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n-Approximately Weak Amenability of Banach Algebras

H. Najafi and T. Yazdanpanah
Department of Mathematics, Persian Gulf University, Boushehr, 75168, Iran

Abstract: We introduce new notions of approximate amenability for a Banach algebra A. A Banach algebra A is n-approximately weakly amenable, for $n\in N$, if every continuous derivation from A into the n-th dual space $A^{(n)}$ is approximately inner. First we examine the relation between m-approximately weak amenability and n-approximately weak amenability for distinct $m,n\in N$. Then we investigate (2n+1)-approximately weak amenability of module extension Banach algebras. Finally, we give an example of a Banach algebra that is 1-approximately weakly amenable but not 3-approximately weakly amenable.

Key words: Banach algebras, amenability, n-weak amenability, approximate amenability, module extension Banach algebras

INTRODUCTION

Let A be a Banach algebra and X a Banach A-bimodule. A derivation from A into X is a bounded linear map satisfying:

$$D(ab) = a.D(b)+D(a).b \quad (a, b\in A)$$

For each $x\in X$ we denote by ad_x the derivation $D(a) = ax-xa$ for all $a\in A$, which is called inner derivation. We denote by $Z^1(A, X)$ the space of all derivations from A into X and by $B^1(A,X)$ the space of all inner derivations from A into X. The first cohomology group of A and X which is denoted by $H^1(A,X)$, is the quotient space $Z^1(A, X)/B^1(A, X)$. A Banach algebra A is amenable if $H^1(A, X^*) = 0$ for each A-bimodule X (X^* is the dual space of X which is an A-bimodule as usual). The concept of amenability for a Banach algebra A, introduced by Johnson (1972). The Banach algebra A is weakly amenable if $H^1(A, A^*) = 0$. Ghahramani and Loy (2004) and Dales *et al*. (1998) introduced several modifications of this notion. We recall the definitions in definitions 1 and 2, below:

Definition 1: A Banach algebra A is called approximately amenable if for each A-bimodule X and for each derivation $D: A\to X^*$ there is a net $(x_\alpha)\subseteq X$ such that $D(a)=\lim_\alpha ad_{x_\alpha}(a)$ for all $a\in A$.

Definition 2: A Banach algebra A is called n-weakly amenable if $H^1(A,A^{(n)}) = 0$, where, $A^{(n)}$ is the n-th dual space of A.

Dales *et al.* (1998) investigated the relation between m-weak amenability and n-weak amenability for distinct $m,n\in N$. They obtained important results on Banach algebras and they characterized large classes of them. Ghahramani and Loy (2004) extensively studied approximate amenability of Banach algebras and they opened a new research field on amenability.

Many other researchers have followed these studies and worked on this topic. For example, Dales *et al.* (2006) investigated this topic on Banach sequence algebras. Lashkarizadeh and Samea (2005) studied the approximate amenability for large classes of semigroup algebras. Ghahramani and Loy (2004) developed valuable results and gave new proofs for the characterization of amenability for Beurling algebras. Also, Choi *et al.* (2008) developed the recent research and as a result, they solved Johnson's (1972) problem which states that for any locally compact group G, the group algebra $L^1(G)$ is n-weakly amenable for each $n\in N$.

The contribution of this study is defining a new notion of amenability which helps to characterize Banach algebras. Such as amenability, n-weak amenability and approximate amenability, present definition determines differences between two Banach algebras.

In this study, we compose two definitions 1 and 2 together and define n-approximately weak amenability and we determine the relations between m-approximately weak amenability and n-approximately weak amenability for distinct $m, n\in N$ (N denotes the set of all positive integers). Then we investigate (2n+1)-approximate weak amenability of module extension Banach algebras. Finally, we give a counter example which shows that approximately weak amenability does not imply 3-approximately weak amenability.

Corresponding Author: Hashem Najafi, Department of Mathematics, Persian Gulf University, Boushehr, 75168, Iran
Tel/Fax: +98(771)4541494

n-APPROXIMATE WEAK AMENABILITY

Definition 3: Let A be a Banach algebra and n∈N. Then A is n-approximately weakly amenable, for n∈N, if every continuous derivation from A into the n-th dual space $A^{(n)}$ is approximately inner. We say that A is approximately weakly amenable if A is 1-approximately weakly amenable.

Proposition 1: Let A be a Banach algebra and n∈N. Assume that A is (n+2)-approximately weakly amenable. Then A is n-approximately weakly amenable.

Proof: Let D: $A\to A^{(n)}$ be a bounded derivation. Then D: $A\to A^{(n+2)}$ is a bounded derivation, so there exists a net $(\Lambda_\alpha)\subseteq A^{(n+2)}$ such that $D(a)=\lim_\alpha ad_{\Lambda_\alpha}(a)$. Let P: $A^{(n+2)}\to A^{(n)}$ be the canonical projection and $\lambda_\alpha = P(\Lambda_\alpha)$. Then $D(a)=P(D(a))=\lim_\alpha ad_{\lambda_\alpha}(a)$ and so D is approximately inner.

Proposition 2: Let A be a Banach algebra and n∈N. Suppose that A is (2n-1)-approximately weakly amenable. Then A^2 is dense in A.

Proof: By proposition 1, it is sufficient to prove the proposition for case n = 1. Let $\phi\in A^*$ and $\phi|_{A^2} = 0$. Then D: $A\to A^*$ with $D(a) = \phi(a)\phi$ is a bounded derivation. Thus there exists a net $(\phi_\alpha)\subseteq A^*$ such that $D(a)=\lim_\alpha ad_{\phi_\alpha}(a)$.

Then for a∈A we have:

$$\phi(a)^2 = D(a)(a) = \lim_\alpha a.\phi_\alpha(a)-\phi_\alpha.a(a) = \lim_\alpha \phi_\alpha(a^2)-\phi_\alpha(a^2)=0$$

Thus $\phi = 0$ and so A^2 is dense in A.

Let A be a non-unital Banach algebra. We denote by $A^\#$ the unitization of A which is $A^\# = A\oplus C$ with the product:

$$(a,\lambda).(b,\mu)=(ab+\mu a+\lambda b,\lambda\mu) \qquad (a,b\in A,\lambda,\mu\in C)$$

It is obvious that $A^\#$ is a Banach algebra as well. We denote by e* the bounded linear functional on $A^\#$ which is zero on A and e*(1) = 1. By these notations we have the following identifications:

$$A^{\#(2n)}=A^{(2n)}\oplus C,\ \ A^{\#(2n+1)}=A^{(2n+1)}\oplus Ce^* \qquad (n\in N)$$

The module actions of $A^\#$ on $A^{\#(2n)}$ are same az the multiplication on $A^\#$ and so $A^{(2n)}$ is a submodule of $A^{\#(2n)}$. The module actions of $A^\#$ on $A^{\#(2n+1)}$ are as follows:

$$(a,\lambda)(\Psi,\mu e^*)=(\lambda\Psi+a.\Psi,\lambda\mu e^*+\Psi(a)e^*)$$
$$(\Psi,\mu e^*).(a,\lambda)=(\lambda\Psi+\Psi.a,\lambda\mu e^*+\Psi(a)e^*)$$

and so $A^{(2n+1)}$ is not a submodule of $A^{\#(2n+1)}$.

Theorem 1: Let A be a non-unital Banach algebra and n∈N.

- If $A^\#$ is (2n)-approximately weakly amenable, then A is (2n)-approximately weakly amenable
- If A is (2n-1)-approximately weakly amenable, then $A^\#$ is (2n-1)-approximately weakly amenable
- Assume that A is commutative. Then $A^\#$ is n-approximately weakly amenable if and only if A is n-approximately weakly amenable

Proof

- Every derivation $D:A\to A^{(2n)}$ can be extended to a derivation $D_0:A^\#\to A^{(2n)}\oplus Ce = A^{\#(2n)}$ with $D_0(1) = 0$. Thus D_0 is approximately inner and so D is approximately inner
- Let $D:A\to A^{(2n-1)}\oplus Ce^*$ be a bounded derivation. Then it is easy to see that D is of the form $D(a) = D_0(a)+\phi(a)e^*$ where, $D_0\in Z^1(A,A^{(2n-1)})$ and $\phi\in A^*$. Thus there exists a net $(\phi_\alpha)\subseteq A^{(2n-1)}$ such that $D_0(a)=\lim_\alpha(a.\phi_\alpha-\phi_\alpha.a)$ for a∈A. Now let a, b∈A. Then we have

$$\phi(ab)=D_0(b)(a)+D_0(a)(b)=\lim_\alpha(b.\phi_\alpha-\phi_\alpha.b)(a)+\lim_\alpha(a.\phi_\alpha-\phi_\alpha.a)(b)=0$$

  and so $\phi|_{A^2} = 0$. By proposition 2, $\phi = 0$ and so $D = D_0$. Thus, D is approximately inner
- Since, A is commutative, n-approximate weak amenability and n-weak amenability are the same (note that every inner derivation is zero in commutative case). Thus this is immediate by proposition 1.4 of Dales *et al.* (1998)

The following example shows that (2n+1)-weak amenability and (2n+1)-approximately weak amenability are two different notions of amenability.

Example 1: Let M_k denote the algebra of k×k matrices over C (the space of complex numberc), with norm:

$$\|(a_{ij})\|_2=\left(\sum_{i,j}|a_{ij}|^2\right)^{1/2}$$

It is easy to see that $M_k^{(n)}\cong M_k$ for each n∈N. Set $A_n = (M_{2^n})^\#$ and

$$A = c_0-\bigoplus_{n=1}^{\infty}A_n$$

Where:

$$c_0(A_n)=\{(a_n)\,|\,a_n\in A_n,\|(a_n)\|=\sup_{n\in N}\|a_n\|<\infty,\|a_n\|\to 0\}$$

By example 6.2 of Ghahramani and Loy (2004) and proposition 1.2 of Dales *et al.* (1998), A is (2n+1)-approximately weakly amenable (approximately amenable) but it is not (2n+1)-weakly amenable.

We denote by $\circ$ and $\Diamond$ the first and second Arens product on A**, respectively. The Banach algebra A is called Arens regular if $\circ = \Diamond$ (Palmer, 1994).

Proposition 3: Let A be an Arens regular Banach algebra and let every $D \in Z^1(A^{**}, A^{**})$ be approximately inner. Then A is 2-approximately weakly amenable.

Proof: Let $D \in Z^1(A, A^{**})$, then by theorem 1.9 of Dales *et al.* (1998) there exists $\tilde{D} \in Z^1(A^{**}, A^{**})$ such that $\tilde{D}(\hat{a}) = D(a)$ where, â denotes the canonical image of $a \in A$ in A**. Thus there exists a net $(\phi_\alpha) \subseteq A^{**}$ such that

$$\tilde{D}(\psi) = \lim_\alpha(\psi \circ \phi_\alpha - \phi_\alpha \circ \psi) \quad (\psi \in A^{**})$$

So that, for $a \in A$, we have $D(a) = \lim_\alpha(a.\phi_\alpha - \phi_\alpha.a)$ and so A is 2-approximately weakly amenable.

Theorem 2: Let A be a Banach algebra such that $A^{(2n-2)}$ is Arens regular for each $n \in N$, ($A^{(0)} = A$). If every $D \in Z^1(A^{(2n)}, A^{(2n)})$ is approximately inner, then A is 2n-approximately weakly amenable for each $n \in N$.

Proof: The case n = 1 is proposition 3. Assume that every Banach algebra with the stated properties is 2k-approximately weakly amenable. We show that A is (2k+2)-approximately weakly amenable. Let $D \in Z^1(A, A^{(2k+2)})$ and let $P: A^{(2k+4)} \to A^{(2k+2)}$ be the canonical projection. By propositions 1.7 and 1.8 of Dales *et al.* (1998), we have $D^{**} \in Z^1(A^{**}, A^{(2k+4)})$ (D^{**} is the second dual of D) and P is an A**-bimodule homomorphism. Let $\tilde{D} = PoD^{**}$, so $\tilde{D} \in Z^1(A^{**}, (A^{**})^{(2k)})$. By applying A** instead of A, since A** is 2k-approximately weakly amenable, there exists a net $(\phi_\alpha) \subseteq A^{(2k+2)}$ such that

$$\tilde{D}(\psi) = \lim_\alpha(\psi.\phi_\alpha - \phi_\alpha.\psi) \text{ for } \psi \in A^{**}$$

Therefore, $D(a) = \lim_\alpha ad_{\phi_\alpha}(a)$ for each $a \in A$ and so A is (2k+2)-approximately weakly amenable.

It has been discussed by Dales *et al.* (1998) that any commutative, weakly amenable Banach algebra (or equivalently commutative, approximately weakly amenable Banach algebra) is n-weak amenable for each $n \in N$.

The next theorem is the partial result for general case of the earlier fact in special case. First, we note that $(A^{(2n+2)}, \circ)$ is the second dual of $(A^{(2n)}, \circ)$ for $n \in N$. Also, $(A^{(2k)}, \circ)$ is a subalgebra of $(A^{(2n)}, \circ)$ for k, $n \in N$ and $k \le n$.

Theorem 3: Let A be an approximately weakly amenable Banach algebra such that A is an ideal in $(A^{**}, \circ)$. Then A is (2n+1)-approximately weakly amenable for each $n \in N$.

Proof: Let A^n be the linear span of $\{a_1 \ldots a_n | a_1, \ldots, a_n \in A\}$. Since, A is an ideal in $(A^{**}, \circ)$, the operators L_a and R_a: $A \to A$ (the left and right multiplication, respectively) are weakly compact for each $a \in A$. Thus $L_a^{(2n)}$ and $R_a^{(2n)}$: $A^{(2n)} \to A^{(2n)}$ (the n-th dual operators of L_a and R_a) are weakly compact. Now we have $A.A^{(2n)}$, $A^{(2n)} A \subseteq A^{(2n-2)}$ for $n \in N$ and so $A^n.A^{(2n)}$, $A^{(2n)}.A^n \subseteq A$. Since, $A^{(2n+1)} = A^* \oplus A^\perp$ and A is approximately weakly amenable, it suffices to show that every derivation D: $A \to A^\perp$ is approximately inner. For such derivation D and for a, $b \in A^n$ and $\Psi \in A^{(2n)}$ we have:

$$D(ab)(\Psi) = D(a).b(\Psi) + a.D(b)(\Psi) = D(a)(b.\Psi) + D(b)(\Psi.a) = 0$$

Since, $\Psi.a$ and $b.\Psi \in A$. Thus D(ab) = 0 and so $D|_{A^{2n}} = 0$. Since, A^{2n} is dense in A, by proposition 2, we have D = 0. Therefore, A is (2n+1)-approximately weakly amenable.

Corollary 1: Let A be a Banach algebra such that A is an ideal in $(A^{**}, \circ)$. Then the following are equivalent:

- A is approximately weakly amenable
- A is (2n+1)-approximately weakly amenable for some $n \in N$
- A is (2n+1)-approximately weakly amenable for each $n \in N$

Dales *et al.* (1998) proved that every C*-algebra is n-weakly amenable for each $n \in N$, so obviously every C*-algebra is n-approximately weakly amenable for each $n \in N$.

(2n+1)-APPROXIMATELY WEAK AMENABILITY OF A⊕X

Let A be a Banach algebra and X be a Banach A-bimodule. Then the module extension Banach algebra corresponding to A and X is $A \oplus X$, the l_1-direct sum of A and X with the algebra product defined as follows:

$$(a, x).(a', x') = (aa', ax' + xa') \qquad (a, a' \in A, x, x' \in X)$$

We investigate (2n+1)-approximately weak amenability of module extension Banach algebra $A \oplus X$. As it has been discussed by Zhang (2002), according to Dales *et al.* (1998), X** is a Banach A**-bimodule, where, A** is equipped with the first Arens product. The module actions are defined as follows:

For x∈X, f∈X*, φ∈X** and u∈A**, define φf, fx∈A* and uf∈X** by

$$\langle a,\phi f\rangle=\langle fa,\phi\rangle,\quad \langle a,fx\rangle=\langle xa,f\rangle,\quad \langle x,uf\rangle=\langle fx,u\rangle \qquad (a\in A)$$

Then, for φ∈X** and u∈A**, define uφ, φu∈X** by

$$\langle f,u\phi\rangle=\langle \phi f,u\rangle,\quad \langle f,\phi u\rangle=\langle uf,\phi\rangle \qquad (f\in X^*)$$

These give the left and right A**-module actions on X**. Also, the definition for uf with u∈A** and f∈X* gives a left Banach A**-module action on X*. When u = a∈A, all the above A**-module actions agree with the A-module actions on the corresponding dual modules X* and X**.

Viewing $A^{(2n)}$ as a new A and $X^{(2n)}$ as a new X, the preceding procedure will successively define $X^{(2n+2)}$ as a Banach $A^{(2n+2)}$-bimodule. The first Arens product is consistently assumed on each $A^{(2n)}$. Now suppose that the bimodule action of $A^{(2n)}$ on $X^{(2n)}$ has been defined, where, n≥1. Then in a natural way, $X^{(2n+k)}$, k≥1, is a Banach $A^{(2n)}$-bimodule with the module multiplications uΛ and Λu∈$X^{(2n+k)}$, for Λ∈$X^{(2n+k)}$ and u∈$A^{(2n)}$, defined by:

$$\langle \gamma,u\Lambda\rangle=\langle \gamma u,\Lambda\rangle,\quad \langle \gamma,\Lambda u\rangle=\langle u\gamma,\Lambda\rangle \qquad (\gamma\in X^{(2n+k-1)})$$

If u = a∈A, these module actions coincide with A-module actions on $X^{(2n+k)}$. Then, for F∈$X^{(2n+1)}$ and φ∈$X^{(2n+2)}$, define Fφ, φF∈$A^{(2n+1)}$ by:

$$\langle u,F\phi\rangle=\langle F,\phi u\rangle,\quad \langle u,\phi F\rangle=\langle Fu,\phi\rangle \qquad (u\in A^{(2n)})$$

For a Banach space Y and an element y∈Y denote by ŷ the image of y in Y** under the canonical mapping. When F∈$X^{(2n+1)}$ and φ∈$X^{(2n)}$, we denote $F\hat{\phi}$ by Fφ and $\hat{\phi}F$ by φF. It is easy to check that:

$$\langle u,F\phi\rangle=\langle \phi u,F\rangle,\quad \langle u,\phi F\rangle=\langle u\phi,F\rangle \qquad (u\in A^{(2n)})$$

By using the canonical image of F or φ in the appropriate 2*l*-th dual space of the space that it belongs to, we can then signify a meaning for Fφ and φF for every F∈$X^{(2n+1)}$ and φ∈$X^{(2m)}$; they are elements of $A^{(2k+1)}$, where, k = max {m-1, n}. Now for μ∈$A^{(2n+2)}$ and F∈$X^{(2n+1)}$, we define μF∈$X^{(2n+1)}$ by:

$$\langle \phi,\mu F\rangle=\langle F\phi,\mu\rangle \qquad (\phi\in X^{(2n)})$$

This actually defines a left Banach $A^{(2n+2)}$-module action on $X^{(2n+1)}$.

Finally, for μ∈$A^{(2n+2)}$ and φ∈$X^{(2n+2)}$, define μφ, φμ∈$X^{(2n+2)}$ by:

$$\langle F,\mu\phi\rangle=\langle \phi F,\mu\rangle,\quad \langle F,\phi\mu\rangle=\langle \mu F,\phi\rangle \qquad (F\in X^{(2n+1)})$$

These actually define the $A^{(2n+2)}$-module actions on $X^{(2n+2)}$.

It is easy to see that $(A\oplus X)^{(2n+1)}$ can be identified with $A^{(2n+1)}\oplus_\infty X^{(2n+1)}$, the l_∞-direct sum of $A^{(2n+1)}$ and $X^{(2n+1)}$. Also, the (A⊕X)-bimodule actions on $A^{(2n+1)}\oplus_\infty X^{(2n+1)}$ are as follows:

$$(a,x).(F,G)=(aF+xG,aG) \text{ and } (F,G).(a,x)=(Fa+Gx,Ga)$$

where, a∈A, x∈X, F∈$A^{(2n+1)}$ and G∈$X^{(2n+1)}$.

It is assume that A is a Banach algebra, X is a Banach A-bimodule and A⊕X is their corresponding module extension Banach algebra.

Lemma 1: Let T:X→$A^{(2n+1)}$ be a continuous A-bimodule homomorphism. Then $\bar{T}:A\oplus X\to(A\oplus X)^{(2n+1)}$ defined by $\bar{T}((a,x))=(T(x),0)$ is a bounded derivation. The derivation $\bar{T}$ is approximately inner if and only if there exists a net $(F_\alpha)\subseteq X^{(2n+1)}$ such that 12158793 598 9797 7979

$$\lim_\alpha(aF_\alpha-F_\alpha a)=0 \text{ and } T(x)=\lim_\alpha(xF_\alpha-F_\alpha x)$$

for all a∈A and x∈X.

Proof: It is routinely checked that $\bar{T}$ is a bounded derivation. Let $\bar{T}$ be approximately inner. Then there are nets $(G_\alpha)\subseteq A^{(2n+1)}$ and $(F_\alpha)\subseteq A^{(2n+1)}$ such that

$$\bar{T}((a,x))=\lim_\alpha(a,x).(G_\alpha,F_\alpha)-(G_\alpha,F_\alpha).(a,x)$$

Thus $(T(x),0)=\lim_\alpha(aG_\alpha-G_\alpha a+xF_\alpha-F_\alpha x,\ aF_\alpha-F_\alpha a)$ and so $\lim_\alpha(aF_\alpha-F_\alpha a)=0$. Since, $\bar{T}$ ((a, 0)) = (0, 0), we have $\lim_\alpha(aG_\alpha-G_\alpha a)=0$ and so $\bar{T}((a,x))=\lim_\alpha(a,x).(0,F_\alpha)-(0,F_\alpha).(a,x)$. Therefore, $T(x)=\lim_\alpha(xF_\alpha-F_\alpha x)$. For the converse let such a net (F_α) exists. Then

$$\bar{T}((a,x))=(T(x),0)=\lim_\alpha(xF_\alpha-F_\alpha x,aF_\alpha-F_\alpha a)=\lim_\alpha(a,x).(0,F_\alpha)-(0,F_\alpha).(a,x)$$

Therefore, $\bar{T}$ is approximately inner.

Lemma 2: Let D: A→$X^{(2n+1)}$ be a bounded derivation and $D^{(2n+1)}$ be the (2n+1)-th dual operator of it. Then $\bar{D}$: A⊕X→$(A\oplus X)^{(2n+1)}$ defined by $\bar{D}((a,x))=(-D^{(2n+1)}(x),D(a))$, for all a∈A and x∈X, is a bounded derivation. Moreover, if $\bar{D}$ is approximately inner, then so is D. Also, if D is approximately inner, then there is a net of bounded derivations $\tilde{D}_\alpha:A\oplus X\to(A\oplus X)^{(2n+1)}$ such that $\tilde{D}_\alpha((a,0))=0$, for all α and for all a∈A and $\bar{D}-\tilde{D}_\alpha$ is inner.

Proof: By lemma 3.4 of Zhang (2002), $\bar{D}$ is a bounded derivation. If $\bar{D}$ is approximately inner, then there are nets $(G_\alpha)\subseteq A^{(2n+1)}$ and $(F_\alpha)\subseteq X^{(2n+1)}$ such that:

$$\bar{D}((a,x)) = \lim_\alpha (a,x).(G_\alpha, F_\alpha) - (G_\alpha, F_\alpha).(a,x)$$

Thus,

$$(0,D(a)) = \lim_\alpha ((a,0).(G_\alpha, F_\alpha) - (G_\alpha, F_\alpha).(a,x)) = \lim_\alpha (aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a)$$

Therefore, $D(a) = \lim_\alpha (aF_\alpha - F_\alpha a)$ for all $a \in A$ and so D is approximately inner. For the converse let $D(a) = \lim_\alpha (aF_\alpha - F_\alpha a)$ for $a \in A$ and $(F_\alpha) \subseteq X^{(2n+1)}$. Let $T_\alpha: X \to A^{(2n+1)}$ be defined by:

$$T_\alpha(x) = -D^{(2n+1)}(x) - xF_\alpha + F_\alpha x \qquad (x \in X)$$

and let $\tilde{D}_\alpha : A \oplus X \to (A \oplus X)^{(2n+1)}$ be defined by:

$$\tilde{D}_\alpha((a,x)) = (T_\alpha(x), 0) \qquad (a \in A, x \in X)$$

Then for $a \in A$ and $x \in X$ we have $\tilde{D}_\alpha((a,0)) = 0$ and

$$(\bar{D} - \tilde{D}_\alpha)(a,x) = (xF_\alpha - F_\alpha x, aF_\alpha - F_\alpha a) = (a,x).(0,F_\alpha) - (0,F_\alpha).(a,x)$$

Thus $\bar{D} - \tilde{D}_\alpha$ is inner and so $(\bar{D} - \tilde{D}_\alpha)$ is the net as required.

Lemma 3: Let D: $A \to A^{(2n+1)}$, $n \geq 0$, be a bounded derivation. Then $\bar{D} : A \oplus X \to (A \oplus X)^{(2n+1)}$ defined by $\bar{D}((a,x)) = (D(a), 0)$ is a bounded derivation. Moreover, $\bar{D}$ is approximately inner if and only if D is approximately inner.

Proof: It is routine to check that $\bar{D}$ is a bounded derivation. Now let D be approximately inner. Thus $D(a) = \lim_\alpha (aF_\alpha - F_\alpha a)$ for some net $(F_\alpha) \subseteq A^{(2n+1)}$ and for all $a \in A$. Then

$$\bar{D}((a,x)) = \lim_\alpha (aF_\alpha - F_\alpha a, 0) = \lim_\alpha (a,x).(F_\alpha, 0) - (F_\alpha, 0).(a,x)$$

and so $\bar{D}$ is approximately inner. Conversely, let $\bar{D}$ be approximately inner. Thus there exist nets $(F_\alpha) \subseteq A^{(2n+1)}$ and $(G_\alpha) \subseteq X^{(2n+1)}$ such that

$$\bar{D}((a,x)) = \lim_\alpha (a,x).(F_\alpha, G_\alpha) - (F_\alpha, G_\alpha).(a,x)$$

Since,

$$\bar{D}((a,x)) = D((a,0)),$$

$$\bar{D}((a,x)) = \lim_\alpha (aF_\alpha - F_\alpha a, aG_\alpha - G_\alpha a)$$

and so $D(a) = \lim_\alpha (aF_\alpha - F_\alpha a)$

Lemma 4: Let T: $X \to X^{(2n+1)}$, $n \geq 0$, be a continuous A-bimodule homomorphism, satisfying $xT(y)+T(x)y = 0$ for all $x,y \in X$. Then $\bar{T} : A \oplus X \to (A \oplus X)^{(2n+1)}$ defined by $\bar{T}((a,x)) = (0, T(x))$ is a bounded derivation. Moreover, $\bar{T}$ is approximately inner if and only if T = 0.

Proof: It is routine to check that $\bar{T}$ is a bounded derivation. Let $\bar{T}$ be approximately inner. Thus $\bar{T}((a,x)) = \lim_\alpha (a,x).(F_\alpha, G_\alpha) - (F_\alpha, G_\alpha).(a,x)$ for some nets

$$(F_\alpha) \subseteq A^{(2n+1)} \text{ and } (G_\alpha) \subseteq X^{(2n+1)}$$

Since, $\bar{T}((a,x)) = \bar{T}((0,x))$, we have $(0,T(x)) = \lim_\alpha (xG_\alpha - G_\alpha x, 0)$. Therefore, T = 0.

Theorem 4: For $n \geq 0$, the module extension Banach algebra $A \oplus X$ is (2n+1)-approximately weakly amenable if and only if the following conditions hold:

(i) A is (2n+1)-approximately weakly amenable;
(ii) Every derivation from A into $X^{(2n+1)}$ is approximately inner;
(iii) For each continuous A-bimodule homomorphism T: $X \to A^{(2n+1)}$, $n \geq 0$, there is a net $(F_\alpha) \subseteq X^{(2n+1)}$ such that $\lim_\alpha (aF_\alpha - F_\alpha a) = 0$ for $a \in A$ and $T(x) = \lim_\alpha (xF_\alpha - F_\alpha x)$ for $x \in X$;
(iv) The only A-bimodule homomorphism T: $X \to X^{(2n+1)}$, $n \geq 0$ for which $xT(y)+T(x)y = 0$, $x,y \in X$ in $A^{(2n+1)}$ is T = 0.

Proof: Let $A \oplus X$ be (2n+1)-approximately weakly amenable. Then by lemmas 2 and 3, A is (2n+1)-approximately weakly amenable and every derivation from A into $X^{(2n+1)}$ is approximately inner. Furthermore, lemma 1 gives condition (iii) and lemma 4 gives condition (iv). For the converse, let:

$$P_1 : (A \oplus X)^{(2n+1)} \to A^{(2n+1)} \text{ and } P_2 : (A \oplus X)^{(2n+1)} \to X^{(2n+1)}$$

be the canonical projections and let $\iota_1 : A \to A \oplus X$ and $\iota_2 : X \to A \oplus X$ are the canonical inclusion maps. Obviously, P_1 and P_2 are A-bimodule homomorphisms and ι_1 is an algebra homomorphism. Let $D : A \oplus X \to (A \oplus X)^{(2n+1)}$ be a bounded derivation. Then $Do\iota_1 : A \to (A \oplus X)^{(2n+1)}$ is a bounded derivation and so $P_1 oDo\iota_1 : A \to A^{(2n+1)}$ and $P_2 oDo\iota_1 : A \to X^{(2n+1)}$ are bounded derivations. Thus they are approximately inner by conditions (i) and (ii). Therefore, $Do\iota_1$ is approximately inner. By lemmas 2-4:

$$\overline{Do\iota_1} = \overline{P_1 oDo\iota_1} + \overline{P_2 oDo\iota_1} : A \to (A \oplus X)^{(2n+1)}$$

is a bounded derivation and there is a net of bounded derivations $\tilde{D}_\alpha : A \oplus X \to (A \oplus X)^{(2n+1)}$ such that $\tilde{D}_\alpha((a,0)) = 0$, for all α and for all $a \in A$ and

$$\overline{Do\iota_1} - \tilde{D}_\alpha$$

is inner. We have

$$(D - \overline{Do\iota_1})((a,0)) = D((a,0)) - \overline{Do\iota_1}((a,0)) = Do\iota_1(a) - Do\iota_1(a) = 0 \qquad (a \in A)$$

Let:

$$\hat{D}_\alpha = D - \overline{Do\iota_1} + \tilde{D}_\alpha$$

Then, for all α, $\hat{D}_\alpha : A \oplus X \to (A \oplus X)^{(2n+1)}$ is a bounded derivation that satisfies $\tilde{D}_\alpha((a,0)) = 0$ for all a$\in$A. Moreover:

$$\hat{D}_\alpha((0,ax)) = \hat{D}_\alpha((a,0).(0,x)) = (a,0).\hat{D}_\alpha((0,x)) = a\hat{D}_\alpha((0,x)) \qquad (a \in A, x \in X)$$

and

$$\hat{D}_\alpha((0,xa)) = \hat{D}_\alpha((0,x).(a,0)) = \hat{D}_\alpha((0,x)).(a,0) = \hat{D}_\alpha((0,x))a \qquad (a \in A, x \in X)$$

Clearly $\hat{D}_\alpha o\iota_2 : X \to (A \oplus X)^{(2n+1)}$ is a continuous A-bimodule homomorphism. By condition (iii), for each α, there exists a net $(F_\beta^\alpha) \subseteq X^{(2n+1)}$ such that:

$$\lim_\beta (aF_\beta^\alpha - F_\beta^\alpha a) = 0$$

for a$\in$A and $P_1 o\hat{D}_\alpha o\iota_2(x) = \lim_\beta (xF_\beta^\alpha - F_\beta^\alpha x)$ for x$\in$X. On the other hand:

$$\begin{aligned}([P_2 o\hat{D}_\alpha o\iota_2(x)]y + x[P_2 o\hat{D}_\alpha o\iota_2(y)],0) &= ([P_2 o\hat{D}_\alpha(0,x)]y,0) + (x[P_2 o\hat{D}_\alpha(0,y)],0)\\ &= \hat{D}_\alpha((0,x)).(0,y) + (0,x).\hat{D}_\alpha((0,y))\\ &= \hat{D}_\alpha((0,x).(0,y))\\ &= \hat{D}_\alpha((0,0))\\ &= (0,0)\end{aligned}$$

Thus

$$[P_2 o\hat{D}_\alpha o\iota_2(x)]y + x[P_2 o\hat{D}_\alpha o\iota_2(y)] = 0 \qquad (x, y \in X)$$

Therefore, $P_2 o\hat{D}_\alpha o\iota_2 = 0$ by condition (iv). Thus we have

$$\begin{aligned}\hat{D}_\alpha((a,x)) = \hat{D}_\alpha((0,x)) &= \hat{D}_\alpha o\iota_2(x)\\ &= (P_1 o\hat{D}_\alpha o\iota_2(x), P_2 o\hat{D}_\alpha o\iota_2(x))\\ &= \lim_\beta (xF_\beta^\alpha - F_\beta^\alpha x, 0)\\ &= \lim_\beta (a,x).(0,F_\beta^\alpha) - (0,F_\beta^\alpha).(a,x)\end{aligned}$$

So, $\hat{D}_\alpha$ is approximately inner. Thus

$$D = \hat{D}_\alpha + (\overline{Do\iota_1} - \tilde{D}_\alpha)$$

is approximately inner.

A COUNTER EXAMPLE

Here, motivated by Zhang (2002), we consider the case that the module action on one side of X is trivial. Then we give a counter example which shows the converse of proposition 1 is not correct.

We denote by X_0 and Y_0 the A-bimodules X with trivial right module action and Y with trivial left module action, respectively. By proposition 2, in case $X = X_0$ it is easy to see that conditions (iii) and (iv) of Theorem 4 are reduced as follow:

$(iii)_0$ for each continuous A-bimodule homomorphism T: $X_0 \to A^{(2n+1)}$, there is a net $(F_\alpha) \subseteq X_0^{(2n+1)}$ such that $\lim_\alpha F_\alpha a = 0$ for a$\in$A and $T(x) = \lim_\alpha xF_\alpha$ for x$\in X_0$;
$(iv)_0$ AX_0 is dense in X_0.

Proposition 4: Let A be a (2n+1)-approximately weakly amenable Banach algebra with bounded approximate identity and let $AA^{(2n)} = A^{(2n)}$. Then $A \oplus X_0$ is (2n+1)-approximately weakly amenable if and only if AX_0 is dense in X_0.

Proof: By proposition 1.5 of Johnson (1972), condition (ii) in theorem 4 always hold. Let T: $X_0 \to A^{(2n+1)}$ be a continuous A-bimodule homomorphism. Then $\langle aF, T(x)\rangle = \langle F, T(xa)\rangle = 0$ for all a$\in$A and F$\in A^{(2n)}$. Thus $T(x)|_{AA^{(2n)}} = 0$ and so T = 0 since $AA^{(2n)} = A^{(2n)}$. So condition $(iii)_0$ holds.

For n = 0, an easy application of Cohen Factorization Theorem (Bonsall and Duncan, 1973) implies that:

Corollary 2: Let A be an approximately weakly amenable Banach algebra with bounded approximate identity. Then $A \oplus X_0$ is approximately weakly amenable if and only if AX_0 is dense in X_0.

A dual result to corollary 2 is as follows:

Corollary 3: Let A be an approximately weakly amenable Banach algebra with bounded approximate identity. Then $A \oplus_0 Y$ is approximately weakly amenable if and only if $A_0 Y$ is dense in ${}_0Y$.

If X and Y are Banach A-bimodules, we denote by $X \oplus_1 Y$ the l_1-direct sum of X and Y.

Proposition 5: Suppose that $A \oplus Y$ and $A \oplus Y$ are approximately weakly amenable. Then the following are equivalent:

(i) $A \oplus (X \oplus_1 Y)$ is approximately weakly amenable;
(ii) There is no nonzero, continuous A-bimodule homomorphism T: $X \to Y^*$;
(iii) There is no nonzero, continuous A-bimodule homomorphism S: $Y \to X^*$;

Proof: The proof is quite similar to the proof of lemma 7.1 of Zhang (2002) and so we omit it's proof.

Corollary 4: The algebra $A\oplus(X\oplus_1 Y)$ is approximately weakly amenable if and only if both $A\oplus X$ and $A\oplus Y$ are approximately weakly amenable and condition (ii) or (iii) in proposition 5 holds.

Proof: This is an immediate consequence of theorem 4 and proposition 5.

Let H be an infinite dimensional Hilbert space. We denote by B(H) and K(H) the space of all bounded linear operators and compact operators on H, respectively. By lemma 7.4 of Zhang (2002), there is an element $a_0 \in B(H)\backslash K(H)$ such that a_0 is not right invertible in B(H) and $a_0K(H)$ is dense in K(H). For such element $a_0 \in B(H)\backslash K(H)$, $a_0B(H)$ is a proper right ideal of B(H). Thus $cl(a_0B(H))$, the closure of $a_0B(H)$, is also a proper right ideal of B(H). So there is $0\neq \Lambda \in B(H)^*$ such that $\Lambda . a_0 = 0$. Then $\Lambda B(H) \neq \{0\}$ is a right B(H)-submodule of $B(H)^*$. Set

$$X_0 = (K(H))_0 \quad \text{and} \quad {}_0Y = {}_0(cl(\Lambda B(H)))$$

Example 2: By above notations, $B(H)\oplus(X_0 \oplus_1 {}_0Y)$ is (approximately) weakly amenable but not 3-approximately weakly amenable.

Proof: By example 7.5 (Zhang, 2002), $B(H)\oplus(X_0 \oplus_1 {}_0Y)$ is weakly amenable. Also, it is shown that $B(H)\oplus(X_0 \oplus_1 {}_0Y)$ fails condition (iv) theorem 2.1 (Zhang, 2002) for m = 1, which is condition (iv) of present theorem 4 for n = 1. Thus $B(H)\oplus(X_0 \oplus_1 {}_0Y)$ is not 3-approximately weakly amenable.

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