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On the Cauchy-Goursat Theorem

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Abstract: In this study, we have presented a simple and un-conventional proof of a basic but important Cauchy-Goursat theorem of complex integral calculus. The pivotal idea is to sub-divide the region bounded by the simple closed curve by infinitely large number of different simple homotopically closed curves between two fixed points on the boundary. Beauty of the method is that one can easily see the significant roll of singularities and analyticity requirements. We suspect that our approach can be utilized to derive simpler proof for Green's theorem, Stoke's theorem, generalization to Gauss's divergence theorem, extension of Cauchy-Goursat theorem to multiply connected regions, critical study of the affects of singularities over a general field with a general domain and a simpler approach for complex integration such as Cauchy integral formula, residue theorem etc. Avoiding topological and rigor mathematical requirements, we have sub-divided the region bounded by the simple closed curve by a large number of different simple closed curves between two fixed points on the boundary and have introduced: $I_i = \int_{p,c_i}^{q} f(z)dz$ where path of integration is from p to q along c_i and for I = 0, $-I_i = \int_{q,c_i}^{p} f(z)dz$ 1, 2, ..., n. Line integral along the boundary of the domain was evaluated via $(\partial f/\partial z)$. Commutation between integration, δ-operation and d-operation was established. Using the vector interpretation of complex number, the area ds of a small parallelogram was established as $|\vec{dz} \times \vec{\delta z}| = (dx\ \delta y - dy\ \delta x) = ds$. Finally, using Cauchy-Riemann equations we have established the well celebrated Cauchy-Goursat theorem, i.e., if a function f(z) is analytic inside and on a simple closed curve c then $\oint_c f(z)dz = 0$.

Key words: Cauchy-Goursat theorem, analytic function, simple closed curve, singular points, Cauchy-Riemann equations

INTRODUCTION

Complex variables open everything what is hidden in the real calculus. Complex integration is central in the study of complex variables. As in calculus, the fundamental theorem of calculus is significant because it relates integration with differentiation and at the same time provides method of evaluating integral so is the complex analog to develop integration along arcs and contours is complex integration. Complex integration is elegant, powerful and a useful tool for mathematicians, physicists and engineers. Cauchy-Goursat theorem is a fundamental, well celebrated theorem of the complex integral calculus. This theorem is not only a pivotal result in complex integral calculus but is frequently applied in quantum mechanics, electrical engineering, conformal mappings, method of stationary phase, mathematical physics and many other areas of mathematical sciences and engineering. It provides a convenient tool for evaluation of a wide variety of complex integration. It also forms the cornerstone of the development of results f'(z) of an analytic function f(z) is analytic, Cauchy's integral formula and many advance topics in complex integration.

Due to its pivotal role and importance, mathematicians have discussed it in all respects (Gario, 1981; Gurtin and Martins, 1976; Mibu, 1959; Segev and Rodnay, 1999).

Historically, it was firstly established by Cauchy in 1789-1857 and Churchill and James (2003) and later on extended by Goursat in 1858-1936 and Churchill and James (2003) without assuming the continuity of f' (z). Consequently, it has laid down the deeper foundations for Cauchy- Riemann theory of complex variables. Its usual proofs involved many topological concepts related to paths of integration; consequently, the reader especially

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the undergraduate students can not be expected to understand and acquire a proof and enjoy the beauty and simplicity of it. Hence, it will not be unusual to motivate Cauchy-Goursat theorem by a simple version (Gurin, 1981; Long, 1989; Tucsnak, 1984).

In this study, we have adopted a simple non-conventional approach, ignoring some of the strict and rigor mathematical requirements. Knowledge of calculus will be sufficient for understanding. For further reading, reader can choose any standard book on complex variables and/or calculus (Churchill and James, 2003; Mathews and Russell, 2006). The pivotal idea is to sub-divide the region bounded by the simple closed curve by infinitely large number of different simple homotopically closed curves between two fixed points on the boundary. Beauty of the method is that one can easily see the significant roll of singularities and analyticity requirements. We suspect that our approach will be useful without any difficulty to derive simpler proof for Green's theorem, Stoke's theorem, generalization to Gauss's divergence theorem, extension of Cauchy-Goursat theorem to multiply connected regions, critical study of the affects of singularities over a general field with a general domain and a simpler approach for complex integration such as Cauchy integral formula, residue theorem etc.

RESULTS AND DISCUSSION

Statement of Cauchy-goursat theorem: If a function f(z) is analytic inside and on a simple closed curve c then $\oint_c f(z)dz = 0$.

Proof: Let f(z) = f(x+iy) = f(x, y) = u(x, y)+iv(x, y) be analytic inside and on a simple closed curve c. Need to prove that $\oint_c f(z)dz = 0$.

Consider the region R enclosed by simple closed curve C as $R = \{(x,y)|p \le x \le q \text{ and } g1(x) \le y \le g2(x)\}$. For the sake of proof, assume C is oriented counter clockwise. Let p and q be two fixed points on C (Fig. 1). Subdivide the region enclosed by C, by a large number of paths c_0, c_1, c_2,..., c_n passing through the points p and q with $c_0 = \{(x, y)|p \le x \le q, y = g_1(x)\}$ and $c_0 = \{(x, y)|p \le x \le q, y = g_2(x)\}$ constituting the boundary of C (Fig. 1).

Define

$$I_i = \int_{p,c_i}^{q} f(z)dz \quad \text{where path of integration is from p to q along } c_i$$

and

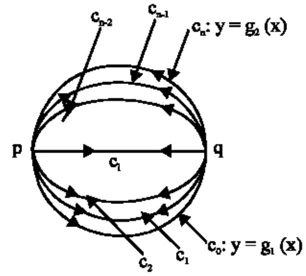

Fig. 1: Region R enclosed by simple closed curve C:

$$-I_i = \int_{q,c_i}^{p} f(z)dz \quad \text{for } i = 0, 1, 2, ---, n$$

Now:

$$\oint_c f(z)dz = \int_{p,c_0}^{q} f(z)dz + \int_{q,c_n}^{p} f(z)dz = I_0 - I_n =$$
$$(I_0 - I_1) + (I_1 - I_2) + (I_2 - I_3) + ---- + (I_{n-1} - I_n) =$$
$$\sum_{i=0}^{n-1} \delta I_i \quad \text{where } \delta I_i = I_i - I_{i+1}$$

Note that for large n, $\delta I_i = I_i - I_{i+1}$ is the small variation in the values of the integrals along two adjacent homotopically close lying paths. Since, f(z) is analytic so we can enjoy commutation between integration and δ-operation as follow:

$$\delta I_i = \delta \int_{p,c_i}^{q} f(z)dz = \int_{p,c_i}^{q} \delta(f\,dz) = \int_{p,c_i}^{q} (\delta f)\,dz + \int_{p,c_i}^{q} f(z)\,\delta\,d\,z \quad (1)$$

From physical interpretation, δ-operation and d-operation on z are commutative, i.e.,

$$\delta\,d\,z = d\,\delta\,z$$

Consequently, integrating by parts the 2nd integral of Eq. 1 and considering the fact that there is no variation in z at the fixed points p and q, Eq. 1 reduces as:

$$\delta I_i = \int_{p,c_i}^{q} (\delta f\,d\,z - d\,f\delta\,z) \quad (2)$$

Hence,

$$\oint_c f(z)dz = \sum_{i=0}^{n-1} \delta I_i = \sum_{i=0}^{\infty} \int_{p,c_i}^{q} (\delta f\,d\,z - d\,f\delta\,z) \quad (3)$$

Now considering the function ds as a function of complex conjugate coordinates, i.e.,

$F(z) = f(z, \bar{z}) \Rightarrow$ Eq. 3 changes as:

$$\oint_C f(z)dz = \sum_{i=0}^{\infty} \int_{p,c_i}^{q} (\delta f dz - df \delta z) = \sum_{i=0}^{\infty} \int_{p,c_i}^{q} \frac{\partial f}{\partial \bar{z}} (dz \delta \bar{z} - \delta z \, d\bar{z}) =$$

$$\sum_{i=0}^{\infty} \int_{p,c_i}^{q} \frac{\partial f}{\partial \bar{z}} [(dx + idy)(\delta x - i\delta y) - (\delta x + i\delta y)(dx - idy)] = \quad (4)$$

$$-2i \sum_{i=0}^{\infty} \int_{p,c_i}^{q} \frac{\partial f}{\partial \bar{z}} (dx\delta y - dy\delta x)$$

Now, using the vector interpretation of complex number, the area ds of a small parallelogram is given by $|\overrightarrow{dz} \times \overrightarrow{\delta z}| = (dx\, \delta y - dy\, \delta x) = ds$ Consequently, Eq. 4 reduces as:

$$\oint_C f(z)dz = -2i \sum_{i=0}^{\infty} \int_{p,c_i}^{q} \frac{\partial f}{\partial \bar{z}} (dx\delta y - dy\delta x) = -2i \sum_{i=0}^{\infty} \int_{p,c_i}^{q} \frac{\partial f}{\partial \bar{z}} ds =$$

$$\sum_{i=0}^{\infty} \int_{p,c_i}^{q} [(\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}) + i(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y})] ds \underset{\text{(Cauchy–Riemann equations)}}{=} 0$$

This completes the proof

CONCLUSION

Cauchy-Goursat theorem is the basic pivotal theorem of the complex integral calculus. The present proof avoids most of the topological as well as strict and rigor mathematical requirements. Instead, standard calculus results are used. Line integral of f(z) around the boundary of the domain, e.g., along C has been evaluated via (∂f/∂z). In general (∂f/∂z) may not be zero but analyticity of f(z) 0plays a pivotal roll for (∂f/∂z). It is also interesting to note the affect of singularities in the process of sub-division of the region and line integrals along the boundary of the regions. I suspect this approach can be considered over any general field with any general domain.

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