# Journal of Applied Sciences

ISSN 1812-5654

science alert

ANSInet
an open access publisher
http://ansinet.com

Journal of Applied Sciences 9 (4): 789-793, 2009
ISSN 1812-5654


# A New Approach to Find the Optimal Solution for Base Stock Policies

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**Abstract:** In this study, the cost function of the defined system is derived first. Then, we prove this function is convex in the system's base stock level. Finally, based on the convexity of the cost function, the optimal solution for the base stock model is determined. For demonstrating the applicability of the proposed method, we resort to solving an example.

**Key words:** Base stock model, inventory system, continuous review, Poisson demand, backordered demand

## INTRODUCTION

In this study, a continuous review inventory system which is controlled by applying a base stock policy is considered. Demands are assumed to be generated by a stationary and independent Poisson process with one unit demand at a time. Unfilled demand is backordered and the cost for a backorder is proportional to the delay period until delivery takes place. The transportation time is constant.

Stochastic inventory models have received considerable attention in inventory literature. We consider one of the most common practical inventory control problems. This stochastic inventory model is known as the base stock type. A base stock policy places a replenishment order to restore the base-stock S, whenever the inventory position (stock on hand plus stock on order minus backordered) is below S. Hence, the reorder point is S−1 and the policy is commonly referred to as a (S−1, S) policy. Consider a single-item inventory system with Poisson demand, negligible set-up cost and constant lead time.

It is common in models of periodic review systems to assume that demands occur only periodically. Such models can be analyzed by direct analogy to continuous review systems; for example, Axsäter (2006). However, as stated by Rosling (2002), demand arriving in continuous time is more realistic. Schultz (1990) derived the optimal base stock policy with continuous review policy, when unsatisfied demands are backlogged.

Hausman *et al*. (1998) considered a periodic review model in which the interdependency of demanded items is described by the means of a multivariate normal distribution. Item stocks are controlled independently by applying base stock policies. The objectives are to determine (within bounds) the order fill rate (within a specified time window) for a given set of base stock policies and also to maximize the order fill rate subject to a budgetary constraint on the (maximum) allowable investment in stock. The methodology and results are most interesting but the size of problem that can be handled and hence its practical applicability, is limited by the computational complexity in using a multivariate normal distribution.

Johansen (2001) considered a periodic review base stock policy. Because the optimal policy is often rather complicated, he introduced modified base stock policies which are easy to implement. Each modified base stock policy is specified by a pair of S, t where, S is the base stock and t is a lower bound for the number of review periods between review epochs in which placing a replenishment order is permitted.

A widespread approach to inventory modeling is to associate costs with the measures of system performance and determine the control policy which minimizes the long run average cost per unit time. This type of approach ignores the impact of a control policy on the timing of the cash flows associated with payments to suppliers and revenue streams from customers. For this reason, Hilla and Pakkalab (2005) concentrated on cash flows and determined the control policy which maximizes the expected net present value of the cash flows associated with a demand, valued at the time when that demand occurs. They assumed a Poisson demand process, a fixed lead time where unsatisfied demand is backordered and the system is controlled using a base stock policy.

Kranenburg and Houtum (2007) studied the (S−1, S) lost sales inventory model with multiple demand classes. A class-dependent penalty is incurred if a demand is not fulfilled from stock. The penalty represents contractual penalty cost, (additional) cost of emergency transportation and/or loss of goodwill of a customer. The

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total inventory holding and penalty cost were minimized and they distinguished between different classes by introducing critical levels. A demand from a certain class is only fulfilled if the physical stock is above the critical level for this class. Also in literature, there are some studies considered evaluation and optimization within a given class of policies (Dekker *et al*., 2002; Deshpande *et al*., 2003).

Hilla and Pakkalab (2007) considered a retail inventory system in which customer orders arrive at random and each order specifies a list of items. There is a fixed cost associated with the immediate dispatch of items included on an order and in stock and an additional fixed cost for dispatching, after a delay, an item included on the order but not in stock. The stock of each item is controlled using a base stock system and the objective is to find the item base stocks that jointly minimize the total cost of the system. The key to achieving this goal, is to replace the dependent arrival of items on an order with an equivalent independent process. They shown the procedure through a wide range of numerical examples.

Johansen and Thorstenson (2008) have extended well-known formula for the optimal base stock of the inventory system with continuous review and constant lead time to the case with periodic review and stochastic, sequential lead times. Their extension uses the notion of the extended lead time. The extended lead time has been defined as the time between a customer demand instant and the delivery instant of the replenishment order triggered by that demand. By applying this concept, exact performance measures for the base-stock inventory system with continuous review have been extended to a system with periodic review and stochastic, sequential lead times. Simple conditions for optimization have also been derived. The crucial assumption for these results is that lead times are sequential so that orders do not cross in time. They claimed the results obtained can be generalized further to accommodate compound Poisson demand.

We formulate this problem mathematically in order to minimize the total holding and shortage costs, where the system's base stock level is as a decision variable. First, the cost function of the defined system is derived. Then, we show the cost function is convex with respect to the decision variable. Finally, based on the convexity of the cost function, the optimal solution for the base stock model is found. For demonstrating the applicability of the proposed method, an example is solved.

## DESCRIPTION AND PROBLEM FORMULATION

To solve the problem, consider an inventory system that consists of one supplier and one retailer, as shown in Fig. 1. The retailer faces stationary and independent Poisson demand with one unit demand at a time. Unfilled demand is backordered and the shortage cost is considered just for the retailer. The retailer and the supplier carry inventory and replenish their stock according to a one-for-one policy; that is, when a demand occurs, one unit is immediately ordered from the supplier and the supplier orders one unit at the same time. It means essentially that it is assumed that ordering costs are low and can be disregarded. Delayed demands and delayed orders are satisfied on a first come, first served basis. The transportation time from the supplier to the retailer and the replenishment lead time from the outside source to the supplier are deterministic and constant (i.e., the outside source has ample capacity).

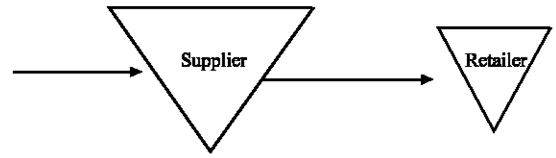


Fig. 1: Inventory system

A one-for-one replenishment policy is completely characterized by determination of the inventory positions at the retailer and at the supplier.

We introduce the following notation:

| | | |
|---|---|---|
| $\bar{S}$ | : | Retailer's base stock level in an inventory system with base stock policy |
| $S_0$ | : | Supplier inventory position in an inventory system with a one-for-one ordering policy |
| S | : | Retailer inventory position in an inventory system with a one-for-one ordering policy |
| L | : | Transportation time from the supplier to the retailer |
| $L_0$ | : | Transportation time from the outside source to the supplier |
| $\lambda$ | : | Demand intensity at both the retailer and the supplier |
| h | : | Holding cost per unit per time unit at the retailer |
| $h_0$ | : | Holding cost per unit per time unit at the supplier |
| $\beta$ | : | Shortage cost per unit per time unit at the retailer |
| $\Pi^S(S_0)$ | : | The expected retailer inventory carrying and shortage cost to fill a unit of demand by applying one-for-one ordering policy when the supplier and the retailer inventory position are $S_0$ and S, respectively |

$\gamma\ (S_0)$ : The average warehouse holding cost per unit by applying one-for-one ordering policy when the supplier inventory positions is $S_0$

$\pi^S\ (L_0)$ : The expected retailer inventory carrying and shortage cost to fill a unit of demand by applying one-for-one ordering policy when the retailer inventory positions is S and $L_0$ is the delay encountered at the supplier level

$c\ (S_0, S)$ : The total holding and shortage costs per time unit by applying one-for-one ordering policy when the supplier and the retailer inventory positions are $S_0$ and S, respectively

$p\ (u, \lambda)$ : Probability mass function of Poisson with mean $\lambda$

$P\ (u, \lambda)$ : Cumulative probability distribution function of Poisson with mean $\lambda$

Considering one-for-one policy, we derive the cost function of the base stock model by setting the inventory position at the supplier to a certain value and then find the optimal solution for this model.

## DERIVING THE COST FUNCTION AND FINDING THE OPTIMAL SOLUTION

In present base stock model, when a demand occurs at the retailer, one unit is immediately ordered from the outside source. We use $(\bar{S}-1,\bar{S})$ notation for this base stock model; that is, when inventory level (minus backorders plus outstanding orders) declines to $\bar{S}-1$, one unit is ordered. This base stock policy is completely characterized by determination of the retailer's base stock level $(\bar{S})$. The ordering cost per time unit in all base stock policies with one unit demand at a time is constant and for this reason it is not considered in the analysis. Here, the optimal value of $\bar{S}$ as a decision variable is determined.

It is easy to conclude that we can disregard policies with negative inventory position for the supplier and the retailer in one-for-one ordering policy when looking for the optimal solution. We therefore confine ourselves to the case where, $S_0\geq 0$ and $S\geq 0$.

In one-for-one ordering policy an order placed by the retailer arrives after $L+\Delta$ time units, where, $\Delta$ is random delay encountered at the supplier level in case the warehouse is out of stock. (Note that $0\leq\Delta\leq L_0$ because we assume that $S_0\geq 0$). Based on Axsäter (1990), the distribution of $\Delta$ for $S_0>0$ is well known and $\Delta$ is equal to $L_0$ where, $S_0 = 0$.

We convert the one-for-one model to the base stock model (Fig. 2). To find the optimal value of $\bar{S}$, present approach uses the optimal retailer inventory position in the inventory system with the one-for-one policy when we set the value of inventory position at the supplier equal to zero.

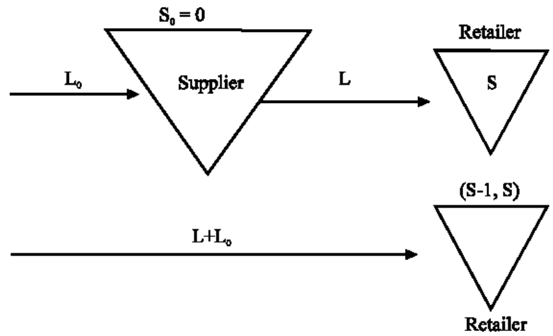


Fig. 2: The equivalent systems based on lemma

**Lemma:** A one-for-one replenishment system with S, $S_0 = 0$, L and $L_0$ is the same as an inventory system with single level with $(S-1, S)$ policy and lead time $L_0+L$; where, the system orders one unit as soon as a demand arrives and the inventory position level drops to $S-1$.

**Proof:** When a demand occurs at the retailer, one unit is immediately ordered from the supplier and the supplier orders one unit at the same time. When demand occurs the supplier is empty, because the inventory position at suppler is zero, the supplier is received the ordered unit after $L_0$ time units (the outstanding orders are assigned before they arrive at the supplier and the supplier must order new one to assign to new order) and it is received by the retailer L time units away. Therefore, an order placed by the retailer arrives after $L_0+L$ time units. □

Based on Lemma, if S* minimizes the cost function of the assumed one-for-one replenishment system, S* minimizes the cost function of the base stock system. As mentioned before, policies with negative inventory position for the retailer by applying one-for-one policy can be disregard. We therefore confine ourselves to the case where, $S\geq 0$ for finding the optimal base stock level. To evaluate the total holding and shortage costs per time unit when applying one-for-one ordering policies (i.e., $c\ (S_0, S)$), we use the method introduced by Axsäter (1990) (Appendix).

The total cost of the assumed one-for-one replenishment system can be calculated by Eq. 20:

$$c(0,S) = \lambda(\Pi^S(0) + \gamma(0)) \qquad (1)$$

Using Eq. 12 and 15, it is easy to simplify expression Eq. 1 as:

$$c(0,S) = \lambda\pi^S(L_0) \qquad (2)$$

It means that if S* minimizes $\pi^S\ (L_0)$, it also minimizes c (0, S). Based on Eq. 13, it can be easily verified that:

$$\pi^1(L_0) - \pi^0(L_0) = e^{-\lambda(L+L_0)}\frac{h+\beta}{\lambda} - \frac{\beta}{\lambda} \tag{3}$$

For S>0 and based on Eq. 13, we have:

$$\pi^{S+1}(L_0) - \pi^S(L_0) = e^{-\lambda(L+L_0)}\frac{h+\beta}{\lambda}\sum_{k=0}^{S}\frac{(s+1-k)}{k!}(L+L_0)^k\lambda^k - e^{-\lambda(L+L_0)} \frac{h+\beta}{\lambda}\sum_{k=0}^{S-1}\frac{(s-k)}{k!}(L+L_0)^k\lambda^k - \frac{\beta}{\lambda} = e^{-\lambda(L+L_0)}\frac{h+\beta}{\lambda}\frac{(L+L_0)^S\lambda^S}{S!} + e^{-\lambda(L+L_0)} \frac{h+\beta}{\lambda}\sum_{k=0}^{S-1}\frac{(s+1-k-(s-k))}{k!}(L+L_0)^k\lambda^k - \frac{\beta}{\lambda} \tag{4}$$

Now, we reformulate some part of expression (4) as:

$$e^{-\lambda(L+L_0)}\frac{h+\beta}{\lambda}\sum_{k=0}^{S-1}\frac{(s+1-k-(s-k))}{k!}(L+L_0)^k\lambda^k = e^{-\lambda(L+L_0)}\frac{h+\beta}{\lambda} \sum_{k=0}^{S-1}\frac{(s-k-(s-1-k))}{k!}(L+L_0)^k\lambda^k = e^{-\lambda(L+L_0)}\frac{h+\beta}{\lambda}\sum_{k=0}^{S-1}\frac{(s-k)}{k!} (L+L_0)^k\lambda^k - e^{-\lambda(L+L_0)}\frac{h+\beta}{\lambda}\sum_{k=0}^{S-1}\frac{(s-1-k)}{k!}(L+L_0)^k\lambda^k = e^{-\lambda(L+L_0)}\frac{h+\beta}{\lambda} \sum_{k=0}^{S-1}\frac{(s-k)}{k!}(L+L_0)^k\lambda^k - e^{-\lambda(L+L_0)}\frac{h+\beta}{\lambda}\sum_{k=0}^{S-2}\frac{(s-1-k)}{k!}(L+L_0)^k\lambda^k \tag{5}$$

Replacing Eq. 5 in Eq. 4 and based on Eq. 13, for S>0, it is possible to show that:

$$\pi^{S+1}(L_0) - \pi^S(L_0) = \pi^S(L_0) - \pi^{S-1}(L_0) + e^{-\lambda(L+L_0)} \frac{h+\beta}{\lambda}\frac{(L+L_0)^S\lambda^S}{S!} \Rightarrow (\pi^{S+1}(L_0) - \pi^S(L_0)) -(\pi^S(L_0) - \pi^{S-1}(L_0)) = e^{-\lambda(L+L_0)}\frac{h+\beta}{\lambda}\frac{(L+L_0)^S\lambda^S}{S!} \geq 0 \tag{6}$$

Therefore, $\pi^S(L_0)$ is convex in S and now, using Eq. 3 and 6, we get:

$$\pi^{S+1}(L_0) - \pi^S(L_0) = \pi^S(L_0) - \pi^{S-1}(L_0) + e^{-\lambda(L+L_0)}\frac{h+\beta}{\lambda} \frac{(L+L_0)^S\lambda^S}{S!} = \pi^{S-1}(L_0) - \pi^{S-2}(L_0) + e^{-\lambda(L+L_0)}\frac{h+\beta}{\lambda}\sum_{k=S-1}^{S}\frac{(L+L_0)^k\lambda^k}{k!} = \ldots = \pi^1(L_0) - \pi^0(L_0) + e^{-\lambda(L+L_0)}\frac{h+\beta}{\lambda}\sum_{k=1}^{S}\frac{(L+L_0)^k\lambda^k}{k!} = e^{-\lambda(L+L_0)}\frac{h+\beta}{\lambda} \sum_{k=0}^{S}\frac{(L+L_0)^k\lambda^k}{k!} - \frac{\beta}{\lambda} = \frac{h+\beta}{\lambda}P(S,\lambda(L+L_0)) - \frac{\beta}{\lambda} \tag{7}$$

Based on convexity specifications and Eq. 7, the optimal value of S that minimizes $\pi^S(L_0)$ is the minimal value of S that is valid for:

$$\pi^{S+1}(L_0) - \pi^S(L_0) \geq 0 \rightarrow P(S,\lambda(L+L_0)) \geq \frac{\beta}{\beta+h} \tag{8}$$

Therefore, the optimal value of $\bar{S}$ in the base stock system with lead time $L+L_0$ will be S* where, S* minimizes the cost function of the assumed one-for-one replenishment system.

**Example:** Here, we show the applicability of the proposed method by solving a test problem. We assume the value of the parameters L, $L_0$, h, $h_0$, β and λ are constant and are as: L = 1, $L_0$ = 1, h = 1, $h_0$ = 1, β = 10 and λ = 1

The optimal value of $\bar{S}$ with lead time $L+L_0$ will be S* where, S* is the minimal value of S that is valid for expression Eq. 8 i.e.,:

$$P(3,1\times(1+1)) \approx 0.857 \leq \frac{10}{10+1}$$
$$P(4,1\times(1+1)) \approx 0.947 \geq \frac{10}{10+1} \rightarrow S^* = 4 \rightarrow \bar{S}^* = 4$$
$$P(5,1\times(1+1)) \approx 0.983 \geq \frac{10}{10+1}$$

It should be noted; to calculate the optimal policy of the defined base stock system, we do not need any expression from Axsäter (1990).

## CONCLUSION

We examined the continuous review inventory system that is controlled by applying a base stock policy. To derive the cost function of the system, a one-for-one model is converted to this base stock model. To find the optimal value of retailer's base stock level, present approach uses the optimal retailer inventory position in the inventory system with one-for-one policy, where, the value of inventory position at the supplier is set to 0.

The earlier approaches focus on characterizing the steady-state behavior of the inventory levels of a stockage policy and then use the steady state distribution (or an approximation thereof) to determine the average costs associated with stockage policy. Present approach uses an inventory cost function that reflects costs incurred on an average unit. For this reason, the proposed approach can be used when the cost is given by a nonlinear function of either the delays experienced by the customer, or the unit's storage time at the retailer.

## APPENDIX: EVALUATION OF THE ONE-FOR-ONE ORDERING POLICIES

This Appendix is a summary of Axsäter (1990). Consider $S_0$ and S as the supplier and the retailer inventory positions, respectively. We define (as in Axsäter's studies for one retailer) the following notations:

$g^{S_0}(t) =$ Density function of the Erlang ($\lambda$, $S_0$) distribution of the time that the warehouse order a unit to it is demanded by the retailer, for the one-for-one corresponding system with Poisson demand and $G^{S_0}(t) =$ Cumulative distribution function of $g^{S_0}$.
thus,

$$g^{S_0}(t) = \frac{\lambda^{S_0} t^{S_0 - 1}}{(S_0 - 1)!} e^{-\lambda t} \quad (9)$$

and

$$G^{S_0}(t) = \sum_{k=S_0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \quad (10)$$

The average warehouse holding cost per unit is:

$$\gamma(S_0) = \frac{h_0 S_0}{\lambda}(1 - G^{S_0+1}(L_0)) - h_0 L_0 (1 - G^{S_0}(L_0)),\ S_0 > 0 \quad (11)$$

and for $S_0 = 0$,

$$\gamma(0) = 0 \quad (12)$$

Given the value of random delay at the warehouse is equal to t, the conditional expected cost per unit at the retailer is:

$$\pi^S(t) = e^{-\lambda(L+t)} \frac{h+\beta}{\lambda} \sum_{k=0}^{S-1} \frac{(s-k)}{k!}(L+t)^k \lambda^k + \beta(L + t - \frac{S}{\lambda}) \quad (13)$$

(0! =1 by definition),

The expected retailer inventory carrying and shortage cost which incurred to fill a unit of demand at the retailer is:

$$\Pi^S(S_0) = \int_0^{L_0} g_0^{S_0}(L_0 - t)\pi^S(t)dt + (1 - G^{S_0}(L_0))\pi^S(0) \quad (14)$$

and

$$\Pi^S(0) = \pi^S(L_0) \quad (15)$$

Furthermore, for large value of $S_0$, we have:

$$\Pi^S(S_0) \approx \pi^S(0) \quad (16)$$

The procedure starts by determining $\overline{S_0}$ such that:

$$G^{\overline{S_0}}(L_0) < \varepsilon \quad (17)$$

where, $\varepsilon$ is a small positive number.

The recursive computational procedure is:

$$\Pi^S(S_0 - 1) = \Pi^{S-1}(S_0) + (1 - G^{S_0}(L_0)) \times (\pi^S(0) - \pi^{S-1}(0)) \quad (18)$$

$$\Pi^0(S_0) = G^{S_0}(L_0)\beta L_0 - G^{S_0+1}(L_0)\beta \frac{S_0}{\lambda} + \beta L \quad (19)$$

Sum of the expected total holding and shortage costs per time unit in an inventory system with a one-for-one ordering policy is:

$$c(S_0, S) = \lambda(\Pi^S(S_0) + \gamma(S_0)) \quad (20)$$

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