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Nonlinear Contraction Theorems in Fuzzy Spaces

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Abstract: In this study, fuzzy metric and normed space are considered and some fixed point theorems in these spaces are proved. In this study at first two fixed point theorems in nonlinear case in the fuzzy metric spaces are proved then an nonlinear contraction theorem in the fuzzy normed spaces is proved.

Key words: Fuzzy sets, fuzzy metric space, fuzzy normed spaces, completeness, fixed point theorem

INTRODUCTION

The concept of fuzzy sets was introduced initially by Zadeh (1965). To use this concept in topology and analysis many researchers have developed a theory of fuzzy sets and applications. George and Veeramani (1994) introduced the concept of fuzzy metric spaces. In this study we state some of the basic facts about fuzzy metric and normed spaces.

Definition 1: A binary operation $*[0.1]\times[0,1]\rightarrow[0,1]$ is ontinuous t-norm if * is satisfying the following conditions:

- * is commutative and associative
- * is continuous
- $a*1 = a$ for all $a\in[0,1]$
- $a*b\leq c*d$ whenever $a\leq c$ and $b\leq d$ and $a,b,c,d \in[0,1]$

Two typical examples of continuous t-norm are $a*b = ab$ and $a*b = \min(a,b)$.

Definition 2: A triple (X, M, *) is called a fuzzy metric space if X is an arbitrary (non-empty) set, * is a continuous t-norm, and M is a fuzzy set on $X^2\times(0,\infty)$, satisfying the following conditions for each x, y $z\in X$ and t.s>0:

- $M(x, y, t)>0$
- $M(x, y, t) = 1$ if and only if $x = y$
- $M(x, y, t) = M(y, x, t)$
- $M(x, y, t) * M(y, z, t+s)\leq M(y, z, t+s)$
- $M(x, y)$:$]0,\infty)\rightarrow]0,1]$ is continuous

Example 1: Let (X.d) be a metric space, define $a*b = ab$ or $a*b = \min(a,b)$ and

$$M(x,y.t)=\frac{t}{t+d(x,y)}$$

which is called the standard fuzzy metric induced by metric d.

Let (X, M, *) be a fuzzy metric space . For t>0, the open ball B(x, r, t) with center $x \in X$ and radius $0<r<1$ is defined by:

$$B(x,r,t)=\{y\in X: M(x,y,t)>1-r\}$$

Let (X, M, *) be a fuzzy metric space. Let τ be the set of all $A\subset X$ with $x \in A$ if and only if there exist t>0 and $0<r<1$ such that $B(x, r, t) \subset A$.

Then τ is a topology on x (induced by the fuzzy metric M). This topology is Hausdorff and first countable. A sequence $\{x_n\}$ in X converges to x if and only $M(x_n, x, t)\rightarrow 1$ as $n\rightarrow \infty$, for each t>0. It is called a Cauchy sequence if for each $0<\varepsilon<1$ and t>0, there exits $n_0 \in N$ such that $M(x_n, x_m, t)>1-\varepsilon$ for each $n, m\geq n_0$. The fuzzy metric space (x, M, *) is said to be complete if every Cauchy sequence is convergent.

Lemma 1: Let (X, M, *) be fuzzy metric space. Then, M (x, y, t) is non-decreasing with respect to t, for all x, y in X.

Definition 3: The triple (X, N, *) is said to be a fuzzy normed space if X is a vector space, * is a continuous t-norm and N is a fuzzy set on $X\times(0,\infty)$ satisfying the following conditions for every $x, y \in X$ and t, s>0:

- $N(x, t)>0$
- $N(x, t) = 1$ if and only if $x = 0$
- $N(\alpha x,t)=N\left(x,\frac{t}{|\alpha|}\right)$ for each $\alpha \neq 0$

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- $N(x, t)*N(y, s) \leq N(x+y, t+s)$
- $N(x) : (0, \infty) \to [0, 1]$ is continuous

where, in (c), α is in the scalar field of X. In this case N is called a fuzzy norm.

Lemma 2: Let (X, N,*) be a fuzzy normed space. If define:

$$M(x, y, t) = N(x-y, t)$$

then, M is a fuzzy metric on X, which is said to be induced by the fuzzy norm N.

The fuzzy normed space (X, N, *) is said to be a fuzzy Banach space whenever X is complete with respect to the fuzzy metric induced by fuzzy norm.

Lemma 3: Let (X, M, *) be fuzzy metric space and define

$X_{\lambda,M}: X^2 \to R^+ \cup \{0\}$ by

$$E_{\lambda,M}(x, y) = \inf\{t>0: M(x, y, t)>1-\lambda\}$$

for each $\lambda \in]0,1[$ and $x, y \in X$. Then

(i) For any $\mu \in]0,1[$ there exists $\lambda \in]0,1[$ such that:

$$E_{\mu,M}(x_1, x_n) \leq E_{\lambda,M}(x_1, x_2)+\ldots+E_{\mu,M}(x_{n-1}, x_n)$$

for any $x_1,\ldots,x_n \in X$

(ii) The sequence $\{x_n\}_{n\in N}$ is convergent with respect to fuzzy metric M if and only if $E_{\lambda,M}(x_n, x) \to 0$. Also the sequence $\{x_n\}$ is a Cauchy sequence with respect to fuzzy metric M if and only if it is a Cauchy sequence with $E_{\lambda,M}$.

Proof: For (i), for every $\mu \in]0,1[$, there is a $\lambda \in]0,1[$ such that $(1-\lambda)*\ldots*(1-\lambda)>(1-\mu)$. By the triangular inequality:

$$\begin{aligned}
&M\left(x, x_n, E_{\lambda,M}(x_1,x_2)+\ldots+E_{\lambda,M}(x_{n-1},x_n)+n\delta\right)\\
&\geq M\left(x_1,x_2,E_{\lambda,M}(x_1,x_2)+\delta\right)*\ldots*M\left(x_{n-1},x_n,E_{\lambda,M}(x_{n-1},x_n)+\delta\right)\\
&\geq (1-\lambda)*\ldots*(1-\lambda)>1-\mu
\end{aligned}$$

for every $\delta>0$, which implies that:

$$E_{\mu,M}(x_1,x_n) \leq E_{\lambda,M}(x_1,x_2)+E_{\lambda,M}(x_2,x_3)+\ldots+E_{\lambda,M}(x_{n-1},x_n)+n\delta.$$

Since, $\delta > 0$, is arbitrary,

$$E_{\mu,M}(x_1,x_n) \leq E_{\lambda,M}(x_1,x_2)+E_{\lambda,M}(x_2,x_3)+\ldots+E_{\lambda,M}(x_{n-1},x_n)$$

For (ii) $M(x_n,x,\eta) > (1-\lambda) \Leftrightarrow E_{\lambda,M}(x_n,x) < \eta$ for every $\eta > 0$.

Definition 4: The fuzzy metric space (X, M, *) said that has the property (C), if it satisfies the following condition:

$$M(x, y, t) = C, \text{ for all } t>0 \text{ implies } C = 1$$

Lemma 4: Let the function $\phi(t)$ satisfies the following condition:
(ϕ) $\phi(t):[0,\infty]$ is nondecreasing and

$$\sum_{n=1}^{\infty}\phi^n(t)<\infty$$

for all $t>0$, when $\phi^n(t)$ denotes the n-th iterative function of $\phi(t)$, then $\phi(t)<t$ for all $t>0$.

THE MAIN RESULTS

Theorem 1: Let $\{A_n\}$ be a sequence of mappings A_i of a complete fuzzy metric space (X, M, *), which this space has the property (C), into itself such that, for any two mappings A_i, A_j:

$$M\left(A_i^m(x), A_j^m(y), \phi_{i,j}(t)\right) \geq M(x,y,t)$$

for some m, $x, y \in X$ and for all $t>0$.

Here, $\phi_{i,j}:[0,\infty) \to [0,\infty)$ is a function such that $\phi_{i,j}(t)<\phi(t)$ for i, j = 1,2,... and the function $\phi(t)$: $[0,\infty) \to$ on to $[0,\infty)$ is strictly increasing and satisfies condition ϕ. Then the sequence $\{A_n\}$ has a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X and define a sequence $\{x_n\}$ in X by:

$$x_1 = A_1^m(x_0), x_2 = A_2^m(x_1), \ldots.$$

Then:

$$\begin{aligned}
M(x_1,x_2,\phi(t)) \geq M(x_1,x_2,\phi_{1,2}(t)) &= M\left(A_1^m(x_0), A_2^m(x_1), \phi_{1,2}(t)\right)\\
&\geq M(x_0,x_1,t)
\end{aligned}$$

and:

$$\begin{aligned}
M(x_2,x_3,\phi^2(t)) &\geq M\left(x_2,x_3,\phi_{2,3}(\phi(t))\right)\\
&= M\left(A_2^m(x_1), A_3^m(x_2), \phi_{2,3}(\phi(t))\right)\\
&\geq M(x_1,x_2,\phi(t))\\
&\geq M(x_0,x_1,t)
\end{aligned}$$

and so on. By induction,

$$M\left(x_n,x_{n+1},\phi^n(t)\right) \geq M(x_0,x_1,t)$$

which implies

$$\begin{aligned} E_{\lambda,M}(x_n,x_{n+1}) &= \inf\{\phi^n(t)>0: M(x_n,x_{n+1},\phi^n(t))>1-\lambda\} \\ &\le \inf\{\phi^n(t)>0: M(x_0,x_1,t)>1-\lambda\} \\ &= \phi^n(\inf\{t>0: M(x_0,x_1,t)>1-\lambda\}) \\ &= \phi^n(E_{\lambda,M}(x_0,x_1)) \end{aligned}$$

for every $\lambda \epsilon]0,1[$.

Now, showed that $\{x_n\}$ is a Cauchy sequence. For every $\mu\epsilon]0,1[$, there exists $\gamma\epsilon]0,1[$such that:

$$\begin{aligned} &E_{\mu,M}(x_n,x_m) \\ &\le E_{\gamma,M}(x_{m-1},x_m)+E_{\gamma,M}(x_{m-2},x_{m-1})+\ldots+E_{\gamma,M}(x_n,x_{n+1}) \\ &\le E_{\gamma,M}(x_{01},x_1)\sum_{j=n}^{m-1}\phi^j(t)\to 0 \end{aligned}$$

as m, n > 1. Since, X is complete, there is $x \epsilon X$ such that

$$\lim_{n\to\infty} x_n = x$$

Now is proved that x is a periodic point of A_i for any i = 1, 2, ... Notice:

$$\begin{aligned} M(x,A_i^m(x)t) &\ge M(x,x_n,t-\phi(t))*M(x_n,A_i^m(x),\phi(t)) \\ &= M(x,x_n,t-\phi(t))*M(A_n^m(x_n-1),A_i^m(x),\phi(t)) \\ &\ge M(x,x_n,t-\phi(t))*M(A_n^m(x_n-1),A_i^m(x),\phi_{n,i}(t)) \\ &\ge M(x,x_n,t-\phi(t))*M(x_{n-1},x,t) \\ &\to 1*1=1 \end{aligned}$$

as $n\to\infty$. Thus $M(x,A_i^m(x),t)=1$ and is got $A_i^m(x)=x$.

To show uniqueness, assume that $y \neq x$ is another periodic point of A_i.
Then:

$$\begin{aligned} M(x,y,\phi^n(t)) \ge M(x,y,\phi_{i,j}(\phi^{n-1}(t))) &= M(A_i^m(x),A_i^m(y),\phi_{i,j}(\phi^{n-1}(t))) \\ &\ge M(x,y,\phi^{n-1}(t)) \\ &\ge M(x,y,\phi_{i,j}(\phi^{n-2}(t))) \\ &= M(A_i^m(x),A_i^m(y),\phi_{i,j}(\phi^{n-2}(t))) \\ &\ge M(x,y,\phi^{n-2}(t)) \ge \ldots \ge M(x,y,t) \end{aligned}$$

On the other hand, by Lemma 1 implies that:

$$M(x,y,\phi^n(t)) \le M(x,y,t)$$

Hence, M(x, y, t) = C for all t>0. Since, M has the property (C), it follows:

That C = 1, i.e., X = Y. Also:

$$A_i(x) = A_i(A_i^m(x)) = A_i^m(A_i(x))$$

i.e., A_i (x) is also a periodic point of A_i. Therefore, x = A_i (x), i.e., x is a unique common fixed periodic point of the mappings A_n for n = 1,2... . This completes the proof.

Theorem 2: Let (X, M, *) be a complete fuzzy metric space, let (X, M, *) has the property (C) and let f, g: $X\to X$ be maps that satisfy the following conditions:

- g (x) $\subseteq$ f(X)
- f is continuous
- M (g(x), g(y), $\phi(t)$)$\ge$M(f(x), f(y),t) for all x, y ϵ X where, the function $\phi(t):[0,\infty]\xrightarrow{\text{onto}}[0,\infty]$ is strictly increasing and satisfies condition ϕ

Then f and g have a unique common fixed point provided f and g commute.

Proof: Let $x_0 \epsilon X$. By (a) there is x_1 such that $f(x_1) = g(x_0)$. By induction, we can define a sequence $\{x_n\}_n$ such that $f(x_n) = g(x_{n-1})$. By induction again:

$$\begin{aligned} M(f(x_n),f(x_{n+1}),\phi^n(t)) &= M(g(x_{n-1}),g(x_n),\phi^n(t)) \\ &\ge M(f(x_{n-1}),f(x_n),\phi^{n-1}(t)) \\ &\ge \ldots \ge M(f(x_0),f(x_1),t) \end{aligned}$$

for n = 12,..., which implies that:

$$\begin{aligned} E_{\lambda,M}(f(x_n),f(x_{n+1})) &= \inf\{\phi^n(t)>0: M(f(x_n),f(x_{n+1}),\phi^n(t))>1-\lambda\} \\ &\ge \inf\{\phi^n(t)>0: M(f(x_0),f(x_1),t)>1-\lambda\} \\ &= \phi^n(\inf\{t>0: M(f(x_0),f(x_1),t)>1-\lambda\}) \\ &= \phi^n(E_{\lambda,M}(f(x_0),f(x_1))) \end{aligned}$$

for every $\lambda\epsilon]0,1[$.

Now, have been showed that $\{f(x_n)\}$ is a Cauchy sequence. For every $\mu\epsilon]0,1[$, there exists $\gamma\epsilon]0,1[$ such that, for $m\ge n$,

$$\begin{aligned} &E_{\mu,M}(f(x_n),f(x_m)) \\ \le\ &E_{\gamma,M}(f(x_{m-1},f(x_m))+E_{\gamma,M}(f(x_{m-2}),f(x_{m-1}))+\ldots+E_{\gamma,M}(f(x_n)f(x_{n+1}))) \\ \le\ &\sum_{j=n}^{m-1}\phi^j(E_{\gamma,M}(f(x_0),f(x_1)))\to 0 \end{aligned}$$

as $m,n\to\infty$. Since, X is complete, there exists y ϵ X such that $\lim_{n\to\infty} f(x_n) = y$. So, $g(x_{n-1}) = f(x_n)$ tends to y. It can be seen from (c) that the continuity of f implies that to g.

Thus $\{g(f(x_n))\}_n$ converges to g(y). However, $g(f(x_n)) = f(g(x_n))$ by the commutativity of f and g. Thus $f(g(x_n))$ converges to f (y). Because the limits are unique, f(y) = g(y). So, f(f(y)) = f(g(y)) by commutativity and:

$$\begin{aligned} M(g(y),g(g(y)),\phi^n(t)) &\geq M(f(y),f(g(y)),\phi^{n-1}(t)) \\ &= M(g(y),g(g(y)),\phi^{n-1}(t)) \\ &\geq \ldots \geq M(g(y),g(g(y)),t) \end{aligned}$$

On the other hand, Lemma 1 implies that:

$$M(g(y),g(g(y)),\phi^n(t)) \leq M(g(y),g(g(y)),t)$$

Hence, M (g(y), g(g(y), t) = C for all t>0. Since, M has the property (C), it follows that C = 1, i.e., g(y) = g(g(y)). Thus, g (y) = g(g(y)) = f (g(y)). So, g(y) is a common fixed point of f and g.

If y and z are two fixed points common to f and g, then:

$$\begin{aligned} &M(y,z,t) \\ = \; &M(g(y),g(z),\phi^n(t)) \geq M(f(y),f(z),\phi^{n-1}(t)) \\ = \; &M(y,z,\phi^{n-1}(t))\ldots \geq M(y,z,t) \end{aligned}$$

On the other hand, by Lemma 1

$$M(y,z,\phi^n(t)) \leq M(y,z,t)$$

Hence, M(y, z, t) = C for all t>0. Since, M has the property (C), it follows that C = 1, i.e., y = z.

Theorem 3: Let W be a closed and convex subset of a fuzzy Banach space (V, N, *) and f: W→W a mapping which satisfies the condition:

$$N(x - fx, \phi(t)) * N(y - fy, \phi(t)) \geq N\left(x - y, \frac{t}{2}\right) \qquad (1)$$

for all x, y∈W and for all t>0. The function $\phi(t):[0,\infty]\xrightarrow{\text{on to}}[0,\infty]$ is strictly increasing and satisfies condition ϕ. Then f has at least a fixed point.

Proof: Let x_0 in W be arbitrary and let a sequence $\{x_n\}$ be defined by:

$$x_{n+1} = [x_n + f(x_n)]/2 \qquad (n = 0,1,2,\ldots)$$

For this sequence,

$$x_n - fx_n = 2[x_n - (x_n + fx_n)/2] = 2(x_n - x_{n+1})$$

and hence,

$$N(x_n - fx_n, \phi(t)) = N\left(x_n - x_{n+1}, \frac{\phi(t)}{2}\right) \quad (n = 0,1,2,\ldots) \qquad (2)$$

Therefore, for x = x_{n-1} and y = x_n the condition (Eq. 1) states:

$$N(x_{n-1} - fx_{n-1}, \phi(t)) * N(x_n - fx_n, \phi(t)) \geq N\left(x_{n-1} - x_n, \frac{t}{2}\right)$$

By condition (Eq. 2):

$$N(x_{n-1} - x_n, \phi(t)/2) * N(x_n - x_{n+1}, \phi(t)/2) \geq N(x_{n-1} - x_n, t/2)$$

Hence, $N(x_n - x_{n+1}, \phi(t)) \geq N(x_{n-1} - x_n, t)$.

By Lemma 2 as proof of Theorem 1 is concluded that $\{x_n\}$ is Cauchy sequence in W and converges to some u∈W. Since:

$$\begin{aligned} N(u - fx_n, t) &\geq N(u - fx_n, \phi(t)) \\ &\geq N(u - x_n, \phi(t)/2) * N(x_n - fx_n, \phi(t)/2) \\ &= N(u - x_n, \phi(t)/2) * N(x_n - x_{n+1}, \phi(t)/4) \end{aligned}$$

Hence,

$$\lim_{n\to\infty} fx_n = u$$

Now, let us put in Eq. 1 x = u and y = x_n and use Eq. 2. Then:

$$N(u - fu, \phi(t)) * N(x_n - x_{n+1}, \phi(t)) \geq N(u - x_n, t/2)$$

If now n tend to infinity one has:

$$N(u\text{-}fu, \phi(t)) = 1$$

which implies that fu = u and this theorem is established.

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