ratio.

In this study, our purpose is to improve the performance measurement of competitive analysis to allow the decision-maker to provide and benefit from a forecast but also allow him to control his risk of performing too poorly. In addition to providing more realism, the introduction of this advanced information is a natural mechanism to increase the power of online decision-maker against all-knowing adversaries in a competitive analysis framework. Note, also, that these risk behaviors provide a natural bridge between online Bahncard problems and their offline versions. It is found that with the introduction

**Corresponding Author:** Lili Ding, College of Economics and Management, Shandong University of Science and Technology, Shandong 266510, China

of competitive risk analysis, the valuable information in decision-making helps the optimal purchasing chance advance as long as the input sequences confirm to the forecast.

## COMPETITIVE ANALYSIS WITHOUT RISK

In this study we propose a two-stage Bahncard replacement model that is motivated by the more complex decision in the real life. For example, the Swiss Federal Railways offers two kinds of Half-Fare cards (cost CHF 150 and CHF 250, respectively) with different discount rates to attract more travelers. Suppose that the online decision-maker wishes to buy a new Bahncard $B_2$ with a greater discount of $\beta_2$, after he has hold a Bahncard $B_1$ with a smaller discount of $\beta_1$. However, the greater discount, the more cost of a Bahncard. One has to decide whether and when to buy this more expensive Bahncard based on his owning information.

**Optimal offline algorithm:** An optimal offline algorithm which knows the entire shopping requests is evident, due to the following observation:

Let σ be a shopping request sequence and OPT be an optimal algorithm for σ. Then we can assume that (a) Never buy the first Bahncard $B_1$ at a discount request. (b) OPT buys the second Bahncard $B_2$ either immediately or never.

**Proof:** (a) At a discount request with discount rate $\beta_1$, OPT would postpone purchasing the same card until it expires without the coming of $B_2$. At a discount request with discount rate $\beta_2$, OPT would never buy $B_1$ for $\beta_2<\beta_1$. (b) Assume that OPT buys the second Bahncard with the cost of $C_2$ at time T. It is clear that $\beta_2<\beta_1$, otherwise there is no reason to buy $B_2$. But it would be advantageous to buy $B_2$ more early, because the price $C_2$ is the same and $\beta_1$-$\beta_2$ discount cost of shopping could be saved. This contradicts optimality.

Let $R_1$ be the accumulative regular cost after purchasing the first Bahncard. Suppose that $R_2$ denotes the total regular cost after Bahncard $B_2$ is issued. According to the observation, the optimal offline cost is

$$\text{Cost}_{\text{OPT}}(\sigma)=\begin{cases}C_1+\beta_1(R_1+R_2) & R_2<R_{\text{crit}}\\ C_1+\beta_1R_1+C_2+\beta_2R_2 & R_2\geq R_{\text{crit}}\end{cases} \tag{1}$$

where, $R_{crit} = (C_2/\beta_1-\beta_2)$ is the critical cost before to buy the second $B_2$ and the break-even point for any algorithm.

**Optimal online algorithm:** In reality, there are two common algorithms for the decision-makers. One is never to buy the second Bahncard. However, such Never-buy Algorithm has the competitive ratio of $\beta_1/\beta_2$. The other is immediately to buy $B_2$ only if it appears, denoted by the Immediately-buy Algorithm. It is evidenced that the competitive ratio of the Immediately-buy Algorithm is $C_2$. Both of these two algorithms can not safeguard against a sequence of several expensive requests or cheap ones, respectively. We consider the following online strategy, hereafter called the critical-sum-shopping (CSS) algorithm.

**Algorithm CSS**

- Set the critical total regular cost to be $\bar{R}=R_{crit}$ after the appearance of $B_2$
- If $R_2<\bar{R}$, then decision-maker never buys $B_2$
- If $R_2\geq\bar{R}$, then he waits the total regular cost up to $\bar{R}$ and buys one

We now show how the optimal algorithm CSS can be derived. Let ALG be the any online algorithm and $r_{ALG}$ be the competitive ratio of ALG. The optimal competitive ratio for an online problem is $r^* = \inf_{ALG}(r_{ALG})$. The following proposition generates the optimal competitive ratio and the optimal online strategy.

**Proposition:** The optimal competitive ratio obtained by algorithm CSS is:

$$1+\frac{(\beta_1-\beta_2)C_2}{(C_1+\beta_1)(\beta_1-\beta_2)+\beta_1C_2}$$

**Proof:** Without loss of generality, assume that the Bahncard $B_2$ is available at time t. Suppose that the decision-maker does not buy the second Bahncard until the total regular cost after t is equal to $\tilde{R}$. Thus, ALG pays:

$$C_1+\beta_1R_1+\beta_1\tilde{R}+C_2+\beta_2(R_2-\tilde{R}) \tag{2}$$

For some $\tilde{R}<R_{crit}$ and consider online algorithm ALG. It is clear that the optimal choice by the offline decision-maker against ALG would not buy $B_2$. For this instance, the competitive ratio (online/offline) is:

$$r_1=\frac{C_1+\beta_1R_1+\beta_1\tilde{R}+C_2+\beta_2(R_2-\tilde{R})}{C_1+\beta_1(R_1+R_2)} \tag{3}$$

Note that $(\partial r_1/\partial\tilde{R})<0$, which is always negative. Therefore, the online decision-maker will take the maximum possible value of $\tilde{R}$, such that $\tilde{R}=R_{crit}$. The offline decision-maker would input such shopping request sequences that $R_2-\tilde{R}=0$ to make the online decision-maker in the worst case. Having assumed ε to be an arbitrarily small constant and $\tilde{R}=R_{crit}-\varepsilon$. We obtain:

$$r_1 = \frac{(\beta_1 - \beta_2)C_2}{(C_1 + \beta_1 R_1)(\beta_1 - \beta_2) + \beta_1 C_2 - \beta_1 \varepsilon} \quad (4)$$

Next we consider ALG with $\tilde{R} \geq R_{crit}$. There are two mutually exclusive cases.

In the first case, if the choice of $R_2$ is such that $R_2 < \tilde{R}$, then for $R_2 < R_{crit}$ the online and offline costs are equal and achieve the competitive ratio of 1. For any other choice of $R_2$ with $R_{crit} < R_2 < \tilde{R}$, the online ALG will not buy $B_2$, incurring a cost of $C_1 + \beta_1 (R_1 + R_2)$. Without loss of generality, assume that $R_2 = \tilde{R} - \varepsilon$. Thus, for this case the best attainable cost ratio is:

$$r_2 = \frac{C_1 + \beta_1 (R_1 + R_2)}{C_1 + \beta_1 R_1 + C_2 + \beta_2 R_2} \quad (5)$$

In the second case, the offline decision-maker chooses $R_2 \geq \tilde{R}$. Without loss of generality assume that $R_2 = \tilde{R}$. Thus, the best attainable ratio for this case is:

$$r_3 = \frac{C_1 + \beta_1 R_1 + \beta_1 \tilde{R} + C_2 + \beta_2 (R_2 - \tilde{R})}{C_1 + \beta_1 R_1 + C_2 + \beta_2 R_2} \quad (6)$$

It is found that $r_3 = r_2$. Therefore, the offline decision-maker will choose $R_2 = \tilde{R}$, enforcing the larger ratio of $r_3$. We can get derivatives $(\partial r_3 / \partial \tilde{R}) > 0$. It follows that $r_3$ is a increasing function of $\tilde{R}$. Therefore, for this case, the best attainable ratio is obtained by setting $\tilde{R} = R_{crit}$. Thus, we obtain:

$$r_3 = 1 + \frac{(\beta_1 - \beta_2)C_2}{(C_1 + \beta_1 R_1)(\beta_1 - \beta_2) + \beta_1 C_2} \quad (7)$$

Based on the above analysis, it is evident that $r_1 > r_3$. Hence, the online decision-maker chooses $\tilde{R} = R_{crit}$ and the best attainable competitive ratio is $r_3$, which is achieved by the optimal algorithm CSS. Namely, if $\bar{R}$ at some time is at least $R_{crit}$, then the decision-maker buys the second Bahncard; otherwise, never buys $B_2$.

**Proposition:** The optimal competitive ratio obtained is 2-$(\beta_2/\beta_1)$, when $\eta \rightarrow 0$.

**Proof:** Set $C_1 + \beta_1 R_1 = \eta$. when $\eta \rightarrow 0$, substituting $\eta$ into the function of $r_3$ and the optimal competitive ratio is at most 2-$(\beta_2/\beta_1)$. This analysis provides a smooth generalization of Fleischer's results, which are the special cases obtained with $\beta_1 = 1$.

## COMPETITIVE ANALYSIS WITH RISK

The above competitive analysis is the most fundamental and significant approach, yet it has been criticized as making too conservative assumption about future input sequences. Especially in the economic issues, many decision-makers do not seek to minimize risk, but to manage it. MacCrimmon *et al.* (1986) introduce a basic risk paradigm as the basis for studying risk. Al-Binali (1999) takes a risk by assuming that input sequence will obey some constraints. We provide the following online risk algorithm displayed in their manners.

**Online risk algorithm:** Given any online deterministic algorithm ALG, define the risk of ALG to be Risk (ALG) = $r_{ALG}/r^*$ and a forecast be F as any subset of the allowable input sequences. It is clear that the risk of any online algorithm is $\geq 1$ and the lower the risk is, the better its performance guarantees. Next, define a forecast denoted by F as any subset of the allowable input sequences. The online decision-maker specifies a risk tolerance $\lambda$. This means that the decision-maker is willing to use the restricted algorithms in $\xi$ = {ALG: Risk (ALG)$\leq \lambda$}. Each of the algorithms in $\xi$ thus has a competitive ratio of at most $\lambda r^*$. Fix any forecast F. An optimal risk algorithm, according to this risk management framework, is an algorithm from $\xi$ that minimizes the competitive ratio, restricted to input sequences from F. Formally, the restricted competitive ratio $\hat{r}$ of any online risk algorithm can be parameterized by the constrains of the total input sequences from F such that:

$$\hat{r}_{ALG} = \sup_{\sigma \in F} \{Cost_{ALG}(\sigma) / Cost_{OPT}(\sigma)\} \quad (8)$$

Thus, the optimal restricted competitive ratio by ALG* with respect to a forecast F can be achieved from:

$$\hat{r}^* = \inf_{ALG} \{\hat{r}_{ALG} : ALG \in \xi\}$$

The reward of the optimal online risk algorithm ALG* denoted by $g_{ALG}$ is measured by the ratio of the optimal competitive ratio to the restricted ratio. The optimal risk algorithm ALG* with respect to a forecast F satisfies:

$$\begin{cases} \max_{ALG} \{g_{ALG} = r^* / \hat{r}_{ALG}\} \\ \text{s.t.} \quad \hat{r}_{ALG} \leq \lambda r^* \end{cases} \quad (9)$$

The steps to use this algorithm can be described as follows:

**Step 1:** Initialize the forecast, $\sigma \in F$
**Step 2:** Set the risk tolerance level to be $\lambda$
**Step 3:** According to definition of Eq. 8, compute the restricted competitive ratio $\hat{r}_{ALG}$, where ALG$\in\xi$ and $\xi$ = {ALG: Risk (ALG)$\leq \lambda$}

**Step 4:** Compare the restricted competitive ratio with the optimal competitive ratio and achieve the reward $g_{ALG}$

**Step 5:** Solve the model Eq. 9 to obtain the optimal online risk algorithm ALG*

We analyze two-stage Bahncard problem in the competitive risk analysis framework based on the two possible forecasts of $R_2 < R_{crit}$ and $R_2 \geq R_{crit}$. For the case of $R_2 < R_{crit}$, the two-stage Bahncard problem has the optimal competitive ratio such that $\hat{r}^* = 1$. It is because that both the offline and online decision-maker will never purchase the second Bahncard with this forecast. For the case of $R_2 \geq R_{crit}$, we present the following strategy:

**Optimal risk tolerance strategy:** Given η, λ, a new Bahncard $B_2$ and a forecast of $R_2 \geq R_{crit}$, the online decision-maker would not buy $B_2$ until the total regular cost of $\tilde{R}$ is up to:

$$\frac{(C_2+\eta)(\lambda r_3-1)}{\beta_1-\beta_2\lambda r_3}$$

otherwise, never purchase $B_2$.

We present the following proposition to show that the risk tolerance strategy is optimal according to the above analysis. The result shows that the introduction of forecast improves the competitive analysis performance of the algorithm CSS, if $R_2 \geq R_{crit}$ is correct.

**Proposition:** If the forecast of $R_2 \geq R_{crit}$ is correct, the optimal restricted competitive is:

$$r_3 = 1+\frac{(\beta_1-\beta_2)(C_2-(\lambda r_3-1)\eta)}{\eta(\lambda r_3-1)(\beta_1-\beta_2)+C_2((\lambda r_3-1)\beta_1+\beta_2)}$$

**Proof:** If $R_2 \geq R_{crit}$ is correct, then the online decision-maker would choose an optimal risk algorithm from ξ to obtain the more reward based on his tolerance. The offline decision-maker would buy the second Bahncard as soon as it appears. Therefore, the restricted competitive ratio of the online risk algorithm is:

$$\hat{r} = \frac{\eta+\beta_1\tilde{R}+C_2+\beta_2(R_2-\tilde{R})}{\eta+C_2+\beta_2 R_2} \quad (10)$$

Since, we want to minimize $\hat{r}$, we want $R_2$ as small as possible subject to $\hat{r} \leq \lambda r^3$ from the following two cases.

**Case 1:** $R_2 < R_{crit}$ According to the preceding risk definition, the online decision-maker can obtain the inequality:

$$\frac{\eta+\beta_1\tilde{R}+C_2+\beta_2(R_2-\tilde{R})}{\eta+\beta_1 R_2} \leq \lambda r^3$$

If $C_2-(\lambda r_3-1)\eta>0$, then we obtain:

$$\tilde{R} \geq \frac{C_2-(\lambda r_3-1)\eta}{(\lambda r_3-1)\beta_1} \quad (11)$$

**Case 2:** $R_2 \geq R_{crit}$. It is shown that the restricted competitive ratio in the false forecast satisfies:

$$\frac{\eta+\beta_1\tilde{R}+C_2+\beta_2(R_2-\tilde{R})}{\eta+C_2+\beta_2 R_2} \leq \lambda r^3$$

from the above competitive risk analysis framework. Thus we get the following result:

$$\tilde{R} \leq \frac{(C_2+\eta)(\lambda r_3-1)}{\beta_1-\beta_2\lambda r_3} \quad (12)$$

For the online decision-maker, he knows the function of $\hat{r}$ in case 2 is monotonically increasing with $\tilde{R}$. The less $\hat{r}$, the smaller $\tilde{R}$. Therefore, substituting the lower bound of Eq. 11 into the function of restricted competitive ratio, we obtain the optimal restricted competitive ratio.

**Proposition:** The optimal restricted competitive ratio is at most:

$$r_3 = 1+\frac{\beta_1-\beta_2}{(\lambda r_3-1)\beta_1+\beta_2}$$

for $\eta \to 0$.

## NUMERICAL EXAMPLE

In general, it is shown that competitive risk analysis can improve the decision-maker's performance significantly. We provide the numerical examples to develop a more intuitive understanding about present analysis and the effect on the competitive ratio when the parameters in this model are varying.

In Table 1, the numerical matters are about the two kinds of Half-Fare cards problem offered by Swiss Federal Railways. Set $C_1 = 150$ CHF, $\beta_1 = 80\%$ and $C_2 = 250$ CHF, $\beta_2 = 50\%$. According the above results, the optimal deterministic online algorithm CSS chooses the total regular cost of $\tilde{R}$ up to 833 CHF to buy the second card and obtain the competitive ratio of 1.375. Using the optimal risk tolerance strategy, the traveler can benefit from his correct forecast to achieve the less ratio of 1.2654 based on $\lambda = 1.10$. It means if the traveler would take a risk

Table 1: Values of $\hat{r}$ ($\beta_1$, $\beta_2$) for different combinations

| ($\beta_1$, $\beta_2$) | $\lambda$ | r* | $\hat{r}^*$ | $r^*/\hat{r}^*$ |
|---|---|---|---|---|
| (0.8, 0.5) | 1.10 | 1.375 | 1.3296 | 1.2841 |
| (0.8, 0.5) | 1.20 | 1.375 | 1.3108 | 1.2989 |
| (0.9, 0.5) | 1.20 | 1.444 | 1.4349 | 1.0739 |
| (0.8, 0.6) | 1.20 | 1.250 | 1.2000 | 1.0416 |

of achieving ratio of 1% larger than optimal one, then he can get a performance improvement of about 57%. It is also found that r* decreases with $\beta_2$ and increases with $\beta_1$.

## CONCLUSIONS

The classical competitive analysis is the most fundamental and important approach to study online problems. But it is not very flexible since it avoids making assumptions about future inputs. In this study, we provide an online risk algorithm, which allows the decision-makers to manage their risk and utilize their forecasts. But how to improve the performance of the competitive algorithm by other methods, such as probability statistics, is a direction. Another direction is the competitive analysis about more discount activities, for example three or more discount cards.

## REFERENCES

Al-Binali, S., 1999. A risk-reward framework for the competitive analysis of financial game. Algorithmica, 25: 99-115.

Albers, S. and H. Bals, 2003. Dynamic TCP acknowledgment: Penalizing long delays. Proceedings of the 14th Annual ACM-SIAM symposium on Discrete Algorithms, Jan. 12-14, Philadelphia, PA., USA., pp: 47-55.

Ding, L.L., Y.F. Xu and S.H. Hu, 2005. The Bahncard Problem with Interest Rate and Risk. In: Internet and Network Economics, Xiaotie, D. and Y. Yinyu (Eds.). Springer-Verlag, Germany, Berlin, Heidelberg, ISBN: 978-3-540-30900-0, pp:307-314.

Edmonds, J., S. Datta and P. Dymond, 2003. TCP is competitive against a limited adversary. Proceedings of the 15h Annual ACM Symposium on Parallel Algorithms and Architectures, June 7-9, ACM New York, USA., pp: 174-183.

Fleischer, R., 2001. On the Bahncard problem. Theor. Comput. Sci., 268: 161-174.

Fujiwara, H. and K. Iwama, 2005. Average-case competitive analysis for ski-rental problems. Algorithmica, 42: 95-107.

Karlin, A., M. Manasse, L. Rudolph and D. Sleator, 1988. Competitive snoopy caching. Algorithmica, 3: 79-119.

Karlin, A.R., C. Kenyon and D. Randall, 2003. Dynamic TCP acknowledgment and other stories about e/(e-1). Algorithmica, 36: 209-224.

MacCrimmon, K.R., W.T. Stanbury and D.A. Wehrung, 1986. Taking risks: The management of uncertainty. 1st Edn., Free Press, New York.

Sleator, D. and R. Tarjan, 1985. Amortized efficiency of list update and paging rules. Commun. ACM, 28: 202-208.

Xu, Y.F., W.J. Xu and H.Y. Li, 2007. On the on-line rent-or-buy problem in probabilistic environments. J. Glob. Optim., 38: 1-20.