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New Method for Finding an Optimal Solution to Quadratic Programming Problems

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Abstract: The aim of this study is to present a new method for finding an optimal solution to quadratic programming problems. The principle of the method is based on calculating the value of critical point. If the critical point belongs to the set of feasible solutions, so the optimal solution to our problem is the critical point itself. If the critical point is not at in the feasible solution set, a new feasible constraint set is built by a homographic transform, in such a way that the projection of the critical point of the objective function onto this set produces the exact solution to the problem on hand. It should be noted here that the objective function may be convex or not convex. On the other hand the search for the optimal solution is to find the hyper plane separating the convex and the critical point. Notice that one does not need to transform the quadratic problem into an equivalent linear one as in the numerical methods; the method is purely analytical and avoids the usage of initial solution. An algorithm computing the optimal solution of the concave function has given.

Key words: Optimization, extreme, interpolation, convex, hyper plane separator

INTRODUCTION

The methods for solving numerical optimization problems are nonlinear in essence iterative routines: a routine cannot find the exact solution in finite time. In fact the method, as treated by Delbos and Gilbert (2005), generates an infinite sequence (x_t) of approximate solutions. The next iteration x_{t+1} is formed according to certain rules, based on local information of the problem, collected on the previous iteration which is a vector consisting of values of the objective, constraints and possibly the gradient or even higher derivatives of these functions. The optimization methods can be classified not only according to the types of problems they solve, but by type of local information they employ. From this point of view of information, the methods are divided into:

- Routines zero-order, using only the values of the objective and constraints but not their derivatives
- Routines-order, using the values and gradients of the objective and constraints
- Routines second order, using the values, gradients and Hessians of the objective and constraints

Optimizing methods of quadratic functions such as the ones presented by Simonnard (1973) transform the initial quadratic problem into an equivalent linear program in order and apply the simplex. Achmanov (1984) only gave approximate solutions by building convergent series which approaches the real solution. De Werra *et al.* (2003) studied feasible direction method with the difficult problem to choose the direction and the parameter. Hillier and Lieberman (2005) transformed the initial problem into a linear one but by adding an important number of constraints.

Bierlaire (2006) has treated a simple example needs at least ten variables. In the method presented here, without any introduction of any variable, the exact solution is reached in the first attempt.

This analytical technique determines an exact optimal solution to the problem. It has to be stressed here that the new method is general to any quadratic optimization problem. The principle of the technique is to project the critical point to a new convex built from the set of feasible solutions. the critical point x^* is transformed to the point y^*; then y^* is projected on one hyper plan separator

An algorithm for finding an optimal solution is given in this study.

SPLITTING THE OBJECTIVE FUNCTION

Consider the following problem:

$$(P) = \begin{cases} f(x) = \sum_{i=1}^{n} \alpha_i x_i + \beta_i x_i^2 \\ \underset{x \in \Omega}{\text{Max}}\, f(x) \end{cases} \tag{1}$$

where, $\Omega = \{x \in R^n : x \geq 0 \text{ and } Ax \leq b\}$, with a $m \times n$ matrix A and vector b of R_+^m. The coefficients α_i and β_i are any real numbers, so that the function f(x) is neither concave nor convex.

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By so doing a decomposition for the function f(x) holds as: f(x) = φ(x)+Ψ(x).

$$\varphi(x)=\sum_{i=1}^{n}\alpha_i x_i+\beta_i x_i^2$$

where, the summation runs over all the indices i for $\beta_i<0$.

and

$$\psi(x)=\sum_{i=1}^{n}\alpha_i x_i+\beta_i x_i^2$$

where, the sum is extended for the indices i for $\beta_i\geq 0$.

In the present study, we study the concave function φ(x). We give, here, the exact optimal solution by projecting the critical point $x^*=\left(\frac{-\alpha_i}{2\beta_i}\right)_i$ onto a new convex Ω'. If the critical point x* is inside the convex Ω, then the optimal solution holds exactly at this point.

We show in theorem 1, that $\operatorname{Max}_{x\in\Omega}\varphi(x)=\varphi(\bar{x})$, where $\bar{x}$ is the projection of the point x* on the convex Ω'.

Let T: $\Omega\subset R^n\to\Omega'\subset R^n$ be the application that transforms the convex Ω into the convex Ω'. $T(x)=\Lambda x$, $\Lambda=\left(\sqrt{-\beta_1},........\sqrt{-\beta_n}\right)$ is non zero, because the $\beta_i<0$, for all i. Then it is conform.

OPTIMIZATION OF THE CONCAVE FUNCTION φ

Let Ω be a closed, bounded convex set of R^n. Let's put:

$$\varphi(x)=\sum_{i=1}^{n}\alpha_i x_i+\beta_i x_i^2. \qquad \alpha_i\in R,\quad \beta_i<0 \quad \text{for all i}$$

$$x^*=(x_i^*)_i=\left(\frac{-\alpha_i}{2\beta_i}\right)_i; \qquad y^*=(y_i^*)_i=\left(\frac{\alpha_i}{2\sqrt{-\beta_i}}\right)_i$$

$$y=(y_i)_i=\left(\sqrt{-\beta_i}x_i\right)_i \qquad \operatorname{Max}_{x\in\Omega}\varphi(x)=\varphi(\bar{x});\ \bar{x}=(\bar{x}_i)_i\in\Omega$$

Theorem: There exists a closed bounded convex set Ω' of R^n and a vector $y_0 = (y_{0i})_i$, such that the following statements are satisfied:

$$\operatorname{Max}_{x\in\Omega}\varphi(x)=\varphi(x^*)-\left\|y^*-y_0\right\|^2 \qquad \text{(Property 1)}$$

$$\left\|y^*-y_0\right\|=\inf_{y\in\Omega'}\left\|y^*-y\right\| \qquad \text{(Property 2)}$$

$$\bar{x}_i=\frac{y_{0i}}{\sqrt{-\beta_i}}, \text{ for each i, i = 1, 2,…..,n.} \qquad \text{(Property 3)}$$

Proof: For every x∈Ω let ,

$$\Delta\varphi_i=\left(\alpha_i x_i^*+\beta_i x_i^{*2}\right)-\left(\alpha_i x_i+\beta_i x_i^2\right)$$

Then

$$\Delta\varphi_i=\alpha_i\left(\frac{-\alpha_i}{2\beta_i}\right)+\beta_i\left(\frac{-\alpha_i}{2\beta_i}\right)^2-\alpha_i x_i-\beta_i x_i^2$$
$$=\beta_i\left(x_i+\frac{\alpha_i}{2\beta_i}\right)^2$$
$$=\beta_i\left(x_i-x_i^*\right)^2$$

But

$$\varphi(x^*)-\varphi(x)=\sum_{i=1}^{n}\Delta\varphi_i$$

$$\varphi(x^*)-\varphi(x)=\sum_{i=1}^{n}-\beta_i\left(x_i-x_i^*\right)^2; \qquad \text{For all } x\in\Omega$$

Therefore,

$$\inf_{x\in\Omega}\left(\varphi(x^*)-\varphi(x)\right)=\inf_{x\in\Omega}\left(\sum_{i=1}^{n}-\beta_i\left(x_i-x_i^*\right)^2\right)$$

can be written in the following form:

$$\varphi(x^*)-\operatorname{Max}_{x\in\Omega}\varphi(x)=\inf_{x\in\Omega}\sum_{i=1}^{n}\left(\sqrt{-\beta_i}\left(x_i^*-x_i\right)\right)^2=\inf_{y\in\Omega'}\sum_{i=1}^{n}\left(y_i^*-y_i\right)^2$$

Let $\Omega'=\left\{y=(y_i)_i\in R^n: y_i=\sqrt{-\beta_i}x_i, x=(x_i)_i\in\Omega\right\}$, because $\inf_{y\in\Omega'}\sum_{i=1}^{n}\left(y_i^*-y_i\right)^2=\left\|y^*-y_0\right\|^2, y_0\in\Omega'$.

Thus $\operatorname{Max}_{x\in\Omega}\varphi(x)=\varphi(x^*)-\left\|y^*-y_0\right\|^2$, hence property 1.

Because $\inf_{y\in\Omega'}\left\|y^*-y\right\|^2=\left\|y^*-y_0\right\|^2$, then $\left\|y^*-y_0\right\|^2\leq\left\|y^*-y\right\|^2$, for every y∈Ω'.

This implies that $\left\|y^*-y_0\right\|\leq\left\|y^*-y\right\|$, for every y∈Ω'.

We have therefore $\left\|y^*-y_0\right\|\leq\inf_{y\in\Omega'}\left\|y^*-y\right\|$.

Because y∈Ω' then $\left\|y^*-y_0\right\|\geq\inf_{y\in\Omega'}\left\|y^*-y\right\|$; then $\left\|y^*-y_0\right\|=\inf_{y\in\Omega'}\left\|y^*-y\right\|$. Property 2 has demonstrated.

The vector y_0 is the projection of the vector y* onto the new convex Ω'.

We have,

$$\operatorname{Max}_{x\in\Omega}\varphi(x)=\varphi(\bar{x})=\varphi(x^*)-\left\|y^*-y_0\right\|^2$$
$$=\varphi(x^*)-\inf_{x\in\Omega}\sum_{i=1}^{n}\left(y_i^*-\sqrt{-\beta_i}x_i\right)^2$$
$$=\varphi(x^*)-\sum_{i=1}^{n}\left(y_i^*-\sqrt{-\beta_i}\bar{x}_i\right)^2$$

Hence property 3.

Remark 1: If x*∈Ω, then $\inf_{x\in\Omega}\sum_i-\beta_i\left(x_i^*-x_i\right)^2=0$ and $\operatorname{Max}_{x\in\Omega}\varphi(x)=\varphi(x^*)$.

Remark 2: The transformation T: $\Omega \subset R^n \rightarrow \Omega' \subset R^n$, for each $x \in \Omega$, associating $T(x) = \Lambda x$, $\Lambda = \left(\sqrt{-\beta_1}, \ldots\ldots, \sqrt{-\beta_n}\right)$ has as Jacobean matrix.
Its determinant is $\prod_{i=1}^{n} \sqrt{-\beta_i} \neq 0$. Then it is conform.

Remark 3: The convex Ω is bounded. We can restrict ourselves to the case $\alpha_i \geq 0$ for each i. In fact, let's assume that we have $\alpha_i x_i + \beta_i x_i^2$ with $\alpha_i < 0$.

Let $\mathrm{Max}(x_i) = \delta_i^*$, $y_i = \delta_i^* - x_i \geq 0$. Therefore,

$$\begin{aligned} \alpha_i x_i + \beta_i x_i^2 &= \alpha_i x_i + \beta_i x_i^2 - \alpha_i \delta_i^* + \alpha_i \delta_i^* \\ &= \beta_i y_i^2 + \left(-\alpha_i - 2\beta_i \delta_i^*\right) y_i + \alpha_i \delta_i^* + \beta_i \delta_i^{*2} \\ &= \beta_i y_i^2 + \alpha_i' y_i + K_i \end{aligned}$$

where $\alpha_i' = -\alpha_i - 2\delta_i^* \beta_i \geq 0$ and $K_i = \alpha_i \delta_i^* + \beta_i \delta_i^{*2}$.

Therefore, it is sufficient to replace x_i with $\delta_i^* - y_i$ and $\alpha_i x_i + \beta_i x_i^2$ with $\beta_i y_i^2 + \alpha_i' y_i + K_i$.

Algorithm of computing the optimal solution of the concave function φ.

Algorithm Ahmed Chikhaoui:

Inputting data matrix A, vectors b, c, alpha and beta.
If all β_i = -1 **then** Ω' = Ω else build Ω' and compute the critical point $x^* = \left(\frac{\alpha_i}{-2\beta_i}\right)_i$ for all $i = 1, 2, \ldots$

If $x^* \in \Omega$ then x^* is the Optimal solution; compute $\varphi(x^*)$. **STOP**.
else // calculating coordinates of y^* and its projection y_0
begin
choose the supporting hyper plane separator (i.e. $ax^* > b$)
for i= 1 to n
begin

$$y^* = (y_i^*)_i = \left(\frac{\alpha_i}{2\sqrt{-\beta_i}}\right)_i$$

$$y_0 = P_\Omega\left(y^*\right) = y^* - \frac{< y^*, a > - b}{\|a\|^2} a$$

$$\bar{x}_i = \frac{y_{0i}}{\sqrt{-\beta_i}}$$

end
If $\bar{x} \in \Omega$ then $\bar{x}$ is the optimal solution; compute $\varphi(\bar{x})$ STOP.
else
Change the supporting hyper plane separator
end

In the algorithm we use the supporting hyperplane. Figure 1 shows a separator supporting hyperplane H_1.

The following example traits the case $x^* \in \Omega$. Hillier and Lieberman (2005).

Example 1:

$$\begin{cases} \max \varphi(x_1, x_2) = 54x_1 + 78x_2 - 9x_1^2 - 13x_2^2 \\ 3x_1 + 2x_2 \leq 18 \\ \quad\quad 2x_2 \leq 12 \\ x_1 \quad\quad \leq 4 \\ x_1, x_2 \geq 0 \end{cases}$$

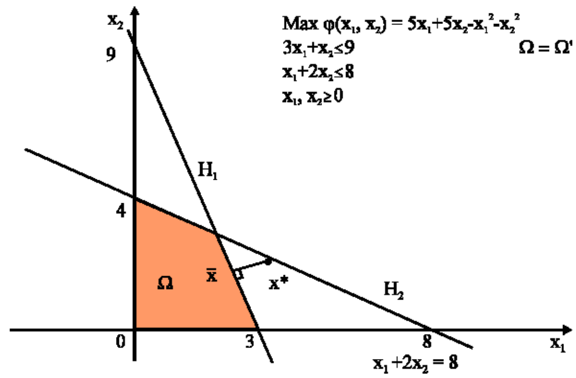

Fig. 1: Supporting hyperplanes H_1 and H_2. The supporting hyperplane H2 does not separate x* of Ω while H1 is separate supporting hyperplane containing $\bar{x}$

In this case $\beta_1 = -9$, then $\Omega <> \Omega'$.

$$\frac{\partial f}{\partial x_1}(x_1, x_2) = 54 - 18x_1 = 0 \Rightarrow x_1^* = \frac{54}{18} = 3;$$

$$\frac{\partial f}{\partial x_2}(x_1, x_2) = 78 - 26x_2 = 0 \Rightarrow x_2^* = \frac{78}{26} = 3$$

The critical point x^* = (3, 3) $\in \Omega$ $\underset{(x_1,x_2)\in\Omega}{\mathrm{Max}} \varphi(x_1, x_2) = \varphi\left(x^*\right) = 198$. Note the rapid method.

The following example traits the case Ω = Ω'. Example treats by Dozzi (2004).

Example 2:

$$\begin{cases} \max \varphi(x_1, x_2) = 5x_1 + 5x_2 - x_1^2 - x_2^2 \\ 3x_1 + x_2 - 9 \leq 0 \\ x_1 + 2x_2 - 8 \leq 0 \\ x_1, x_2 \geq 0 \end{cases}$$

In this case $\beta_1 = \beta_2 = -1$, then $\Omega <> \Omega'$.

$$\frac{\partial f}{\partial x_1}(x_1, x_2) = 5 - 2x_1 = 0 \Rightarrow x_1^* = 5/2$$

$$\frac{\partial f}{\partial x_2}(x_1, x_2) = 5 - 2x_2 = 0 \Rightarrow x_2^* = 5/2$$

The critical point x^* = (5/2, 5/2) $\notin \Omega$ because $3\frac{5}{2} + \frac{5}{2} - 9 = 1 > 0$, we compute the projection of x^*

$$\bar{x} = P_\Omega\left(x^*\right) = x^* - \frac{< x^*, a > - b}{\|a\|^2} a$$

where, a = (3, 1), b = 9

$$< x^*, a >= \frac{5}{2}*3+\frac{5}{2}*1 = 10,\ \|a\|^2 = 10.\ \bar{x} = \left(\frac{5}{2},\frac{5}{2}\right) - \frac{1}{10}(3,1) = (2.2,\ 2.4).$$

$$\underset{(x_1,x_2)\in\Omega}{\text{Max}}\ \varphi(x_1,x_2) = \varphi(\bar{x}) = 12.40$$

Here,

$$\Omega = \left\{(x_1,x_2)\in R^2 / 3x_1 + x_2 - 9 \le 0, x_1 + 2x_2 - 8 \le 0, x_1 \ge 0, x_2 \ge 0\right\}$$

The following example traits the case $\Omega <> \Omega'$.

Hillier and Lieberman (2005) has treated the following example. We can compare the two methods.

Example 3:

$$\begin{cases} \max \varphi(x_1,x_2) = 5x_1 + 8x_2 - x_1^2 - 2x_2^2, \\ 3x_1 + 2x_2 \le 6 \\ x_1, x_2 \ge 0 \end{cases}$$

in this case $\beta_1 = -1$ and $\beta = -2$, then $\Omega \ne \Omega'$.

$$\frac{\partial f}{\partial x_1}(x_1,x_2) = 5 - 2x_1 = 0 \Rightarrow x_1^* = 5/2$$

$$\frac{\partial f}{\partial x_2}(x_1,x_2) = 8 - 4x_2 = 0 \Rightarrow x_2^* = 2$$

The critical point $x^* = \left(\frac{5}{2},2\right) \notin \Omega,$ because $3\frac{5}{2} + 2.2 > 6$,

The critical point $x^* = (5/2,\ 2) \notin \Omega$ because $3\frac{5}{2} + 2.2 > 6,$ we compute the projection of y* (y* is the translation of x*).

$$\bar{x} = P_\Omega(x^*) = x^* - \frac{< x^*, a > - b}{\|a\|^2} a$$

where, a = (3, 2), b = 6.

$$< x^*, a >= \frac{5}{2}*3 + 2*2 = \frac{23}{2}, \|a\|^2 = 13.$$

$$\bar{x} = \left(\frac{5}{2}, 2\right) - \frac{11}{2*13}(3,2) = \left(\frac{5}{2} - \frac{33}{26},\ 2 - \frac{11}{13}\right) = \left(\frac{16}{13}, \frac{15}{13}\right)$$

$$\Omega' = \left\{(y_1,y_2)\in R^2 / 3y_1 + \frac{2}{\sqrt{2}} y_2 \le 6, y_1 \ge 0, y_2 \ge 0\right\}$$

Besides $y^* = \left(\frac{5}{2}, 2\sqrt{2}\right)$. The projection of the point y* onto Ω' gives the following point y_0:

$$y_0 = \left(1, \frac{3}{\sqrt{2}}\right) \Rightarrow \bar{x} = \left(1, \frac{3}{2}\right) \text{ and } \underset{x\in\Omega}{\text{Max}}\ \varphi(x) = \varphi\left(1, \frac{3}{2}\right) = 11{,}500$$

Note that the projection of point x* onto Ω gives the point $\left(\frac{16}{13}, \frac{15}{13}\right)$ and $\varphi\left(\frac{16}{13}, \frac{15}{13}\right) = 11,207.$

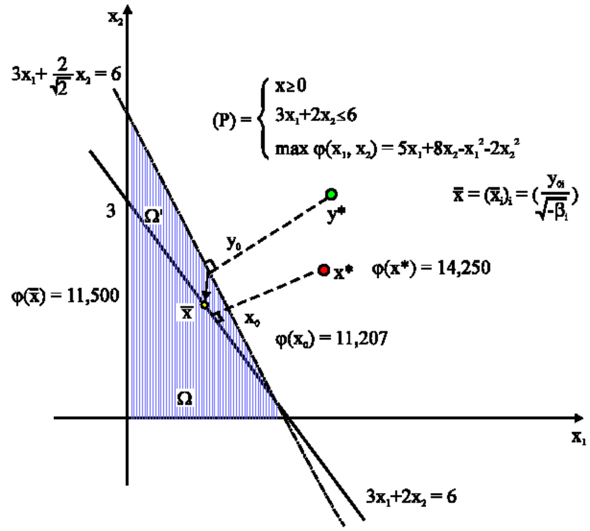

Fig. 2: The points: x*, y*, y_0 and $\bar{x}$

Obviously,

$$\underset{x\in\Omega}{\text{Max}}\ \varphi(x) \le \varphi(x^*) = \varphi\left(\frac{5}{2}, 2\right) = 14{,}250$$

The point $\left(\frac{16}{13}, \frac{15}{13}\right)$ of convex Ω is the nearest point of the critical point x* but it is not the optimal solution of φ.

Figure 2 shows the transformation of point x* in y*, the projection of y* in y_0 there and finally the calculation of the point $\bar{x}$ (optimal solution).

RESULTS AND DISCUSSION

After the decomposition of the objective function into two functions, one concave φ and the other convex Ψ where if the critical point belongs to omega (feasible constraint set), the optimal solution of f is the optimal solution of φ.

If the critical point does not belong to omega then the optimal solution of f is the projection of the critical point onto Ω' (new convex obtained by the translation T).

Convex Ω is replaced by a closed and lower bounded convex in R^n.

Unlike the previous techniques either analytical producing exact solution by going through linearization process or numerical producing approached solutions by Bierlaire (2006), Hillier and Lieberman (2005), Dong (2006) and De Klerk and Pasechnik (2007), the technique discussed here, produces the exact solution without going through linearization, thus it has enlarged the field of applications.

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