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An Analytical Solution for the Typical Energy Cascade in Turbulence

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Abstract: In this study, we derived the Fokker-Planck equation which is based on the Langevin equation for the typical energy cascade in turbulence. We show that the typical energy cascade in turbulence is well described by a linear Langevin equation and obtain the drift and diffusion coefficients. We exactly solve the Fokker-Planck equation for a certain case and find an explicit form of the probability distribution function. Present theoretical results from the Langevin equation is shown the Gaussian probability distribution function. This probability distribution function is known to be a good approximation of the statistics of the energy cascade in turbulent flows. We arise naturally as solution of stochastic processes for the variable $x = \ln(\varepsilon_r)$.

Key words: Energy cascade, turbulence, Fokker-Planck equation, Probability distribution function, Langevin equation

INTRODUCTION

Turbulent flow, which is a very complex flow phenomenon, abounds in nature, science and engineering. Turbulence is caused by increasing injected energy from the outside into a fluid system with the velocity of the fluid exceeding some value (the Reynolds number), Re, at which the fluid state translates from a laminar flow to a turbulent one (Chanal *et al*., 2000). If the fluid is a stochastically stationary process, which means that the global statistic properties of the flow state do not vary from the translation of the starting point when time is measured, then, a large energy eddy splits into middle-size eddies which then split into smaller-size eddies (Ingraham, 1992; Celani *et al*., 2002) and so on. This series cascade process of energy dissipation is a deterministic process with conditional probability as shown in Fig. 1. Not long after this, turbulence intermittency was discovered by Frisch (1995), which means that turbulence and non-turbulence occur alternately in a flow field in the time domain. However, they both coexist and interweave together in the flow field in the space domain. In order to describe the turbulence intermittency, many researchers modify the classical cascade model in Fig. 1a into another model as shown in Fig. 1b (Frisch, 1995). The statistical properties of isotropic and homogeneous turbulent flows have been reconsidered recently within the context of continuous, additive, Markov stochastic processes (Pearson *et al*., 2004). Analysis of experimental data from a turbulent flow has shown that the scale-dependence of the probability

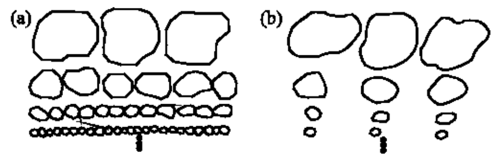

Fig. 1: (a) Classical cascade model: a large energy eddy splits into middle-size eddies which split into smaller-size eddies and so on. This is a series cascade process of energy dissipation. (b) The modified model showing the classical cascade according to turbulence intermittency (Frisch, 1995)

distribution functions of velocity increments (Schmiegel, 2005) and of the logarithm of the energy dissipation rate (Sreenivasan *et al*., 2003) may be described by Fokker-Planck equations. The typical energy cascade corresponds to an Ornstein-Uhlenbeck process for the stochastic variable $x = \ln(\varepsilon_r)$, since, the drift and diffusion terms of the Fokker-Planck equation is respectively found linear and constant. This typical model is exactly solvable. The probability function is found Gaussian . In this study the theoretical results are compatible on the data, recorded in a typical low emperature turbulent jet flow at high Reynolds number (Cleve *et al*., 2004). We aim at building a faithful, yet simple model of available experimental data, from a phenomenological perspective. Fokker-Planck equations, which govern the evolution of the probability distribution are equivalent mathematically to

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stochastic differential equations with an additive noise term (Langevin equation) (Mazzi and Vassilicos, 2004).

Recently Naert *et al.* (1997) presented experimental evidence that the Probability Distribution Function (PDF) obeys a Fokker-Planck Equation (FPE) as follows:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}(D^{(1)}(x,t)P(x,t)) + \frac{\partial^2}{\partial x^2}(D^{(2)}(x,t)P(x,t)) \quad (1)$$

where, the drift and diffusion coefficients $D^{(1)}$ (x, t) and $D^{(2)}$ (x, t) are derived by analysis of experimental data of a fluid dynamical experiment (Friedrich and Peinke, 1997a). In this study we consider the application of the Langevin equation to obtain the FPE with simplified assumption that drift and diffusion coefficients are time-independent.

HOW TO DERIVE THE FOKKER-PLANCK EQUATION FOR THE TYPICAL ENERGY CASCADE IN TURBULENCE

Consider the Langevin equation for the energy cascade in turbulence:

$$\dot{x} = \gamma x - De^{2\gamma t} + \sqrt{D}\Gamma(t) \quad (2)$$

With, $\Gamma(t)$ Gaussian white-noise $\langle\Gamma(t)\Gamma(t')\rangle = 2\delta(t-t')$, γ,D are constant and $-De^{2\gamma t}$ is derived from an energy conservation condition (Songnian *et al.*, 2005; Delour *et al.*, 2001). In general a formal solution of the stochastic differential Eq. 2 can not be given simply, then we should derive FPE from Eq. 2. In the Fokker-Planck equation the following Keramers-Moyal expansion coefficients (Drazin, 1992) $D^{(1)}$(x,t) and $D^{(2)}$(x,t) are obtained from Eq. 3.

$$D^{(n)}(x,t) = \frac{1}{n!}\lim_{t\to 0}\frac{1}{t}\left\langle [x(t)-x(0)]^n \right\rangle_{t=t_0} \quad (3)$$

To derive these Keramers-Moyal expansion coefficients, first we write the differential Eq. 2 in the form of an integral equation:

$$x(t) - x(0) = \int_0^t [\gamma x(t') - De^{2\gamma t} + \sqrt{D}\Gamma(t')]dt' \quad (4)$$

assume that *x* can be expanded according to:

$$x(t) = x(0) + t\dot{x}(t) \quad (5)$$

To insert Eq. 5 into Eq. 4 leads to:

$$x(t) - x(0) = \int_0^t [\gamma x(0) + \gamma t\dot{x}(t') - De^{2\gamma t} + \sqrt{D}\Gamma(t')]dt' \quad (6)$$

$$x(t) - x(0) = \int_0^t \gamma x(0)dt' + \int_0^t \gamma t[\gamma x(t') - De^{2\gamma t} + \sqrt{D}\Gamma(t)]dt' + \int_0^t [-De^{2\gamma t}]dt' + \int_0^t \sqrt{D}\Gamma(t')dt' \quad (7)$$

To calculate the average value of Eq. 7, we obtain:

$$\langle x(t) - x(0)\rangle = \left\langle \int_0^t \gamma x(0)dt' + \int_0^t \gamma t[\gamma x(t) - De^{2\gamma t} + \sqrt{D}\Gamma(t)]dt' - \int_0^t De^{2\gamma t}dt' \right\rangle \quad (8)$$

In the limit $t\to 0$, the drift coefficient of the FPE is derived from Eq. 8:

$$D^{(1)}(x,t) = \gamma x(t) - De^{2\gamma t} \quad (9)$$

Now, let us derive diffusion coefficient $D^{(2)}$(x,t) from Eq. 3, then we can write :

$$D^{(2)}(x,t) = \frac{1}{2}\lim_{t\to 0}\frac{1}{t}\left\langle [x(t)-x(0)]^2 \right\rangle_{t=t_0} \quad (10)$$

$$D^{(2)}(x,t) = \frac{1}{2}\lim_{t\to 0}\frac{1}{t}\left\langle \int_0^t\int_0^t D\Gamma(t')\Gamma(t'')dt'dt'' \right\rangle \quad (11)$$

$$D^{(2)}(x,t) = \frac{1}{2}\lim_{t\to 0}\frac{1}{t}\int_0^t\int_0^t D2\delta(t'-t'')dt'dt'' = D \quad (12)$$

As we have shown in Eq. 9 and 12, we will consider the drift coefficient $D^{(1)}$ and diffusion coefficient $D^{(2)}$ time-independent, where, $D^{(1)}$ is linear in x and $D^{(2)}$ is constant. A linear drift and constant diffusion coefficients are characteristic of an Ornstein-Uhlenbeck process (Feller, 1991). One may note that here, unlike the examples of Ornstein-Uhlenbeck processes usually discussed, the slope γ of the drift is positive (Feller, 1991). By using Eq. 1, 9 and 12 we derive:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}(\gamma x(t) - De^{2\gamma t}P(x,t)) + \frac{\partial^2}{\partial x^2}(DP(x,t)) \quad (13)$$

Equation 13 is the Fokker-Planck equation which Naert *et al.* (1997) experimentally obtained for the typical energy cascade.

Exact solution of the Fokker-Planck equation: Now, we are going to solve the FPE analytically with the mathematical method (Risken *et al.*, 1996). By using the Fourier transformations between P(x,t) and $Z(\lambda,t)$ we can write:

$$P(x,t) = \int_{-\infty}^{+\infty} e^{i\lambda x}Z(\lambda,t)d\lambda \quad (14)$$

$$Z(\lambda, t) = \int_{-\infty}^{+\infty} e^{-i\lambda x} P(x,t) dx \quad (15)$$

where, $z(\lambda,t)$ is the generating function as defined as follows :

$$Z(\lambda, t) = \langle e^{-i\lambda x} \rangle \quad (16)$$

By differentiating $z(\lambda,t)$ with respect to t, we obtain:

$$Z_t = -i\lambda \langle \dot{x} e^{-i\lambda x} \rangle = -i\lambda \langle (\gamma x - De^{2\gamma t}) e^{-i\lambda x} + (\sqrt{D}\Gamma(t)) e^{-i\lambda x} \rangle \quad (17)$$

Since, $\Gamma(x,t)$ is a Gaussian white-noise, we can use the standard trick of the Langevin equation, it is called Novikov theorem (Polyakov, 1995; Masoudi and Azimi Anaraki, 2006).

$$\left\langle \Gamma(x,t) \exp\left(-i \sum_j \lambda_j X(x_j, t) \right) \right\rangle = -i \sum_j \kappa(x - x_j) \lambda_j Z \quad (18)$$

By using Eq. 17 and 18, we derive:

$$Z_t = \gamma \lambda Z_\lambda + iDe^{2\gamma t} \lambda Z - \sqrt{D} \lambda^2 Z \quad (19)$$

In the stationary state, we can solve Eq. 19 and we obtain:

$$Z = e^{\frac{-\sqrt{D}\left(\lambda - i\sqrt{D}e^{2\gamma 1}\right)^2}{2\gamma}} e^{-\frac{\sqrt{D}}{2\gamma} De^{4\gamma 1}} \quad (20)$$

By using Eq. 14 and 20, the PDF is defined as:

$$P(x,t) = \sqrt{\frac{2\pi\gamma}{D^{\frac{1}{2}}}} e^{-\frac{\sqrt{D}}{2\gamma} De^{4\gamma 1}} e^{-\sqrt{D}e^{2\gamma 1}} e^{\frac{-\gamma x^2}{\sqrt{D}}} \quad (21)$$

Now, we consider the probability distribution function for the typical energy cascade in turbulence has been obtained. The PDF showing different properties of this system.

CONCLUSION

In this study, the Probability Distribution Function (PDF) is analytically calculated as a Gaussian function in terms of stochastic variables for a typical energy cascade in turbulence by using the mathematical method. Recently Friedrich and Peinke calculated the PDF from the experimental data of a low temperature helium jet at R_e = 20000, R_λ = 328 (Chabaud *et al.*, 1994) that is compatible with present results, their results about drift and diffusion coefficients are in good agreement with Eq. 13. To conclude, by a detailed analysis of fluid

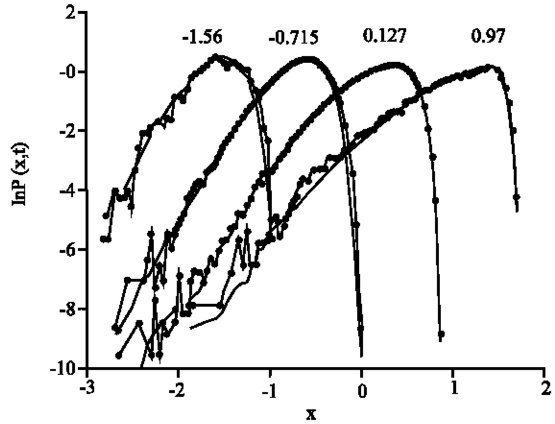

Fig. 2: Experimental verification of the PDF. The PDF calculated directly (dotted line) from Eq. 21, (solid line) is plotted for four values of x (written above the histograms)

dynamical experiments, they were able to obtain a phenomenological description of the statistical properties of a typical energy cascade in turbulence, using a FPE as an evolution equation for the PDF of stochastic variable (Friedrich and Peinke, 1997b). This equation contains the information on the changing shape of the PDF as a function of the scale η (Kolmogorov viscous scale) in Fig. 2 (Naert *et al.*, 1997). As we have shown in Eq. 21, The calculation is in good agreement with the experimental results in Fig. 2.

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