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Dynamics Response of Railway Under a Moving Load

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Abstract: In this study, a new method for the dynamic analysis of railway, as a beam with limited length, lying on a viscoelastic bed and subjected to a moving load is presented. The dynamic analysis was carried out for Pasternak-viscoelastic bed having shear layers. The aim was to obtain deflection, slope, bending moment and shear forces of the beam under moving load. By utilizing the theory of dynamic response of Timoshenko beam and using modal superposition method, the governing equations of motion were obtained. Given the fact that conventional methods are incapable of showing discontinuities in shear diagram and break point in moment diagram which arise from the concentrated moving load, this study introduces a new method based on summing modes for handling the aforementioned discontinuities. A main advantage of this method is in the accurate and efficient evaluation of the bending moment and shear force of a beam under moving loads. Numerical results are presented to show the rapid convergence of responses using the proposed method.

Key words: Dynamics analysis, viscoelastic bed, Timoshenko beam, moving load

INTRODUCTION

The analysis of dynamic behavior of structures under moving loads has been one of the research interests in recent years. For dynamic analysis of beams under moving loads depending on beam parameters, several methods are used. Analytical methods such as using separation of variations, integral transforms or numerical methods such as finite difference or finite element methods can be mentioned. Fryba (1999) showed that for the beams of finite length the problem can be solved by expanding the deflection function as an infinite series. Otherwise the problem can be solved by transformations which are compatible with boundary conditions (Kargarnovin and Younesian, 2004). Fryba (1999) presented one of the most comprehensive references for moving load problems, in which the beams with finite length are dynamically analyzed for different load equations such as concentrated, distributive, vibrational, random, 2-D moving system and also for different velocity status such as constant or inconstant.

Biondi *et al*. (2004) and Pesterev and Bergman (2000) have divided the moving load problems in three groups: first group includes the problems in which load inertia is negligible in comparison with the beam inertia. In second group the beam inertia is negligible in comparison with load inertia. Third group are the problems in which both load inertia and beam inertia are sizeable and should be taken into account. The latter leads to more complicated problem.

Pesterev and Bergman (2000) presented a method to improve the response of an Euler-Bernoulli beam which is exposed to a moving vibrating system. Furthermore, Pesterev *et al.* (2001) presented a method to improve the response especially for bending moment and shear force along an Euler-Bernoulli beam exposed to a moving load with constant value. The basis of this method is summing vibrational response of Euler-Bernoulli beam by modal analysis with quasi-static response of the system. Evaluating of the quasi-static solution of Euler-Bernolli beam under the moving loads, dynamic equations are solved by modal analysis method.

Mei and Mace (2005) evaluated the vibration of the beams and structures by using Timoshenko theorem and wave propagation in elastic ambient-Dalamber solution. He also presented different shapes of the modes in a Timoshenko beam due to different frequencies.

In this study, the dynamic response analysis of a rail of as a Timoshenko beam with finite length on a Pasternak-type viscoelastic bed and fully defined supports and subjected to a moving load is investigated. The speed of the moving load is assumed to be constant A new method for handling the aforementioned discontinuities is proposed. This method not only can

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appropriately show these continuities, but leads to quick convergence of the responses. Moreover, considering first three modes of the systems, the obtained response is more accurate. Results can be used by railway engineers in dynamic analysis of the rails.

THEORETICAL ANALYSIS

Motion equations: In this study, the rail is considered as a Timoshenko beam with finite length on a Pasternak-type viscoelastic bed. The concept of Pasternak viscoelastic bed is shown in Fig. 1, in which instead of elastic shear layer (which is used in Pasternak-type elastic bed) a viscose shear layer is used and the linear viscose elements are arranged parallel to the spring elements. This arrangement by considering the interaction of viscoelastic elements and also the shear forces between the elements due to the beam deflection, provides a more real simulation of the rail.

In the present study, the following assumptions are used in analysis of the beam on Pasternak bed under moving load:

- The behavior of the beam can be described by Timoshenko theorem
- Geometrical and mechanical parameters of the beam such as density, modulus of elasticity, cross section and moment of inertia are constant along the beam
- The mass of moving load is negligible in comparison with the beam mass
- The material of the beam is uniform, isotropic and linear elastic
- The speed of moving load is constant
- Structure depreciation is negligible
- The beam is always under the moving load

Considering Fig. 1 governing differential equations of the motion of a Timoshenko beam are (Timoshenko, 1990):

$$\rho A\frac{\partial^2 w(x,t)}{\partial t^2}+k^*AG\left[\frac{\partial\psi(x,t)}{\partial x}-\frac{\partial^2 w(x,t)}{\partial x^2}\right]+ kw(x,t)+\eta\frac{\partial w(x,t)}{\partial t}-\mu\frac{\partial^3 w(x,t)}{\partial t\partial x^2}=-F_0\delta(x-\zeta)\chi(t) \quad (1a)$$

$$\rho I\frac{\partial^2\psi(x,t)}{\partial t^2}+k^*AG\left[\psi(x,t)-\frac{\partial w(x,t)}{\partial x}\right]-EI\frac{\partial^2\psi(x,t)}{\partial x^2}=0 \quad (1b)$$

where, w(x, t) and ψ(x, t) are deflection and slope of the beam due to bending moment, respectively.

Equation 1a, b describe differential equations of a Timoshenko beam supported by a Pasternak-type viscoelastic foundation subjected to moving load with a constant speed. In Eq. 1a, b, $\delta(x-\zeta)$ is Dirac delta-function and ζ is instantaneous position of the moving load.

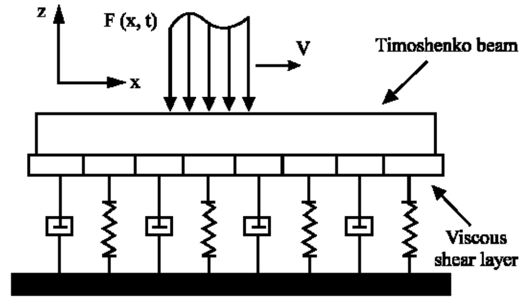

Fig. 1: A Timoshenko beam on Pasternak-viscoelastic bed having shear layers under moving load

$$\zeta = Vt \quad (2)$$

where, V is the velocity of the moving load.

Moreover, $\chi(t)$ is a function which is equal to unity when the beam is under load and otherwise it is equal to zero:

$$\chi(t)=u(t)-u(t-L/V) \quad (3)$$

where, u(t) is unit step function.

For sake of simplicity by introducing the notations:

$$\frac{EI}{\rho A}=C_b \quad \frac{k^*AG}{\rho A}=C_s \quad \frac{\rho I}{\rho A}=C_r \quad \frac{k}{\rho A}=C_L \quad \frac{\eta}{\rho A}=C_d \quad \frac{\mu}{\rho A}=C_v \quad (4)$$

and substituting into Eq. 1, we can get:

$$\ddot{w}(x,t)+C_s\left(\psi'(x,t)-w''(x,t)\right)+C_L w(x,t)+ C_d\dot{w}(x,t)-C_v\dot{w}''(x,t)=-\frac{F_0\delta(x-\zeta)\chi(t)}{\rho A} \quad (5a)$$

$$C_r\ddot{\psi}(x,t)+C_s\left(\psi(x,t)-w'(x,t)\right)-C_b\psi''(x,t)=0 \quad (5b)$$

where, (.) and (') are derivatives related to time and position, respectively.

Evaluating the vibration modes of the system: Modal analysis method is based on the separation of variables to solve differential equations of the free vibration motion and calculate eigenfunctions which satisfy the boundary equations (Fryba, 1999). For this purpose, by considering w(x, t) and ψ(x, t) as:

$$w(x,t)=W(x)e^{i\omega t} \quad (6a)$$

$$\psi(x,t) = \Psi(x)e^{i\omega t} \tag{6b}$$

and substituting them into Eq. 5 and neglecting the viscoelasticity of the bed and the motion of the load, the following eigenvalue problem appears:

$$-\omega^2 W(x) + C_s(\Psi'(x) - W''(x)) = 0 \tag{7a}$$

$$-C_r\omega^2\Psi(x) + C_s(\Psi(x) - W'(x)) - C_b\Psi''(x) = 0 \tag{7b}$$

In order to simplify the solution of the above differential equations, by neglecting $\Psi(x)$ and its derivatives, the following forth degree equation according to W(x) appears:

$$C_b W^{IV} + \left(C_r + \frac{C_b}{C_s}\right)\omega^2 W'' + \left(\frac{C_r}{C_s}\omega^4 - \omega^2\right)W = 0 \tag{8}$$

Considering $e^{\beta x}$ as the base response and substituting it into Eq. 8 one can get:

$$C_b\beta^4 + \left(C_r + \frac{C_b}{C_s}\right)\omega^2\beta^2 + \left(\frac{C_r}{C_s}\omega^4 - \omega^2\right) = 0 \tag{9}$$

To solve this equation for β^2, Δ will be in this form:

$$\Delta = \left(C_r - \frac{C_b}{C_s}\right)^2 \omega^4 + 4C_b\omega^2 \tag{10}$$

Δ is always positive, therefore the roots of Eq. 9 can be obtained as:

$$\beta_1^2 = \frac{-1}{2}\omega^2\left[\frac{C_r}{C_b} + \frac{1}{C_s}\right] - \sqrt{\frac{1}{4}\left[\frac{C_r}{C_b} - \frac{1}{C_s}\right]^2 \omega^4 + \frac{\omega^2}{C_b}} \tag{11a}$$

$$\beta_2^2 = \frac{-1}{2}\omega^2\left[\frac{C_r}{C_b} + \frac{1}{C_s}\right] + \sqrt{\frac{1}{4}\left[\frac{C_r}{C_b} - \frac{1}{C_s}\right]^2 \omega^4 + \frac{\omega^2}{C_b}} \tag{11b}$$

Because β_1^2 is always negative, two sine and cosine responses are obtained for Eq. 7. Mei and Mace (2005) by introducing critical frequency as:

$$\omega_{cr}^2 = \frac{C_s}{C_r} \tag{12}$$

showed that for frequencies lower than critical frequency ($\omega<\omega_{cr}$), β_2^2 is positive and thus the other base response of the system would be of type hyperbolic sine and cosine:

$$W(x) = A_1 \sin\beta_1 x + A_2 \cos\beta_1 x + A_3 \sinh\beta_2 x + A_4 \cosh\beta_2 x \tag{13}$$

For frequencies higher than critical frequency ($\omega>\omega_{cr}$), the deflection mode of the beam is in the form:

$$W(x) = A_1 \sin\beta_1 x + A_2 \cos\beta_1 x + A_3 \sin\beta_2 x + A_4 \cos\beta_2 x \tag{14}$$

For more simplicity β_1 and β_2 are calculated from these equations:

$$\beta_1 = \sqrt{\frac{1}{2}\omega^2\left[\frac{C_r}{C_b} + \frac{1}{C_s}\right] + \sqrt{\frac{1}{4}\omega^4\left[\frac{C_r}{C_b} - \frac{1}{C_s}\right]^2 + \frac{\omega^2}{C_b}}} \tag{15a}$$

$$\beta_2 = \sqrt{\left|-\frac{1}{2}\omega^2\left[\frac{C_r}{C_b} + \frac{1}{C_s}\right] + \sqrt{\frac{1}{4}\omega^4\left[\frac{C_r}{C_b} - \frac{1}{C_s}\right]^2 + \frac{\omega^2}{C_b}}\right|} \tag{15b}$$

Usually, a large part of energy is carried by primary modes. In other words, the participation factor of higher modes in comparison with lower modes is negligible. Thus the forms of primary modes are used in modal analysis. Hence, in this study Eq. 13 is used in the analysis. By substituting W(x) from Eq. 13 into Eq. 7a and introducing notations:

$$P = \beta_1 - \frac{\omega^2}{\beta_1 C_s} \tag{16a}$$

$$Q = \beta_2 + \frac{\omega^2}{\beta_2 C_s} \tag{16b}$$

$\Psi(x)$ can be written in this form:

$$\Psi(x) = A_1 P\cos\beta_1 x - A_2 P\sin\beta_1 x + A_3 Q\cosh\beta_2 x + A_4 Q\sinh\beta_2 x \tag{17}$$

The constants A_1, A_2, A_3 and A_4 and values ω, β_1 and β_2 depend on beam boundary conditions. Using boundary conditions leads to following algebraic equations:

$$H_{4\times4}\begin{bmatrix}A_1\\A\\A_3\\A_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{18}$$

where, ω and then β_1, β_2 are calculated by taking determinant of H equal to zero. Constants A_2, A_3 and A_4 can be calculated from Eq. 18 according to an arbitrary coefficient A_1. This coefficient will be obtained from following equations which indicate the perpendicularity of vibration modes:

$$\int_0^L (W_m(x)W_n(x) + C_r\Psi_m(x)\Psi_n(x))dx = \delta_{mn} \tag{19a}$$

$$\int_0^L \{C_s W_m(x)(\Psi_n'(x) - W_n''(x)) + C_s \Psi_m(x)(\Psi_n(x) - W_n'(x)) - C_b \Psi_m(x)\Psi_n''(x)\}dx = \omega_n^2 \delta_{mn} \quad (19b)$$

METHODS AND PROCEDURES

Solving motion equations by summing modes method: Generally, vertical deflection and slope of the beam as a system response to forced vibration are approximated by first n terms of following series:

$$w(x,t) = \sum_{j=1}^{n} W_j(x) y_j(t) = W(x)^T Y(t) \quad (20a)$$

$$\psi(x,t) = \sum_{j=1}^{n} \psi_j(x) y_j(t) = \psi(x)^T Y(t) \quad (20b)$$

where, W(x) and ψ(x) are vectors consist of first n terms of eigenfunctions and Y(t) is a vector consists of n passive time-dependent functions so-called generalized coordinates.

By substituting Eq. 20 into Eq. 5, pre-multiplying both sides by W(x) and ψ(x), respectively, integrating over the length *l* of the beam and summing the results we get to:

$$M\ddot{Y}(t) + C\dot{Y}(t) + KY(t) = f(t) \quad (21)$$

in which:

$$M = I_{n\times n} \quad (22a)$$

$$C = C_d \int_0^L W(x)W(x)^T dx - C_v \int_0^L W(x)W''(x)^T dx \quad (22b)$$

$$K = \Omega^2_{n\times n} + C_L \int_0^L W(x)W(x)^T dx \quad (22c)$$

$$f(t) = -\int_0^L \tilde{F}(x,t)W(x)dx \quad (22d)$$

By solving Eq. 21 and calculating Y(t) from Eq. 20 vertical deflection and slope of the beam can be obtained. Indeed the shear force and bending moment can be written in the form:

$$M(x,t) = EI\psi'(x,t) = EI\sum_{j=1}^{n} \psi_j'(x) y_j(t) = EI\psi'(x)^T Y(t) \quad (23a)$$

$$V(x,t) = k^* AG(\psi(x,t) - w'(x,t)) = k^* AG \sum_{j=1}^{n} (\psi_j(x) - W_j'(x))\, y_j(t) = k^* AG(\psi(x)^T - W'(x)^T)Y(t) \quad (23b)$$

Improved solution for the evaluation of shear force and bending moment: In the shear force diagram of a beam under concentrated moving load, there is a jump equal to the applied load in the instantaneous point of the load. Also in the bending moment diagram there is a discontinuity due to the concentrated moving load (Pesterev *et al*., 2001). The eigenfunctions and their derivatives are continuous functions related to x and are not able to capture these discontinuities. Increasing the number of modes does not improve the accuracy due to Gibbs phenomenon. In this study, an improved method is described which can coverage the discontinuities and improve the accuracy.

For this purpose, in order to calculate shear force, taking into account Eq. 1a instead of Eq. 23a one can write:

$$V'(x,t) = k^* AG\left[\frac{\partial \psi(x,t)}{\partial x} - \frac{\partial^2 w(x,t)}{\partial x^2}\right] = -F_0\delta(x-\zeta)\chi(t) - \rho A\frac{\partial^2 w(x,t)}{\partial t^2} - kw(x,t) - \eta\frac{\partial w(x,t)}{\partial t} + \eta\frac{\partial^3 w(x,t)}{\partial t \partial x^2} \quad (24)$$

By substituting Eq. 20-22 into Eq. 24 and integrating related to x, shear force is obtained in the following form:

$$V(x,t) = k^* AG\left[\psi(x,t) - \frac{\partial w(x,t)}{\partial x}\right] = -F_0\delta(x-\zeta)\,\chi(t) + k^* AG[\psi(x) - W'(x)]^T\, Y(t) + k^* AG[\psi(x) - W'(x)]^T r(t) - k^* AG[\psi(x) - W'(x)]^T \Omega^{-2}[C_L\, Y(t) + C_d\dot{Y}(t)] + \mu W'(x)T\dot{Y}(t) + V_0(t) \quad (25)$$

where, r(t) is:

$$r(t) = \Omega^{-2}\left[CY(t) - f(t) + C_L\int_0^L W(x)W(x)^T dx Y(t)\right] \quad (26)$$

and $V_0(t)$ is integration constant. In order to calculate bending moment taking into account Eq. 1b one can write:

$$M'(x,t) = EI\frac{\partial \psi(x,t)}{\partial x^2} = \rho I\frac{\partial^2 \psi(x,t)}{\partial t^2} + k^* AG\left[\psi(x,t) - \frac{\partial w(x,t)}{\partial x}\right] = \rho I\,\psi(x)^T\ddot{Y}(t) + V(x,t) \quad (27)$$

Therefore, by using Eq. 21 upto Eq. 23, 26 and 27 bending moment is given by:

$$M(x,t) = EI\frac{\partial \psi(x,t)}{\partial x^2} = EI\psi'(x)^T Y(t) - F(t)(x-\zeta)u(x-\zeta)\chi(t) - k^* AG\int[\psi(x) - W'(x)]^T dx\,\Omega^{-2}[C_L Y(t) + C_d\dot{Y}(t)] + EI\,\psi'(x)^T r(t) + \mu W(x)^T\dot{Y}(t) + V_0(t)x + M_0(t) \quad (28)$$

By continuing the integration from Eq. 28 and 25 the equations for ψ(x, t) and w(x, t) can be defined. Integration

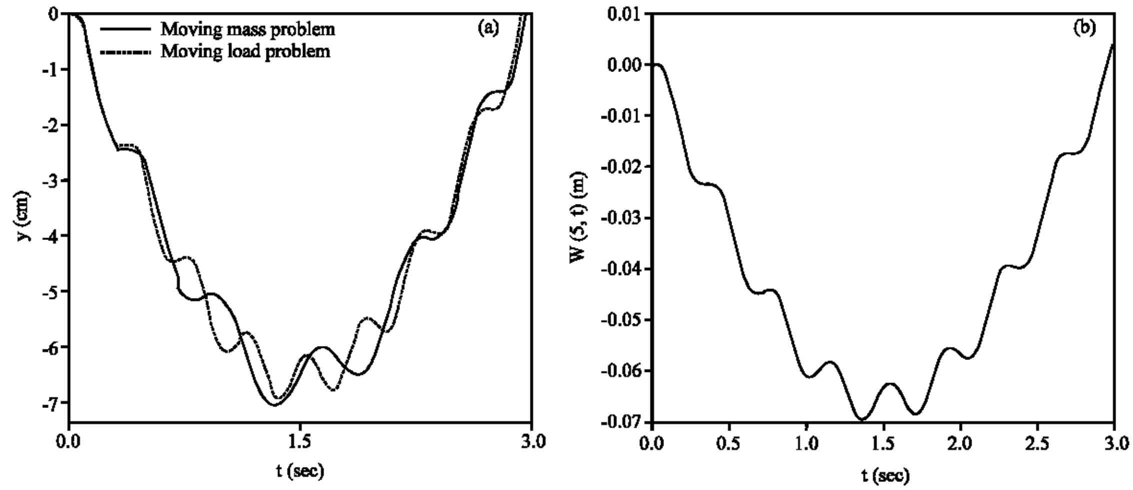

Fig. 2: Deflection time history at the midspan of beam, (a) an Euler-Bernolli beam (Esmailzadeh and Ghorashi, 1995) and (b) a Timoshenko beam, moving load problem obtained in this study

Table 1: Beam and load parameters (Esmailzadeh and Ghorashi, 1995)

Beam and load parameters	Values
L (m)	10
E (GPa)	207
I (m^4)	1.04×10^{-6}
A (m^2)	0.001
ρ (kg m^{-3})	7040
V (km h^{-1})	12
F_0 (N)	700

constants such as $V_0(t)$ and $M_0(t)$ can be calculated from boundary conditions.

Numerical example: The aim of the numerical example is to examine the convergence of the presented method to calculate deflection, slope, shear force and bending moment of a beam. The proposed method is programmed in Matlab 6.5 in which parameters of the beam, load and bed are inputted. Natural frequencies and vibration modes are first calculated in the software and then by solving differential Eq. 21 and calculating the generalized coordinates, desired outputs such as shear force and bending moment are plotted.

Due to lack of references for analysis of Timoshenko beam with limited length lying on a Pasternak viscoelastic bed under moving loads, the results are compared with results obtained from Esmailzadeh and Ghorashi (1995) which has studied the vibration of an Euler-Bernolli beam without bed and with parameters shown in Table 1. Therefore in the software the bed parameters are assumed equal to zero.

Figure 2a shows deflection time history at the midspan of an Euler-Bernolli beam with parameters shown in Table 1. For comparison a Timoshenko beam with the same parameters is considered. Other parameters which are not shown in Table 1 such as k* and G, are considered corresponding with other parameters for a steel beam with rectangle cross section area. Figure 2b shows the deflection of midspan of a Timoshenko beam calculated by software. As can be shown it is similar to Fig. 2a.

RESULTS AND DISCUSSION

A simply supported Timoshenko beam with parameters shown in Table 2 is considered in order to compare the results.

Figure 3a shows the effect of considered number of modes on vertical displacement of the beam in t = 0.3 sec, when the moving load is in midspan. The rotational angle of the cross section area of the beam versus number of the modes in the same time is shown in Fig. 3b. As can be shown, increasing the number of modes leads to convergence of responses. Also because of the influence of the bed, the effect of the moving load is limited to nearby points and at points far enough from the point of load application, the dynamic response is approaching to zero. Figure 3c, d show the effect of considered number of modes on the bending moment and shear force distribution along the beam by applying the improved method. Additional to fast convergence (in n = 3), it can be observed that using this method increases the accuracy of shear force and bending moment diagrams and also the discontinuities in these diagrams can be captured. Also it can be shown that in Fig. 3c, d like Fig. 3a, b the dynamic response at far enough points from the load approaches to zero.

Figure 4a and b show, respectively the distribution of the bending moment and the shear force along the beam by means of the conventional and proposed method.

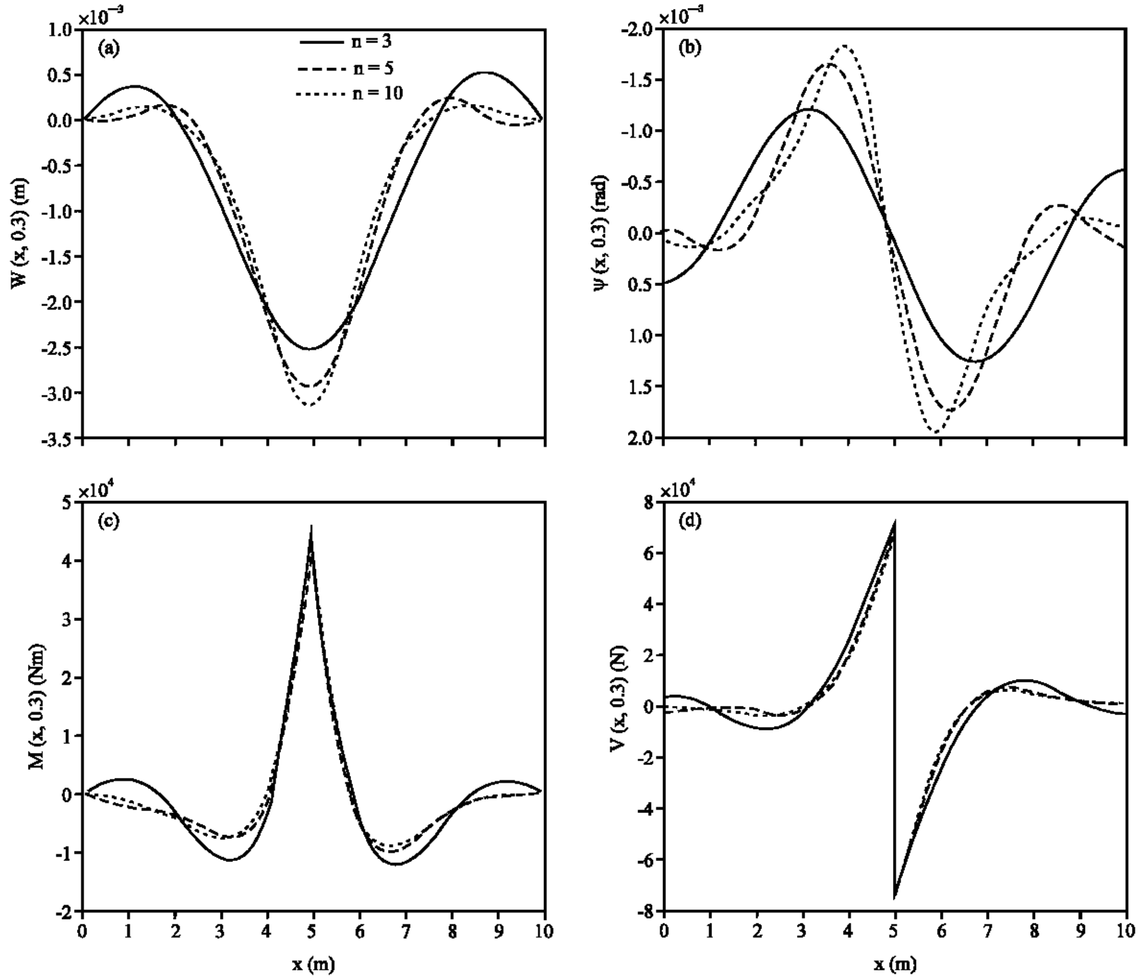

Fig. 3: The effect of considered number of modes on the dynamic response of the beam when the moving load is in midspan, (a) vertical displacement (b) slope, (c) bending moment distribution along the beam and (d) shear force distribution along the beam

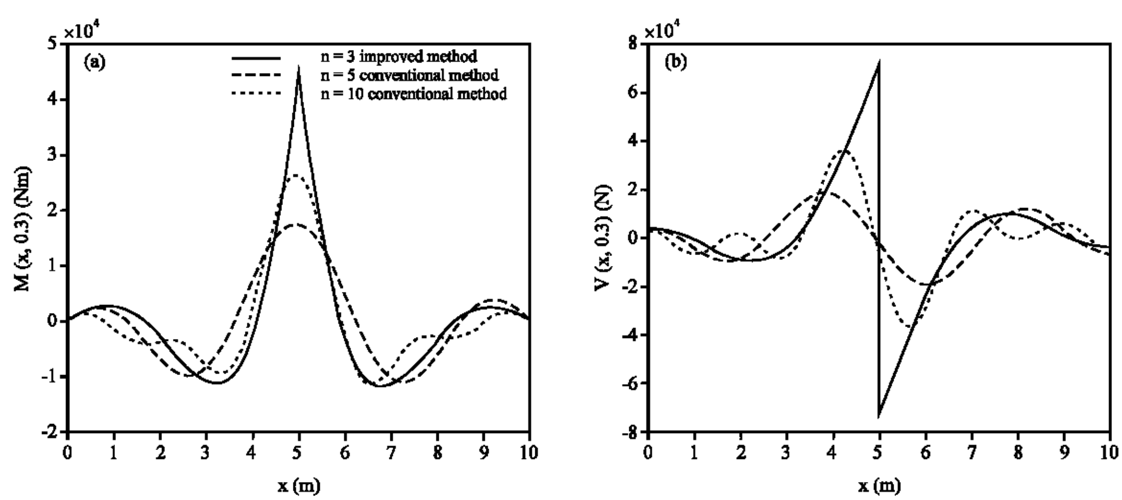

Fig. 4: A comparison of conventional method with proposed improved method in dynamic response of the beam when the moving load is in midspan, (a) bending moment distribution and (b) shear force distribution

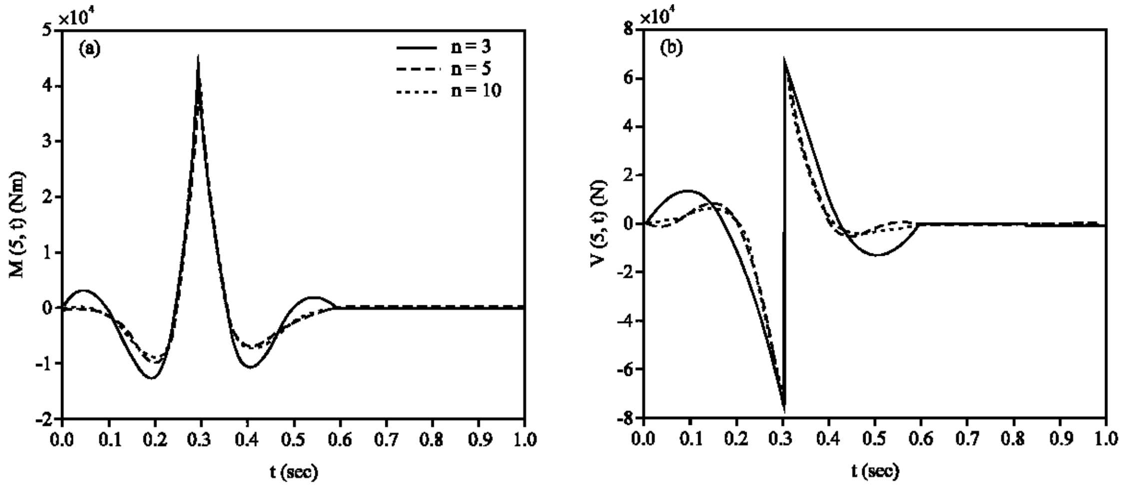

Fig. 5: (a) Bending moment time history and (b) shear force time history at the midspan of the Timoshenko beam under concentrated moving load for different mode

Table 2: Beam and load parameters as a numerical example

Beam and load parameters	Values
L (m)	10
E (GPa)	207
I (m^4)	39.5×10^{-6}
A (m^2)	86.13×10^{-4}
ρ (kg m^{-3})	7820
V (km h^{-1})	60
k*	0.85
k_f (MPa)	20
μ (kN sec)	69
η (kPa sec)	138
ν	0.3
F_0 (kN)	144

Conventional method is not able to capture the discontinuities in the bending moment because the modes used in conventional method are continuous functions. Increasing the number of modes, the response gradually approaches to the result of new method. For the same reason, conventional methods according to Dirichlet theorem can not capture the discontinuities in the shear force diagram and passes through the average of left and right limits (Fig. 4b), while improved method shows a discontinuity equal to moving load value, F_0 = 144 kN.

In Fig. 5a, b the time evaluation of the bending moment and shear force at the midspan of the beam (x = 5 m) are presented respectively. Figure 5a, b show the improved convergence of the new method. The results show that in the new method considering first three modes, first five modes or and even first ten modes does not affect grately on the results. In other words, the convergence takes place so quickly that the responses related to n = 3 are near to n = 5 and n = 10 responses. Furthermore, because of the viscoelastic bed, after passage of the moving load (t≥0.6 sec), dynamic response of the beam approaches to zero.

CONCLUSIONS

In this study, a Timoshenko beam on the Pasternak viscoelastic bed subjected to moving load is investigated by mode summations modal analysis method. First, vibration modes of a Timoshenko beam are calculated and then the solution of forced vibration of the beam is represented. Continuous modes (such as Sin, Cos, Sinh and Cosh) are used in conventional modal analysis which although can show deflection and slope of the beam, but due to using continuous function, they are not able to capture discontinuities and jumps in the shear force and the bending moment diagrams. Therefore, an improved method proposed which is able to capture the discontinuities in the shear force and the bending moment in moving load problems. The results obtained by conventional and improved methods are evaluated. A numerical example has been presented which demonstrate the improved convergence of the new presentation. The results show that the convergence of the responses is so fast that considering first three modes of the beam in new method leads to results similar to those obtained from conventional method by considering more than first five modes.

The main achievement of this study is presenting a new method to solve the problems of moving loads on a Timoshenko beam. Generally the proposed method includes following advantages:

- Capability of capturing the discontinuities and jumps in the shear force and the bending moment diagrams
- Quick convergence of the responses
- Improvement the accuracy of the response especially when considering first three modes of the system

NOTATIONS

A : Section area
E : Elastic modulus
F : Load
G : Shear modulus
I : Inertia moment
K : Bed modulus
K* : Correction coefficient
L : Length of the beam
V : Velocity of the load
w : Deflection
η : Damping coefficient
μ : Viscose coefficient
ρ : Density
ω : Frequency
ψ : Slope of the beam
ζ : Instantaneous position of the load

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