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# Module Approximate Amenability for Semigroup Algebras

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**Abstract:** In this study, we introduce module approximate amenability. Indeed, we extend the concept of approximate amenability of Banach algebra A to the case that there is an extra U-module structure on A and we show that $l^1(S)$ is module approximately amenable if and only if S is amenable, where, S is an inverse semigroup with subsemigroup E of idempotents and $l^1(S)$ has $l^1(E)$-module structure.

**Key words:** Semigroup algebra, module amenability, approximate amenability

## INTRODUCTION

New notions of amenability was introduced by Amini (2004) and Ghahramani and Loy (2004). In this study, we mix these different notions and introduce module approximate amenability for a Banach algebra and then we investigate this notion on semigroup algebra $l^1$ (S).

Let A be a Banach algebra and X be a Banach A-bimodule. A derivation from A into X is a bounded linear map D:A→X satisfying:

$$D(ab) = a.D(b)+D(a).b$$

For each x∈X we denote by $ad_x$ the derivation D(a) = ax-xa for all a∈A, which is called an inner derivation. If X is a Banach A-bimodule, X* (the dual space of X) is an A-bimodule as usual. A Banach algebra A is called amenable if for any Banach A-bimodule X, every derivation D : A→X* is inner. The celebrated Johnson's Theorem (in discrete case) asserts that a discrete group G is amenable if and only if the Banach algebra $l^1(G)$ is amenable (Johnson, 1972).

A Banach algebra A is called approximately amenable if for any Banach A-bimodule X, every derivation D : A→X* is approximately inner, that is, there exists a net $(f_i)\subseteq X^*$ such that for every a∈A, $D(a) = \lim_i (af_i - f_i a)$ in norm topology. Approximate amenability was introduced by Ghahramani and Loy (2004). One of motivations for definition of approximate amenability comes from Gourdeau (1992), where the researchers has shown that the assumption of existence of a bounded net $(x_i)$ is in fact equivalent to amenability of A.

Let U and A be Banach algebras such that A is a Banach U-bimodule with compatible actions, that is:

$$\alpha.(ab) = (\alpha.a)b, \quad a(\alpha.b) = (a.\alpha)b \quad (a, b \in A, \alpha \in U)$$

Let X be a Banach A-bimodule and a Banach U-bimodule with compatible actions, that is:

$$\alpha.(a.x) = (\alpha.a).x,\ (a.\alpha).x = a.(\alpha.x),\ (\alpha.x).a = \alpha.(x.a) \quad (a, b \in A, \alpha \in U, x \in X)$$

and the same for right or two-sided action. Then we say that X is a Banach A-U-module. Moreover, if:

$$\alpha.x = x.\alpha \quad (\alpha \in U, x \in X)$$

then X is called a commutative A-U-module. A bounded map D : A→X is called a module derivation if:

$$D(a \pm b) = D(a) \pm D(b),\ D(ab) = a.D(b) + D(a).b \quad (a, b \in A)$$

$$D(\alpha.a) = \alpha.D(a),\ D(a.\alpha) = D(a).\alpha \quad (\alpha \in U, a \in A)$$

When, X is commutative, each x∈X defines a module derivation as follows:

$$ad_x(a) = a.x - x.a \quad (a \in A)$$

that is called inner derivation. Now we define module approximate amenability.

**Definition 1:** A Bananch algebra A, which is a U-bimodule, is called module approximately amenable (as a U-bimodule) if for any commutative Banach A-U-module X, each module derivation D : A→X* is approximately inner.

## MODULE APPROXIMATE AMENABILITY FOR SEMIGROUP ALGEBRA

Here, we investigate module approximate amenability of $l^1(S)$ as a $l^1(E)$-module, where, S is an inverse semigroup with idempotents E.

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A discrete semigroup S is called an inverse semigroup if for each s∈S there is a unique s*∈S such that s*ss* = s* and ss* s = s. An element e ∈ S is called an idempotent if $e^2$ = e* = e. The set of idempotent elements of S is denoted by E. It is easy to see that E is a commutative subsemigroup of S and $l^1(E)$ could be regarded as a subalgebra of $l^1$ (S) (Howie, 1976).

We consider $l^1(S)$ as a $l^1(E)$-module with the following module actions:

$$\delta_e.\delta_s = \delta_s,\ \delta_s.\delta_e = \delta_{se} = \delta_s * \delta_e \quad (s\in S, e\in E)$$

Consider the congruence relation ~ on S where, s~t if and only if there is an e∈E such that se = te. The quotient semigroup $G_S$:=S/~ is then a group. The inverse semigroup S is amenable if and only if the discrete group $G_S$ is amenable (Duncan and Namioka, 1978). With this notation, $l^1(G_S)$ is a quotient of $l^1(S)$ and so the earlier action of $l^1(E)$ on $l^1(S)$ lifts to an action of $l^1(E)$ on $l^1(G_S)$ and making it a Banach $l^1(E)$-module.

**Lemma 1:** With above notations $l^1(G_S)$ is module approximately amenable if and only if it is approximately amenable.

**Proof:** Consider the quotient map π:S→$G_S$, that maps each s∈S into congruence class of s. For each s∈S and e∈E we have π(s) = π(se) and so:

$$\delta_e.\delta_{\pi(s)} = \delta_{\pi(s)} = \delta_{\pi(se)} = \delta_{\pi(s)}.\delta_e$$

Thus action of $l^1(E)$ on $l^1(G_S)$ is trivial. Therefore, we can take all Banach $l^1(G_S)$-modules as a commutative $l^1(G_S)$- $l^1(E)$-module with trivial action of $l^1(E)$. Let X be a Banach $l^1(G_S)$-module and D : $l^1(G_S)$→X* be a module derivation. For each λ∈C (the field of complex numbers), f∈ $l^1(G_S)$ and e∈E we have:

$$D(\lambda f) = D(\lambda(\delta_e.f)) = D((\lambda\delta_e).f) = \lambda\delta_e.D(f) = \lambda(\delta_e.D(f)) = \lambda D(f)$$

This shows that D is C-linear. Hence, $l^1(G_S)$ is module approximately amenable if and only if it is approximately amenable.

**Theorem 1:** Let A and B be Banach algebras and Banach U-modules with compatible actions and let φ:A→B be a continuous Banach algebra homomorphism and U-module homomorphism with dense range. If A is module approximately amenable, then so is A.

**Proof:** If X is a commutative B-U-module, it could be regarded as a commutative A-U-module by:

$$a.x = \varphi(a).x,\ x.a = x.\varphi(a) \quad (a\in A, x\in X)$$

Also each module derivation D:B→X* gives a module derivation Doφ:A→X*. Since, A is module approximately amenable, there is a net $(f_i)$⊆X* such that:

$$D(\varphi(a)) = Do\varphi(a) = \lim_i (a.f_i - f_i.a) \quad (a\in A)$$

Since, $a.f_i = \varphi(a).f_i$ and $f_i.a = f_i\varphi.(a)$, we have:

$$D(\varphi(a)) = \lim_i (\varphi(a).f_i - f_i.\varphi(a)) \quad (a\in A)$$

Now density of φ(A) in B and continuity of D imply that:

$$D(b) = \lim_i (b.f_i - f_i.b) \quad (b\in B)$$

Hence, B is module approximately amenable.

**Theorem 2:** Let S be an inverse semigroup with idempotents E. Consider $l^1(S)$ as a Banach module over $l^1(E)$ with the multiplication right action and the trivial left action. Then $l^1(S)$ is module approximately amenable if and only if S is amenable.

**Proof:** If S is amenable, then $l^1(S)$ is module amenable by Theorem 3.1 (Amini, 2004). Thus $l^1(S)$ is module approximately amenable. Conversely, let $l^1(S)$ be module approximately amenable. Consider the quotient map π:S→$G_S$, that maps each s∈S into congruence class of s. Then π induces the continuous epimorphism $\tilde{\pi}: l^1(S) \to l^1(G_S)$ with $\tilde{\pi}(\delta_s) = \delta_{\pi(s)}$. The earlier theorem shows that $l^1(G_S)$ is module approximately amenable and so $l^1(G_S)$ is approximately amenable by Lemma 1. Hence $G_S$ is amenable group by Theorem 3.2 by Ghahramani and Loy (2004). Therefore, S is amenable by Theorem 1 by Duncan and Namioka (1978).

**Remark:** Lashkarizadeh and Samea (2005) have shown that if S is a cancellative semigroup such that $l^1(S)$ is approximately amenable, then S is amenable.


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