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Assessment of HAM and PEM to Find Analytical Solution for Calculating Displacement Functions of Geometrically Nonlinear Prestressed Cable Structures with Concentrated Mass

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Abstract: In this study, two powerful analytical methods, called He's Parameter-Expanding Methods (PEM) and Homotopy Analysis Method (HAM) are used to calculating displacement functions of geometrically nonlinear prestressed cable structures. In this study, the results of two methods are compared and it is shown that one term in series expansions is sufficient to obtain a solution by using the PEM. Comparison of the obtained solutions with those obtained using numerical method shows that two methods are effective and convenient for solving this problem. These two methods introduce a capable tool for solving this kind of nonlinear problems.

Key words: Analytical technique, parameter-expanding method, homotopy analysis method, prestressed cable structure

INTRODUCTION

Liao (1992) employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely the homotopy analysis method and then modified it, step by step (Liao, 2003, 2004). This method does not need small/large parameters and has been successfully applied to solve many types of nonlinear problems in solid and fluid mechanics (Cheng *et al*., 2005; Rahimpour *et al*., 2008; Kimiaeifar, 2008; Kimiaeifar and Saidai, 2008; Sajid *et al*., 2008).

Recently, considerable attention has been directed towards analytical solutions for nonlinear equations based on homotopy technique. Homotopy theory becomes a powerful mathematical tool, when it is successfully coupled with the perturbation theory (He, 1998, 2000; Hillermeier, 2001; Kimiaeifar, 2008). He's Parameter-Expanding Method (PEM) is one of the most effective and convenient method for analytical solving of nonlinear differential equations. PEM has been shown to effectively, easily and accurately solve a large class of linear and nonlinear problems with components converging rapidly to accurate solutions. PEM was first proposed by He (2006) and was successfully applied to various engineering problems. It is worth mentioning that there are a few works on using parameter-expanding method in the literature; Xu (2007) suggested He's parameter-expanding methods for strongly nonlinear oscillators. Tao (2008) proposed frequency-amplitude relationship of nonlinear oscillators using PEM.

Prestressed cable structures and their continuous counterparts in membrane or fabric structures, are often perceived as architecturally elegant structural forms; particularly for large clear span coverings. The extremely low weight to plan area ratio of such structures and the associated curved surfaces, often resent cable and fabric structures as refreshing alternatives to the more common bulky rectangular forms. The use of prestressed mechanisms as structural forms also tends to give clients and the general public the impression of utilizing the most modern of available technology. However, the same three properties of low-weight, unusual curved surfaces and nonlinear response to load, combine to form challenging problems to the structural engineer charged with ensuring a cable or fabric structure has safe dynamic characteristics, especially under wind loading. Nonlinear vibration has several phenomena not found in linear vibration and, in particular, any displacement-time relationship is dependent on initial conditions. Thus different values of so-called natural frequencies can be obtained for a given system simply by altering the initial velocity or displacement.

In this study He's parameter-expanding method and homotopy analysis method are used to calculate the displacement functions of geometrically nonlinear

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prestressed cable structures. It is shown that the HAM solution is very accurate for whole domain and for all effective parameters by using high number of series solutions. In PEM solution only one term in series expansions is sufficient to obtain an accurate solution but increasing the coefficients of nonlinear term, the error of PEM solution increases.

GOVERNING EQUATION OF VIBRATION OF TWO-LINK STRUCTURE

The symmetrical prestressed two-link structure is shown in Fig. 1 which has a single degree of freedom. It can be shown (Kwan, 1998) that a central load P for this structure is related to its corresponding static deflection x by:

$$\frac{EA}{L_0}x(t) + 2t_0L_0x(t) - pL_0^{\ 2} = 0 \tag{1}$$

where, EA is the axial stiffness, t_0 is the initial pretension and L_0 is the original undeformed length, of the two-links.

Consider now the vibration of the two links such that they remain straight throughout in which case the acceleration of a small element of length dx at a distance x from the support is xy/L_0, where, y is the acceleration of joint B. The D'Alembert forces D for one link are thus given by:

$$D = \int_0^{L_0} \rho\, dy \left(\frac{y}{L_0}\ddot{x}(t)\right) = \frac{\rho \ddot{x}(t) L_0}{2} \tag{2}$$

where, ρ is the mass per unit length of the links. If we isolate the portion BC and take free body moment of the portion BC about B, it is obtained:

$$\int_0^{L_0} \rho\, dy \left(\frac{y}{L_0}\ddot{x}(t)\right)(L_0 - y) = R\, L_0 \tag{3}$$

and

$$\frac{\rho \ddot{x}(t) L_0}{6} = R \tag{4}$$

where, R is the vertical dynamic reaction at the supports.

Substitution of Eq. 1-3 into the overall vertical equilibrium relationship,

$$R = \frac{P}{2} + D$$

leads to:

$$\ddot{x}(t) + \frac{3t_0}{\rho L_0^{\ 2}}x(t) + \frac{3EA}{2\rho L_0^{\ 4}}x(t)^3 = 0 \tag{5}$$

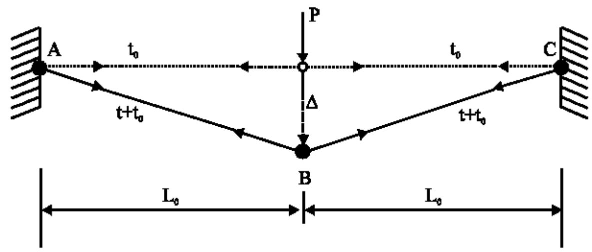

Fig. 1: Geometry of problem: The symmetrical two-link structure (Bars AB and BC Joint at point B)

which is the equation describing the free undamped vibration of the prestressed two-link. If the two-link structure had a concentrated mass M at B, then Eq. 5 would be altered slightly to:

$$\ddot{x}(t) + \frac{6t_0}{3ML_0 + \rho L_0^{\ 2}}x(t) + \frac{3EA}{3ML_0^3 + 2\rho L_0^{\ 4}}x(t)^3 = 0 \tag{6}$$

By definition:

$$\alpha = \frac{6t_0}{3ML_0 + \rho L_0^{\ 2}} \text{ and } \beta = \frac{3EA}{3ML_0^3 + 2\rho L_0^{\ 4}}$$

Eq. 6 reduce to:

$$\ddot{x}(t) + \alpha x(t) + \beta x(t)^3 = 0 \tag{7}$$

APPLICATION OF HAM

The governing equation for the nonlinear prestressed cable structures is expressed by Eq. 7. Nonlinear operator is defined as follow:

$$N[x(t;q)] = \frac{\partial^2 x(t;q)}{\partial t^2} + \alpha x(t;q) + \beta x(t;q)^3 \tag{8}$$

where, q∈[0,1] is the embedding parameter. As the embedding parameter increases from 0 to 1, U(t; q), varies from the initial guess, $U_0(t)$, to the exact solution, U(t):

$$x(t;0) = U_0(t), \qquad x(t;1) = U(t) \tag{9}$$

Expanding x(t;q) in Taylor series with respect to q results in:

$$x(t;q) = U_0(t) + \sum_{m=1}^{\infty} U_m(t) q^m \tag{10}$$

Where:

$$U_m(t) = \frac{1}{m!}\left.\frac{\partial^m x(t;q)}{\partial q^m}\right|_{q=0} \tag{11}$$

Homotopy analysis method can be expressed by many different base functions (Liao, 2003), according to the governing equations; it is straightforward to use a base function in the form of:

$$U(t) = \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} b_{kpm} t^k \cos^m(t) \sin^p(t) \tag{12}$$

where, b_{kpm} are the coefficients to be determined. When the base function is selected, the auxiliary functions H(t), initial approximations $U_0(t)$ and the auxiliary linear operators L must be chosen in such a way that the corresponding high-order deformation equations have solutions with the functional form similar to the base functions. This method referred to as the rule of solution expression (Liao, 2003).

The linear operator L is chosen as:

$$L[x(t;q)] = \frac{\partial^2 x(t;q)}{\partial t^2} + x(t;q) \tag{13}$$

From Eq. 15 and 16 results in:

$$L[c_1 \sin(t) + c_2 \cos(t)] = 0 \tag{14}$$

where, c_1 to c_2 are the integral constants. According to the rule of solution expression and the initial conditions, the initial approximations, $U_0(t)$ as well as the integral constants, c_1 to c_2 are formed as:

$$U_0(t) = c_1 \sin(t) + c_2 \cos(t), \quad c_1 = 0, c_2 = \lambda \tag{15}$$

The zeroth order deformation equation for U(t) is:

$$(1-q)L[x(t;q) - U_0(t)] = q\hbar H(t) N[x(t;q)] \tag{16}$$

$$x(0;q) = \lambda, \qquad \frac{\partial x(0;q)}{\partial t} = 0 \tag{17}$$

According to the rule of solution expression and from Eq. 12, the auxiliary function H(t) can be chosen as follows:

$$H(t) = 1 \tag{18}$$

Differentiating Eq. 16, m times, with respect to the embedding parameter q and then setting q = 0 in the final expression and dividing it by m!, it is reduced to:

$$U_m(t) = \chi_m U_{m-1}(t) + \hbar\left(\sin(t)\int_0^t H(t) R_m(U_{m-1})\cos(t)\,dt + \cos(t)\int_0^t H(t) R_m(U_{m-1})\sin(t)\,dt\right) + c_1 \sin(t) + c_2 \cos(t) \tag{19}$$

$$U_m(0) = 0, \qquad U'_m(0) = 0 \tag{20}$$

Equation 19 is the mth order deformation equation for x(t), where:

$$R_m(U_{m-1}) = \frac{d^2 U_{m-1}(t)}{dt^2} + \alpha U_{m-1}(t) + \beta\left[\sum_{j=0}^{m-1} U_{m-1-j}(t) \sum_{z=0}^{j} U_{j-z}(t) U_z(t)\right] \tag{21}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases} \tag{22}$$

As a result, the first and second terms of the solution's series are as follows:

$$U_0(t) = \lambda \cos(t) \tag{23}$$

$$U_1(t) = \frac{1}{8}\sin(t) h \lambda(-4t + 4\alpha t + 3\lambda^2\beta t + \sin(t)\cos(t)\lambda^2\beta) \tag{24}$$

The solution's series U(t) is developed up to 12th order of approximation.

CONVERGENCE OF HAM SOLUTION

The analytical solution should converge. It should be noted that the auxiliary parameter ħ controls the convergence and accuracy of the solution series (Liao, 2003). The analytical solution represented by Eq. 12 contains the auxiliary parameter ħ, which gives the convergence region and rate of approximation for the homotopy analysis method In order to define a region such that the solution series is independent on ħ, a multiple of ħ-curves are plotted. The region where the distribution of x' and x versus ħ is a horizontal line is known as the convergence region for the corresponding function. The common region among the x(t) and its derivatives are known as the overall convergence region.

To study the influence of ħ on the convergence of solution, the ħ-curves of x' and x(1) are plotted for different values of constant parameters, as shown in Fig. 2. Moreover, increasing the order of approximation increases the range of the convergence region (Fig. 3).

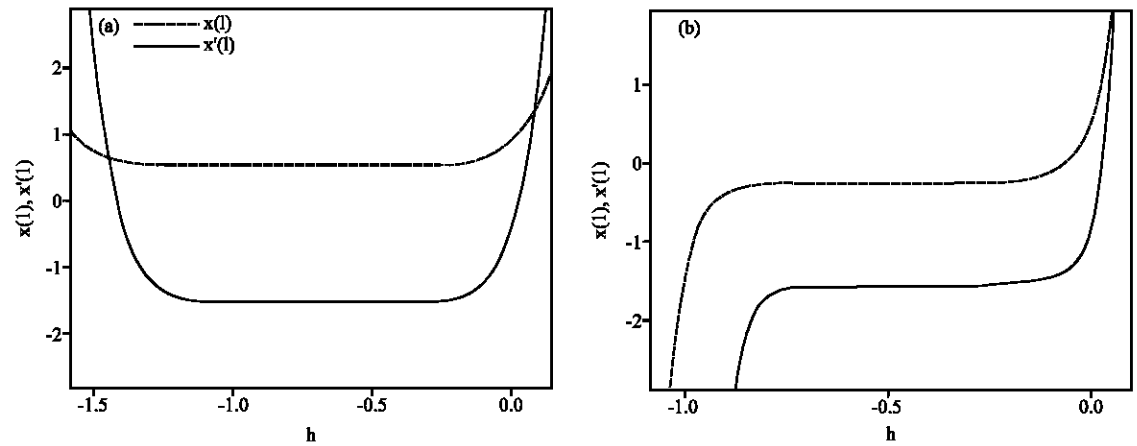

Fig. 2: The ħ curves to indicate the convergence region, EA = 556 kN, L_0 = 1.143 m, ρ = 4.6416×10^{-2} kg m^{-1} and M = 100 kg m^{-1} and M = 100 kg: (a) t_0 = 3558.6 N, (b) t_0 = 4448 N

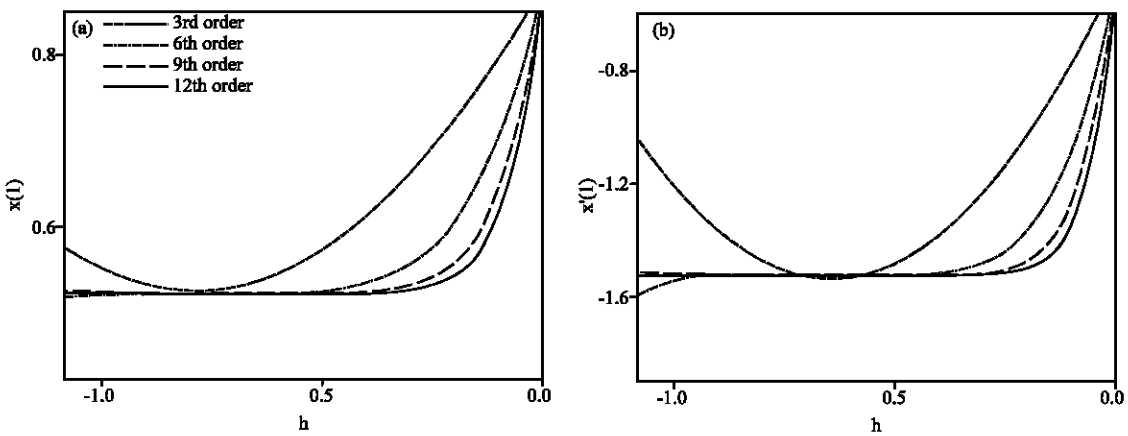

Fig. 3: The effect of order of approximation on convergence region, EA = 556 kN, L_0 = 1.143 m, ρ = 4.6416×10^{-2} kg m^{-1} and M = 100 kg m^{-1} and t_0 = 4448 N: (a) x (1), (b) x' (1)

PEM FOR SOLVING THE PROBLEM

According to the PEM (He, 2006), Eq. 7 can be rewritten as:

$$\frac{d^2x(t)}{dt^2}+\alpha x(t)+\beta x(t)^3=0 \qquad (25)$$

and the initial conditions are as follows:

$$x(0)\ \lambda, \qquad x'(0)=0 \qquad (26)$$

The form of solution and the constants one in Eq. 25 can be expanded as:

$$x(t)=x_0(t)+p\,x_1(t)+p^2x_2(t)+\cdots \qquad (27)$$

$$\alpha=\omega^2+p\,b_1+p^2b_2+\cdots \qquad (28)$$

$$\beta=pc_1+pc_2+\cdots \qquad (29)$$

Substituting Eq. 27-29 into Eq. 25 and processing as the standard perturbation method, we have:

$$x_0''(t)+\omega^2x_0(t)=0,\ x_0(0)=\lambda, \qquad x_0'(0)=0 \qquad (30)$$

$$\frac{d^2x_1(t)}{dt^2}+\omega^2x_1(t)+b_1x_0(t)+c_1x_0{}^3=0.\quad x_1(0)=0,\quad x_1'(0)=0 \qquad (31)$$

The solution of Eq. 30:

$$x_0(t)=\lambda\cos(\omega t) \qquad (32)$$

Substituting $x_0(t)$ from the above equation into Eq. 31 results in:

$$\frac{d^2x_1(t)}{dt^2}+\omega^2x_1(t)+b_1\lambda\cos(\omega t)-c_1\lambda^3\cos^3(\omega t)=0 \qquad (33)$$

But from Eq. 28 and 29:

$$b_1 = \frac{\alpha - \omega^2}{\varepsilon}, \qquad c_1 = \beta \tag{34}$$

Based on trigonometric functions properties we have:

$$\cos^3(\omega t) = 1/4\cos(3\omega t) + 3/4\cos(\omega t) \tag{35}$$

Replacing Eq. 32 into 31 and eliminating the secular terms yields:

$$\frac{\alpha - \omega^2}{p} + \beta\lambda^2 = 0 \tag{36}$$

Set p = 1 then two roots of this particular equation can be obtained as:

$$\omega = \pm\sqrt{\alpha - \omega^2 + \beta\lambda^2} \tag{37}$$

Replacing ω from Eq. 37 into 32 yields:

$$x(t) = x_0(t) = A\cos(\pm\sqrt{\alpha - \omega^2 + \beta\lambda^2}\, t) \tag{38}$$

Finally, x(t) is the answer of above problem.

RESULTS AND DISCUSSION

In this study, the usefulness of the presented parameter-expanding method and homotopy analysis method are investigated by considering above problem. To validate the results, convergence studies are carried out and the results are compared with those obtained using numerical results base on fourth order Runge Kutta method (Hoffman, 1992) and shown in Table 1 and 2 in the case of EA = 556 kN, L_0 = 1.143 m, ρ = 4.6416×10^{-2} kg m^{-1}, M = 500 kg and t_0 = 3558.6 N. It is worth mentioning that the relative error is defined as follows:

$$E_{rel} = 100\times\left|1 - \frac{\text{Results of HAM or PEM}}{\text{Numerical results}}\right| \tag{40}$$

The effect of constant parameters has been studied in Fig. 4 and 5 that are compared with the numerical results. Also, in the Fig. 7 the percentage of relative error has been shown to indicate the accuracy of the procedure. In addition in the Fig. 7 it has been shown that the maximum error is about 2.5%, that it is very small error for PEM solution.

Table 1: Compression between results of x(t) predicted by PEM, HAM and numerical method

t	x(t) PEM	HAM	Numerical	E_{rel} (PEM)	E_{rel} (HAM)
0.1	0.995287	0.995392	0.995085	0.02026	0.03083
0.2	0.980982	0.980799	0.980483	0.05082	0.03216
0.3	0.957212	0.956978	0.956609	0.06305	0.03857
0.4	0.924889	0.924523	0.924115	0.08372	0.04413
0.5	0.884983	0.884456	0.883838	0.12953	0.06992
1	0.605643	0.600543	0.600170	0.91176	0.06207
2	-9.88E-02	-9.72E-02	-9.71E-02	1.73499	0.07446
5	-0.421876	-0.413786	-0.413404	2.04952	0.09253

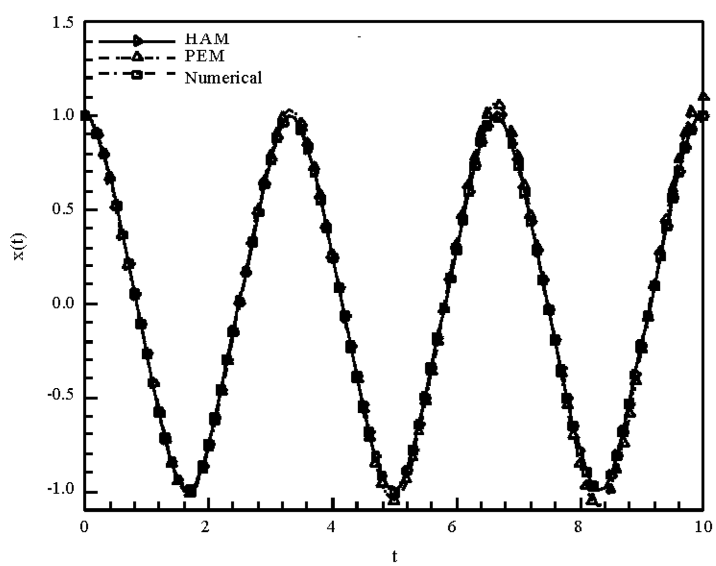

Fig. 4: Displacement-time plot for the two-link structure, predicted by PEM and HAM, EA = 556 kN, L_0 = 1.143 m, ρ = 4.6416×10^{-2} kg m^{-1}, M = 100 kg and t_0 = 4448 N

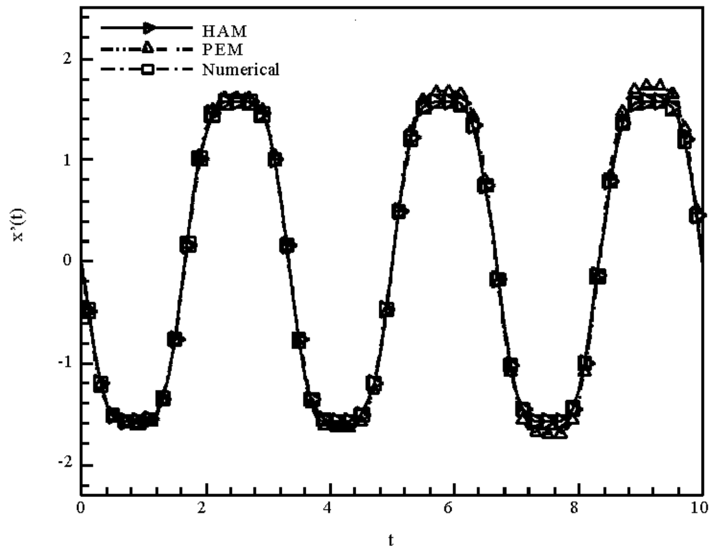

Fig. 5: Velocity-time plot for the two-link structure, predicted by PEM and HAM, EA = 556 kN, L_0 = 1.143 m, $\rho = 4.6416\times10^{-2}$ kg m^{-1}, M = 100 kg and t_0 = 4448 N

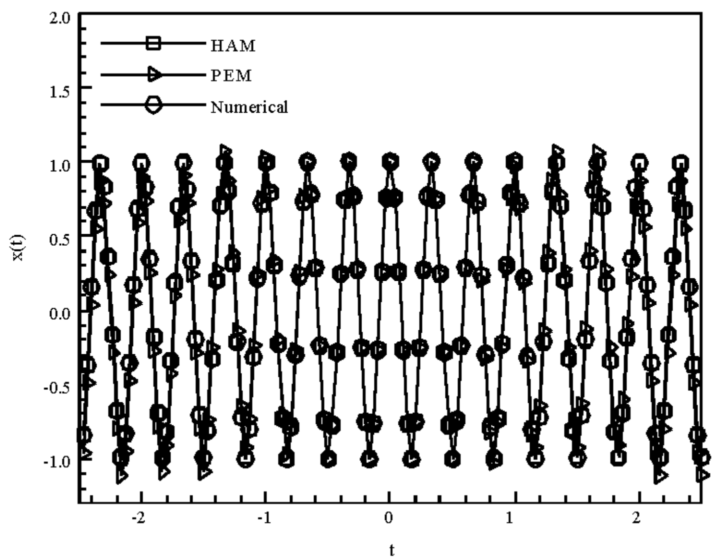

Fig. 6: Displacement-time plot for the two-link structure, predicted by PEM and HAM, EA = 556 kN, L_0 = 1.143 m, $\rho = 4.6416\times10^{-2}$ kg m^{-1}, M = 100 kg and t_0 = 4448 N

Table 2: Compression between results of x'(t) predicted by PEM, HAM and numerical method

	x'(t)				
t	PEM	HAM	Numerical	E_{rel} (PEM)	E_{rel} (HAM)
0.1	-9.81E-02	-9.81E-02	-9.81E-02	0.01759	0.05949
0.2	-0.193323	-0.193323	-0.193282	0.02120	0.02113
0.3	-0.283202	-0.283233	-0.283101	0.03553	0.04650
0.4	-0.365505	-0.365588	-0.365374	0.03584	0.05849
0.5	-0.441956	-0.438796	-0.438549	0.77691	0.05641
1	-0.667986	-0.656323	-0.656728	1.71430	0.06179
2	-0.726554	-0.706434	-0.706209	2.88086	0.03186
5	7.16E-01	6.95E-01	6.95E-01	3.00881	0.03078

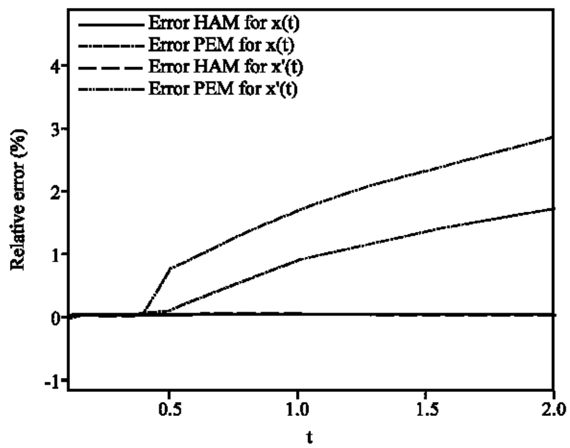

Fig. 7: The percentage of relative error for the two-link structure, predicted by PEM and HAM, EA = 556 kN, L_0 = 1.143 m, ρ = 4.6416×10^{-2} kg m^{-1}, M = 100 kg and t_0 = 4448 N

Based on Table 1, 2 and Fig. 4-6, it can be concluded that only one term in series expansions is sufficient to obtain a highly accurate solution, which is valid for the whole solution domain.

CONCLUSION

In this study, homotopy analysis method and a new method called He's parameter-expanding method has been studied. In the numerical methods, stability and convergence should be considered so as to avoid divergence or inappropriate results. In the analytical perturbation method, we should exert the small parameter in the equation. Therefore, finding the small parameter and exerting it into the equation are deficiencies of these methods. Two of the semi-exact methods which don't need small/large parameters are Homotopy Analysis Method and Parameter Expanding method. In addition, to comprise the obtained results, the governing equation was solved numerically by authors based on fourth order Runge Kutta method. Some remarkable virtues of the methods were studied and their applications for obtaining the displacement functions of geometrically nonlinear prestressed cable structures analytically, have been illustrated. The obtained results have a good agreement with those obtained using numerical method. It is clear HAM is a generalized Taylor series method, searching for an infinite series solution, PEM is clearly a new perturbation method, searching an asymptotic solution with only one term and no convergence theory is needed. Moreover, increasing the domain of independent parameter or increasing the coefficients of nonlinear term, increases the error of PEM solution, whereas, the HAM solution is very accurate for whole domain of solution, as shown in Table 1-2. Also, as shown in Table 3 the equation was solved by Maple software to be convincing about authors' numeric solution. The results show that the methods are promising for solving this type of problems and might find wide applications.

Table 3: Compression between results of x(t) predicted by PEM, HAM and Maple software

	x(t)				
t	PEM	HAM	Numerical	E_{rel} (PEM)	E_{rel} (HAM)
0.1	-0.935133	-0.934539221	-0.9343988	0.078574314	0.0150277
0.2	0.756481	0.757211	0.758731978	0.296676277	0.20046312
0.3	-0.517793	-0.517801	-0.5180807	0.055532695	0.05398853
0.4	0.24998	0.250792	0.250861791	0.351504746	0.02782054
0.5	2.34E-02	2.32E-02	2.33E-02	0.472828174	0.19342971
1	-1.01164078	-0.99473848	-0.99818848	1.34767133	0.34562611
2	9.68E-01	9.99E-01	9.94E-01	2.579573407	0.54349186

NOMENCLATURE

D	: D'Alembert forces
EA	: Axial stiffness
H(t)	: Auxiliary function
L	: Linear operator
L_0	: Original undeformed length
N	: Nonlinear operator
q	: Embedding parameter
R	: Vertical dynamic
R_m (U_{m-1})	: Reminder term
t	: Time (Independent dimension less parameter)
t_0	: Initial pretension
ρ	: Mass per unit length of the links
ħ	: Auxiliary parameter

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