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Novel Method for Determining the Maximally Productive Units using DEA

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Abstract: In this study, we propose a new method for determine maximally productive units based on input-output orientation data envelopment analysis. In this method, we find that reduce inputs and improve outputs units without regard to any factor weights is possible. The new method is a linear mathematical programming technique that determines the intensities of units. No assumptions are required on the internal transformation processes of the units. Decision making depends critically on the way excellent units are frequently described. Present findings have implications for the monitoring and financing of units. Some units with the maximal productivity should be considered as a guide for the other units to reduce inputs and improve outputs units. Numerical illustrations are provided for 15 hospitals dataset of Sherman and 12 hospitals in Tehran, Iran.

Key words: Maximally productive units, data envelopment analysis, mathematical programming

INTRODUCTION

Data Envelopment Analysis (DEA) is a fractional mathematical programming technique that has been developed by Charnes *et al*. (1978). It evaluates the relative efficiency of homogeneous units by considering multiple inputs and outputs. The efficiency is defined as a ratio of the weighted sum of outputs to the weighted sum of inputs. Assuming that there are n Decision Making Units (DMUs) each with m inputs and s outputs, the relative efficiency of a particular DMU_o is obtained by solving the following fractional programming problem:

$$\max z_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}$$

Subject to:

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \qquad j = 1,\dots,n \tag{1}$$

$$u_r \ge 0,\ r = 1,\dots,s$$

$$v_i \ge 0,\ i = 1,\dots,m$$

where, j is the DMU index, j = 1,...n, r, the output index, r = 1,...,s; i the input index, i = 1,...,m; y_{rj} the value of the rth output for the jth DMU, x_{ij} the value of the ith input for the jth DMU, u_r the weight given to the rth output, v_i the weight given to the ith input and z_o is the relative efficiency of DMU_o, the DMU_o is efficient if and only if $z_o = 1$.

A fully rigorous development would replace u_r, $v_i \ge 0$ with u_r, $v_i \ge \varepsilon > 0$ where ε is a non-Archimedean element smaller than any positive real number (Cooper *et al*., 2004).

This fractional program can be converted into a linear programming problem using the Charnes-Cooper transformation, where the optimal value of the objective function indicates the relative efficiency of DMU_o. The linear programming problem known as the CCR model, the CCR model with constant return to scale can be formulated as follows to obtain a score of technical efficiency:

$$\max z_o = \sum_{r=1}^{s} u_r y_{ro}$$

Subject to:

$$\sum_{i=1}^{m} v_i x_{io} = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \qquad j = 1,\dots,n \tag{2}$$

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$$u_r \geq 0, \; r = 1,...,s$$

$$v_i \geq 0, \; i = 1,...,m$$

Definition 1 (Efficiency): Full (100%) efficiency is attained by any DMU if and only if none of its inputs or outputs can be improved without worsening some of its other inputs or outputs.

Definition 2 (Relative Efficiency): A DMU is to be rated as fully (100%) efficient on the basis of available evidence if and only if the performances of other DMUs does not show that some of its inputs or outputs can be improved without worsening some of its other inputs or outputs (Cooper *et al*., 2004).

Since, 1978, several different versions of DEA models have been proposed in the literature, showing different useful properties which make them apt to different practical situations. The reader is referred to the several references (Banker *et al*., 1984; Bowlin *et al*., 1985; Charnes *et al*., 1978; Cooper *et al*., 2004, 2006; Emrouznejad and Victor, 2004).

In DEA, sometimes by application those models yielding many numbers of DMUs as efficient in this case the drawbacks of lack of discrimination among DMUs. The objective of this paper is to investigate this problem by proposing a new method to discrimination among technically efficient units and find maximally productive units. We call the optimization principle the winner(s) criterion. It is based on the idea that if the technology of a maximally productive unit or composite of such units, called the winner(s), is used to process all available input vectors, then the total value of these input resources must be maximized. Similarly, in the cost minimization orientation, the winner or winning composite is required to produce all required outputs at minimum imputed relative cost.

STRUCTURE OF NEW METHOD

We take the point of view of a real or hypothetical entity considered to be the owner of all the inputs and productive units. We suppose this entity would wish to choose the most productive unit(s) among them in a sense to be explained. If unit-p, say, were determined to be most productive, then the owner would prefer that particular unit to process or transform all the input vectors to outputs to the extent that it has capacity to do so. This will be the case if the valuation rates, for which unit-p is technically efficient, also maximize the value to the owner of the total of all the input vectors. In that case, we call unit-p the winner and it is awarded, in principle, the job of transforming all the input vectors to outputs. More generally, it may not be possible for a single unit to process all available inputs, so that a composite set of such winning units is required. These maximally productive units are called the winners.

We may distinguish two orientations. The owner might seek to determine maximally productive unit(s) in such a way that cost is minimized subject to the production of at least the amounts given by a specified target output vector. Alternatively, the owner may wish to determine maximally productive unit(s) so that output value is maximized subject to available input resources. Thus, in the cost minimization orientation, we pose the following problem. What composite of the units should be used and what relative factor valuations should be assigned, so that at least the same or more outputs as observed could be produced at minimum relative cost? Similarly, in the output value maximization orientation, the problem posed is to determine what composite of units and relative factor valuations should be chosen so the composite would consume no more than the observed input resources while yielding maximal relative value?

For the input cost minimization orientation, the above considerations produce the following model which seeks to produce at least the required outputs at minimum cost:

$$\min z = \sum_{i=1}^{m} \frac{\sum_{j=1}^{n} \zeta_j x_{ij}}{X_i}$$

Subject to:

$$\frac{\sum_{j=1}^{n} \zeta_j y_{rj}}{Y_r} \geq 1, \quad r = 1,...,s \tag{3}$$

$$\zeta_j \geq 0, \quad j = 1,...,n$$

Where, $X_i = \sum_{j=1}^{n} x_{ij}$ and $Y_r = \sum_{j=1}^{n} y_{rj}$, for i = 1,....,m and r = 1,....,s

As a measure of solution effectiveness for the (Eq. 3) model, we use the optimal cost reduction factor that we denote by ρ^*. This measure is given by:

$$\rho_i^* = \frac{\sum_{j=1}^{n} \zeta_j^* x_{ij}}{X_i}, \quad i = 1,...,m \tag{4}$$

Similarly, for the output value maximization orientation we obtain model:

$$\max z = \sum_{r=1}^{s} \frac{\sum_{j=1}^{n} \zeta_j y_{rj}}{Y_r}$$

Subject to:

$$\frac{\sum_{j=1}^{n} \zeta_j x_{ij}}{X_i} \le 1, \quad i = 1,...,m \tag{5}$$

$$\zeta_j \ge 0, \; j = 1,...,n$$

For the output value maximization model, the optimal value gain factor $\gamma^*$is given by:

$$\gamma^* = \frac{\sum_{j=1}^{n} \zeta_j^* y_{rj}}{Y_r}, \; r = 1,...,s \tag{6}$$

In the both orientation models (3) and (5) intensities $\zeta_j \ge 0$, are related to each respective unit j and may be interpreted that ζ^*_j is the number of replications of unit j.

Definition 3: A unit is said maximally productive if its corresponding ζ^*_j is positive in some solution of the associated models (3) and (5).

NUMERICAL EXAMPLES

We provide tow numerical examples to illustrate applications of tow orientation method.

Example 1: In this example, we apply the new method to the data in Table 1.

This table shows a data testing and comparing various technical efficiency models and provides an example of the general case of multiple inputs and multiple outputs. Construction of these data is also discussed in Bowlin *et al*. (1985) and Troutt *et al*. (2007).

The new models were applied to the data of Table 1. The resulting ζ^*_j-values are shown in Table 2. The corresponding ρ^*_i and γ^*_r for i = 1,2,3 and r = 1,2,3 are computed from Eq 4 and 6, respectively as follows:

$\rho^*_1 = 0.930233$, $\rho^*_2 = 0.956454$, $\rho^*_3 = 0.935201$,
$\gamma^*_1 = 1.853896$, $\gamma^*_2 = 1.054803$, $\gamma^*_3 = 0.948504$.

Example 2: In Table 3, we consider a group of 12 hospitals in Tehran, Iran with three inputs (m = 3) and three outputs (s = 3). Both two orientations method proposed in section previous are applied to the data in Table 3.

Table 1: Sherman's (1981) 15 hospitals dataset

	Inputs			Outputs		
Unit_j	FTE	Bed days	Supply ($)	Teaching units	Regular patients	Server patients
H1	23.5	41050	130000	50	3000	2000
H2	24.5	43160	140000	50	2000	3000
H3	26.0	43160	150000	100	2000	3000
H4	25.0	41050	140000	100	3000	2000
H5	28.5	50530	160000	50	3000	3000
H6	36.0	62105	210000	100	2000	5000
H7	51.5	92630	270000	50	10000	2000
H8	25.0	49475	140000	100	3000	2000
H9	24.5	43160	165000	50	2000	3000
H10	77.0	92630	340000	100	10000	2000
H11	44.5	65260	265000	50	5000	3000
H12	30.0	60000	170000	100	3000	3000
H13	43.5	81110	245000	50	4000	5000
H14	30.0	60000	170000	100	3000	3000
H15	26.5	47370	160000	50	3000	2000
Total	516.0	872690	2855000	1100	58000	43000

Table 2: Estimation results for the Sherman (1981) dataset using the tow orientations new method

Unit_j	Minimization orientation (ζ^*_j)	Technical efficiency	Maximization orientation (ζ^*_j)
H1	17.454550	1.000000	0
H2	0	1.000000	0
H3	0	1.000000	0
H4	1.090909	1.000000	20.392857
H5	0	1.000000	0
H6	1.181818	1.000000	0
H7	0	1.000000	0
H8	0	1.000000	0
H9	0	1.000000	0
H10	0	1.000000	0
H11	0	1.000000	0
H12	0	1.000000	0
H13	0	1.000000	0
H14	0	1.000000	0
H15	0	0.886793	0

We apply the Eq. 3 and 5 models to this dataset with the following context in mind. We assume that the same corporation owns all units and wishes to consider transferring resources of less efficient ones to more efficient ones.

In minimization orientation approach (Eq. 3) the resulting ζ^*_j-values are shown in Table 4. It indicates that the given output levels Y_r would be produced most efficiently by employing approximately 4.976878 of DMU number one (H1) and 7.936443 of DMU number ten (H10). Noting that the total cost of the status quo solution is 12 hospitals, we obtain $\rho^*_1 = 0.707856$, $\rho^*_2 = 0.482572$ and $\rho^*_3 = 0.713970$, using Eq. 4.

The maximization orientation (5) is applied to the dataset of Table 3 and the resulting ζ^*_j-values are shown in Table 4. It shows that H1 and H10 are maximally productive units with $\gamma^*_1 = 1.395664$, $\gamma^*_2 = 2.105165$ and $\gamma^*_3 = 1.444643$, using Eq. 6.

Table 3: Dataset 12 hospitals in city Tehran, Iran

	Inputs			Outputs		
$Unit_j$	No.of beds	No. of physicians	Expense IRR	No. of bedridden	No. of outpatient	Teaching units
H1	201	42	87429719000	22897	273162	106
H2	487	140	152485670578	28507	164424	627
H3	515	256	244494788599	21202	164000	810
H4	314	91	109049435093	13305	71000	2115
H5	383	107	110000000000	13967	96183	597
H6	151	53	46000000000	12355	127733	95
H7	171	83	72697793274	9859	102881	264
H8	108	45	30320842488	6614	90441	21
H9	349	65	48465400955	12865	183917	480
H10	183	40	47533623909	10063	120185	874
H11	247	55	78728831277	15251	60611	450
H12	356	114	110621073122	26935	79166	1025
Total	3465	1091	1137827178295	193820	1533703	7464

Table 4: Estimation results for the dataset 12 hospitals in city Tehran, Iran using the tow orientations new method

$Unit_j$	Minimization orientation (ζ^*_j)	Technical efficiency	Maximization orientation (ζ^*_j)
H1	4.976878	1.000000	6.751870
H2	0	0.733104	0
H3	0	0.494144	0
H4	0	1.000000	0
H5	0	0.528263	0
H6	0	1.000000	0
H7	0	0.718900	0
H8	0	0.903496	0
H9	0	1.000000	0
H10	7.936443	1.000000	11.51844
H11	0	0.784505	0
H12	0	1.000000	0

The two orientations of this model agree on the same set of maximally productive units, H1 and H10. The composites formed from these units can uniformly reduce inputs by a factor of ρ^*_1 i = 1,2,3 or improve outputs uniformly by a factor of γ^*_1 i = 1,2,3 without regard to any factor weights. We note that the maximally productive units are all ratio efficient, as is intuitively expected (technical efficiency of H1 and H10 in Table 4).

RESULTS

In the previous section, the new method was applied to the data of example 1 and data of example 2. In the first example (Table 2) since $\zeta^*_4>0$ this method suggests that for both orientations unit H4 is a maximally productive unit. The minimization orientation also included H1 and H6. The composites formed from these units can uniformly reduce inputs by a factor of 0.9406293 (average value for ρ^*_i, i = 1,2,3) or improve outputs uniformly by a factor of 1.285734 (average value for γ^*_i, i = 1,2,3) without regard to any factor weights. From Table 4 in example 2 we see that the two orientations of this method agree on exactly the same set of maximally productive units, namely units H1 and H10. The composites formed from these units can uniformly reduce inputs by a factor of 0.634799 or improve outputs uniformly by a factor of 1.648491, without regard to any factor weights. Thus these units are signaled as benchmark performers. Managers of these systems of units might consider moving resources from the poorer performing units to those indicated by this method.

DISCUSSION

By regarding the input and output vector pairs, $\{x^j, y^j\}$ as representing transformation processes, where $x_j = x_{ij}$, i = 1,...,m, j = 1,..,n and $y_j = y_{rj}$, r = 1,...s, j = 1,...,n. Such a vector pair suggests that the jth-productive unit was observed to process a bundle of inputs x_j and transformed them to outputs y^j. We assume that resources, the x-vectors, consist only of consumable ones by which we mean that in the process $\{x^j, y^j\}$, all input resources are completely consumed. We have assumed no knowledge about the internal transformation processes of inputs to outputs in any of the productive units. Thus, it cannot be known whether an observed input-output unit can transform some other distinct bundle of inputs, or if it can, what output vector would be produced. We assume only that if it were possible to replicate a productive unit {x, y} by a factor of ζ, then the resulting unit would consume an input vector of exactly ζx and yield an output vector of exactly ζy. A nonnegative optimal intensity ζ^*_j in a composite unit does not require or imply that the jth-unit can or should be physically operated at scale of ζ^*_j times its observed scale of operation.

CONCLUSION

In this study, a novel maximally productive input-output units model have been proposed in order to find efficiency set of best decision-making units. This approach results to reduce inputs and improve outputs units without regard to any factor weights. These models were first applied to Sherman's 15 hospitals dataset and were then applied to 12 hospitals in Tehran, Iran dataset. We believe these results have good face validity in terms of the way excellent hospitals are frequently described.

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