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Stability of the Cubic Functional Equation in Menger Probabilistic Normed Spaces

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Abstract: In this study, the stability of the cubic functional equation: f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+12f(x) in the setting of Menger probabilistic normed spaces is proved.

Key words: Hyers-Ulam-Rassias stability, cubic mappings, generalized normed space, Banach space, normed space

INTRODUCTION

The functional equation:

$$f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+12f(x)$$

is said to be the cubic functional equation. Skof (1983) by proving that if f is a mapping from a normed space X into a Banach space Y satisfying:

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\|\leq\varepsilon$$

for some $\varepsilon>0$, then there is a unique quadratic function $g:X\to Y$ such that:

$$\|f(x)-g(x)\|\leq\frac{\varepsilon}{2}$$

Cholewa (1984) extended Skofs theorem by replacing X by an abelian group G. Skof's result was later generalized by Czerwik (1992) in the spirit of Hyers-Ulam-Rassias (Ulam, 1964). The stability problem of the quadratic equation has been extensively investigated by a number of mathematicians and references therein. In addition, Alsina (1987) and Mihet and Radu (2008) investigated the stability in the settings of fuzzy, probabilistic and random normed spaces.

In the sequel, the usual terminology, notations and conventions of the theory of random normed spaces shall be adopted, as by Schweizer and Sklar (1983). Throughout present study, the space of all probability distribution functions (briefly, d.f.'s) is denoted by:

$$\Delta^+=\{F:R\cup\{-\infty,+\infty\}\to[0,1]:F(0)=0\ \text{ and }\ F(+\infty)=1\}$$

where, F is left continuous and non decreasing on R. Also the subset is the set:

$$D^+=\{F\in\Delta^+: l^-F(+\infty)=1\}$$

where, $l^-f(x)$ denotes the left limit of the function f at the point x, $l^-f(x)=\lim_{t\to x} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $D\leq G$ if and only if $F(t)\leq G(t)$ for all t in R. The maximal element for Δ^+ in this order is the d.f. given by:

$$\varepsilon_0(t)=\begin{cases}0, & \text{if } t\leq 0\\ 1, & \text{if } t>0\end{cases}$$

Definition 1: A mapping $T:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if T satisfies the following conditions:

- T is commutative and associative
- T is continuous
- $T(a,1)=a$ for all $a\in[0,1]$
- $T(a,b)\leq T(c,d)$ whenever $a\leq c$ and $b\leq d$ and $a,b,c,d\in[0,1]$

Two typical examples of continuous t-norm are $T(a,b)=ab$ and $T(a,b)=\min(a,b)$

Now t-norms are recursively defined by $T^1=T$ and

$$T^n(x_1,\dots,x_{n+1})=T(T^{n-1}(x_1,\dots,x_n),x_{n+1})$$

for $n\leq 2$ and $x_i\in[0,1]$, for all $I\in\{1,2,\dots,n+1\}$

The t-norm T is Hadzic type if for given $\varepsilon\in(0,1)$ there is $\delta\in(0,1)$ such that:

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$$T^m(1-\delta,\ldots,1-\delta) > 1-\varepsilon, \quad m \in N$$

A typical example of such t-norms is T (a, b) = min (a, b).

Recall that if T is a t-norm and $\{X_n\}$ is a given sequence of numbers in [0,1], $T_{i=1}^{n} x_i$ is defined recursively by:

$$T_{i=1}^{1} x_i = x_i$$

and

$$T_{i=1}^{n} x_i = T\left(T_{i=1}^{n-1} x_i, x_n\right) \text{ for } n \geq 2$$

is defined as

$$\lim_{n\to\infty} T_{i=1}^{n} x_i$$

Definition 2: A Menger Probabilistic normed space (briefly, Menger PN space) is a triple (X, μ,T), where X is a nonempty set, T is a continuous t-norm and μ is a mapping from X into D^+ such that, the following conditions hold:

- (PN1) μ_x for all t>0 if and only if x = 0
- (PN2) μ_{x-y} (t) = ε_0 for all x, y in X and t≥0
- (PN3) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all x in X, $\alpha \neq 0$ and $t \geq 0$
- (PN4) μ_{x+y} (t+s)≥T(μ_x(t), μ_y(s)) for all x, y, z ∈ X and t, s≥0

Clearly every Menger PN-space is a probabilistic metric space having a metrizable uniformity on X if $\sup_{a<1}$T(a, a) = 1.

Definition 3: Let (X, μ, T) be a Menger PN-space:

- A sequence $\{x_n\}$ in X is said to be convergent to x in X if, for every t>0 and ε>0, there exists positive integer N such that μ_{x_n-x} (t)>1-ε whenever n≥N
- A sequence $\{x_n\}$ in X is called Cauchy sequence if, for every t>0 and ε>0, there exists positive integer N such that μ_{x_n, x_m} (t)>1-ε whenever ≥m≥N
- A Menger PN-space (X, μ, T) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X

Theorem 1: If (X, μ, T) is a Menger PN-space and $\{x_n\}$ is a sequence such that $x_n \to x$ then $\lim_{n\to\infty} \mu_{xn}(t) = \mu_x(t)$

In this study, the stability of the quadratic functional equation in the setting of Menger probabilistic normed spaces is established.

MAIN RESULTS

Definition 4: Let X,Y be vector spaces. The functional equation f: X→Y defined by:

$$f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+12f(x) \quad (1)$$

is called cubic functional equation.

Theorem 2: Let (X, ν, R) be Menger PN space and (Y, μ, T) be a complete Menger PN space. If f: X→Y be a mapping such that:

$$\mu_{f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)}(t) \geq \xi_{x,y}(t) \quad (2)$$

for t>0 in which ξ: $X^2 \to D^+$ and

$$\lim_{n\to\infty} T_{i=1}^{\infty}\left(\xi_{2^{n+i-1}x,0}\left(2^{3n+2i+1}t\right)\right) = 1 \quad (3)$$

Then there exists a unique quadratic mapping Q: X→Y such that:

$$\mu_{f(x)-Q(x)}(t) \geq T_{i=1}^{\infty}\left(\xi_{2^{i-1}x,0}\left(2^{2i+1}t\right)\right) \quad (4)$$

Proof: Putting y = 0 in Eq. 2, then:

$$\mu_{\frac{f(2x)}{8}-f(x)}(t) \geq \xi_{x,0}\left(2^4 t\right) \geq \xi_{x,x}\left(2^3 t\right) \quad (5)$$

Replacing x by 2x in Eq. 5, then:

$$\mu_{\frac{f(4x)}{8}-f(2x)}(t) \geq \xi_{2x,0}\left(2^4 t\right) \quad (6)$$

Triangular inequality implies that:

$$\begin{aligned} \mu_{\frac{f(2^2x)}{8^2}-f(x)}\left(\frac{t}{8^2}\right) &\geq T\left(\mu_{\frac{f(2^2x)}{8^2}-f(2x)}\left(\frac{t}{2^4}\right), \mu_{\frac{f(2x)}{8^2}-f(x)}\left(\frac{t}{2^7}\right)\right) \\ &\geq T\left(\xi_{2x,0}(t), \xi_{x,0}\left(\frac{t}{2^3}\right)\right) \end{aligned} \quad (7)$$

Thus:

$$\begin{aligned} \mu_{\frac{f(2^2x)}{8^2}-f(x)}(t) &\geq T\left(\xi_{2x,0}\left(2^6 t\right), \xi_{x,0}\left(2^3 t\right)\right) \\ &\geq T\left(\xi_{2x,0}\left(2^5 t\right), \xi_{x,0}\left(2^3 t\right)\right) \end{aligned} \quad (8)$$

Replacing x by 4x in Eq. 5 and triangular inequality implies that:

$$\mu_{\frac{f(2^3x)}{8^3}-f(x)}(t) \geq T^2\left(\xi_{4x,0}\left(2^8t\right), \xi_{2x,0}\left(2^5t\right), \xi_{x,0}\left(2^3t\right)\right)$$
$$\geq T^2\left(\xi_{4x,0}\left(2^7t\right), \xi_{2x,0}\left(2^5t\right), \xi_{x,0}\left(2^3t\right)\right) \quad (9)$$

Using the induction on n, is obtained that:

$$\mu_{\frac{f(2^nx)}{8^n}-f(x)}(t) \geq T_{i=1}^{n}\left(\xi_{2^{i-1}x,0}\left(2^{2i+1}t\right)\right) \quad (10)$$

In order to prove convergence of the sequence

$$\left\{\frac{f(2^nx)}{8^n}\right\}$$

replace x with $2^m x$ in Eq. 10 to find that for m, n>0:

$$\mu_{\frac{f(2^{n+m}x)}{8^{(n+m)}}-\frac{f(2^mx)}{8^m}}(t) \geq T_{i=1}^{n}\left(\xi_{2^{i+m-1}x,2^{i+m-1}x}\left(2^{2i+m+1}t\right)\right) \quad (11)$$

Since, the right hand side of the inequality tend to 1 as m tends to infinity, the sequence

$$\left\{\frac{f(2^nx)}{2^{3n}}\right\}$$

is a Cauchy sequence. Therefore, define:

$$Q(x) = \lim_{n\to\infty}\frac{f(2^nx)}{2^{3n}}$$

for all $x \in X$.

Now, it is showed that Q is a quadratic map. Replacing x, y with $2^n x$ and $2^n y$, respectively in Eq. 2. Then it follows that:

$$\mu_{\frac{f(2^nx+2^ny)}{2^{3n}}+\frac{f(2^nx-2^ny)}{2^{3n}}-2\frac{f(2^nx)}{2^{3n}}-2\frac{f(2^ny)}{2^{3n}}}(t) \geq \xi_{2^nx,2^ny}\left(2^{3n}t\right) \quad (12)$$

Taking the limit as m→∞, can be found that Q satisfies Eq. 2 for all x, y ϵ X.

To prove Eq. 4, take the limit as n→∞ in Eq. 10.

To prove the uniqueness of the quadratic function Q subject to Eq. 4, assume that there exists a quadratic function Q' which satisfies Eq. 4. Obviously;

$$Q(2^nx) = 2^{3n}Q(x) \text{ and } Q'(2^nx) = 2^{3n}Q'(x) \text{ for all } x \in X \text{ and } n \in N$$

Hence, it follows from Eq. 4 that:

$$\mu_{Q(x)-Q'(x)}(t)$$
$$\geq \mu_{Q(2^nx)-Q'(2^nx)}\left(2^{3n}t\right)$$
$$\geq T\left(\mu_{Q(2^nx)-f(2^nx)}\left(2^{3n-1}t\right), \mu_{f(2^nx)-Q'(2^nx)}\left(2^{3n-1}t\right)\right)$$
$$T\left(T_{i=1}^{\infty}(\xi_{2^{n+i-1}x,0}\left(2^{3n+2i+1}t\right)), T_{i=1}^{\epsilon}\left(\xi_{2^{n+i-1}x,0}\left(2^{3n+2i+1}t\right)\right)\right)$$

for all x ϵ X. By letting n→∞ in Eq. 4, implies that the uniqueness of Q.

REFERENCES

Alsina, C., 1987. On the stability of a functional equation arising in probabilistic normed spaces. General Inequalities, 5: 263-271.

Cholewa, P.W., 1984. Remarks on the stability of functional equations. Aequationes Math., 27: 76-86.

Czerwik, S., 1992. On the stability of the quadratic mapping in normed spaces. Abh. Math. Sem. Univ. Hamburg, 62: 59-64.

Mihet, D. and V. Radu, 2008. On the stability of the additive Cauchy functional quation in random normed spaces. J. Math. Anal. Applied, 343: 567-572.

Schweizer, B. and A. Sklar, 1983. Probabilistic Metric Spaces. 1st Edn., Elsevier, North Holand, New York, USA., ISBN: 0-444-00666-4.

Skof, F., 1983. Local properties and approximations of operators. Rend. Sem. Mat. Fis. Milano, 53: 113-129.

Ulam, S.M., 1964. Problems in Modern Mathematics. Chapter VI, Science Editions, Wiley, New York, USA., ISBN: 0471928666.